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Equations in HNN-extensions.

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Let $\mathbb{H}$ be a cancellative monoid and let $\mathbb{G}$ be an HNN-extension of $\mathbb{H}$ with finite associated subgroups $A, B \leq \mathbb{H}$. We show that, if equations are algorithmically solvable in $\mathbb{H}$, then they are also algorithmically solvable in $\mathbb{G}$. The result also holds for equations with rational constraints and for the existential first order theory. Analogous results are derived for amalgamated product with finite amalgamated subgroups.

Keywords: Equations; groups and monoids; HNN-extensions; amalgamated product.

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1. Introduction

We prove some tranfer theorems for the algorithmic solvability of equations in monoids. Our main result states that the satisfiability problem for a monoid $\mathbb{G}$, which is an HNN-extension of a cancellative monoid $\mathbb{H}$, with finite associated subgroups, is Turing-reducible to the same problem over the base monoid $\mathbb{H}$.
We then derive several corollaries, variations and extensions of this result:
- systems of equations (without rational constraints)
- amalgamated product instead of HNN-extension
- equations and inequations i.e. first-order existential logic

Contents

2. Preliminaries

We recall in this section all the needed definitions and classical results concerning partial semi-groups, semi-groups, monoids and groups.

2.1. *Partial semi-groups*

Let $(P, \cdot)$ be a set endowed with a function $\cdot : P \times P \to P$. By $D(\cdot) \subseteq P \times P$ we denote the domain of the law $\cdot$. The structure

$$\langle P, \cdot \rangle$$

is a *partial semi-group* iff, for every $p, q, r \in P$,

$$(p, q) \in D(\cdot) \wedge ((p \cdot q), r) \in D(\cdot) \Leftrightarrow (q, r) \in D(\cdot) \wedge (p, (q \cdot r)) \in D(\cdot)$$

and, in the case where $((p \cdot q), r) \in D(\cdot)$

$$(p \cdot q) \cdot r = p \cdot (q \cdot r).$$

Let us notice that, when P is a partial semi-group, the following structure is a semi-group:

$$\langle \mathcal{P}(P), \cdot \rangle$$

where the product is defined by , for every $R, S \in \mathcal{P}(P)$

$$R \cdot S = \{r \cdot s \mid (r, s) \in R \times S \cap D(\cdot)\}.$$

Let A, B be two groups. For every $K, K' \in \{A, B\}$, we denote by $\mathrm{PIs}(K, K')$ the set of all group isomorphisms φ with $\mathrm{dom}(\varphi) \subseteq K, \mathrm{im}(\varphi) \subseteq K'$.We then define

$$\mathrm{PGI}(A, B) := \mathrm{PIs}(A, A) \cup \mathrm{PIs}(A, B) \cup \mathrm{PIs}(B, A) \cup \mathrm{PIs}(B, B).$$

The pair $\langle \mathrm{PGI}(A, B), \circ \rangle$ is a partial semi-group.

2.2. *Monoids and groups*

2.2.1. *Subgroups of monoids*

Cancellative monoids
Subgroups of monoids

Induced partial isomorphisms, under the hypothesis that the monoid is cancellative.

2.2.2. *HNN-extensions*

Let us fix throughout this section, a monoid $\mathbb{H}$ (the base monoid), two finite subgroups $A \leq \mathbb{H}, B \leq \mathbb{H}$ and an isomorphism $\varphi : A \to B$. We then consider the HNN-extension

$$\mathbb{G} = \langle \mathbb{H}, t; t^{-1}at = \varphi(a)(a \in A)\rangle \tag{1}$$

We denote by

$$\pi_G : \mathbb{H} * \{t, \bar{t}\}^* \to \mathbb{G}$$

the homomorphism sending every $h \in \mathbb{H}$ on itself in $\mathbb{G}$ and mapping t to t (resp. $\bar{t}$ to t^{-1}).

The kernel of π_G coincides with the congruence $\approx$ over $\mathbb{H} * \{t, \bar{t}\}^*$ generated by the set of rules

$$t\bar{t} \approx \bar{t}t \approx 1, \tag{2}$$

$$at \approx t\varphi(a), \quad \text{for all } a \in A \text{ and} \tag{3}$$

$$b\bar{t} \approx \bar{t}\varphi^{-1}(b), \quad \text{for all } b \in B. \tag{4}$$

An element of $s \in \mathbb{H} * \{t, \bar{t}\}^*$ can be viewed as a word over the alphabet $\mathbb{H} \cup \{t, \bar{t}\}$ which has the form:

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n, \tag{5}$$

where $n \in \mathbb{N}, \alpha_i \in \{+1, -1\}$, t^{-1} means the letter $\bar{t}$ and $h_i \in \mathbb{H}$.
We name t-sequence every such $s \in \mathbb{H} * \{t, \bar{t}\}^*$. The t-sequence s is said to be a *reduced sequence* iff it does not contain any factor of the form $\bar{t}at$ (with $a \in A$) nor $tb\bar{t}$ (with $b \in B$). We denote by $\mathrm{Red}(\mathbb{H}, t)$ the subset of $\mathbb{H} * \{t, \bar{t}\}^*$ consisting of all reduced sequences. Let us denote by $\sim$ the congruence over $H * \{t, \bar{t}\}^*$ generated by all the rules of type (3),(4) above. The set $\mathrm{Red}(\mathbb{H}, t)$ is saturated by the congruence $\sim$. The following lemma is fundamental.

Lemma 1. *Let s, s' be some reduced sequences. Then $s \approx s'$ if and only if $s \sim s'$.*

This lemma could be named "Britton's lemma for monoids". (See [LS05], for example, for a proof). We define the norm of a given sequence $s \in \mathbb{H} * \{t, \bar{t}\}^*$ by:

$$\|s\| = |s|_{\{t, \bar{t}\}}. \tag{6}$$

One can easily check that, for every $s, s' \in \mathbb{H} * \{t, \bar{t}\}^*$

$$\|s \cdot s'\| = \|s\| + \|s'\|; \quad \|s\| = 0 \Leftrightarrow s \in H. \tag{7}$$

The boolean norm of a t-sequence s is the boolean defined by

$$|||s||| = 1 \Leftrightarrow \|s\| \geq 1. \tag{8}$$

2.2.3. *Amalgamated products*

Standard definition.
Let us consider two monoids $\mathbb{H}_1, \mathbb{H}_2$, two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, and an isomorphism $\varphi : A_1 \to A_2$. The corresponding amalgamated product

$$\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1) \rangle$$

is defined by ... We name a-sequence every element s of $\mathbb{H}_1 * \mathbb{H}_2$. It has a unique decomposition under the form:

$$s = h_0 k_1 h_1 \cdots k_i h_i \cdots k_n h_n, \tag{9}$$

where $n \geq 0, h_1, \cdots, h_i, \cdots, h_{n-1} \in \mathbb{H}_2 - \{1\}$, $k_1, \cdots, k_i, \ldots, k_n \in \mathbb{H}_1 - \{1\}$ and $h_0, h_n \in \mathbb{H}_2$. We name *reduced* a-sequence every $s \in \mathbb{H}_1 * \mathbb{H}_2$ of the form:

$$s = h_0 k_1 h_1 \cdots k_i h_i \cdots k_n h_n, \tag{10}$$

where $n \geq 0, h_1, \cdots, h_i, \cdots, h_{n-1} \in \mathbb{H}_2 - A_2$, $k_1, \cdots, k_i, \ldots, k_n \in \mathbb{H}_1 - A_1$ and $h_0, h_n \in \mathbb{H}_2$. We denote by $\mathrm{Red}(\mathbb{H}_1 * \mathbb{H}_2)$ the set of all reduced a-sequences.
It is well-known that $\mathbb{G}$ is embedded into the HNN-extension

$$\hat{\mathbb{G}} = \langle \mathbb{H}_1 * \mathbb{H}_2, t; t^{-1} a t = \varphi(a) (a \in A_1) \rangle$$

by the map

$$h_1 \in \mathbb{H}_1 \mapsto t^{-1} h_1 t; \;\; h_2 \in \mathbb{H}_2 \mapsto h_2 \tag{11}$$

(see [LS77, Th. 2.6. p. 187]).

2.3. *Rational subsets of a monoid*

Let $\mathbb{M} = (M, \cdot, 1_M)$ be some monoid. The set

$$\mathrm{Rat}(\mathbb{M}) \in \mathcal{P}(\mathcal{P}(M))$$

is the smallest element of $\mathcal{P}(\mathcal{P}(M))$ which posseses the finite subsets of M and which is closed under the operations $\cup$ (the union operation), $\cdot$ (the product operation) and $*$ (the star operation, associating with a subset P the smallest submonoid of $\mathbb{M}$ containing P).

We introduced in [LS05], in the particular case where $\mathbb{M}$ is an HNN-extension, a kind of finite automata, that we call *t-automata*.
`Normalized t-automaton`:
- 3 commutative diagrams of [LS05] (stronger than just $\sim$-compatible.
- for every vertex-type θ,

$$\mu_{\mathcal{A}} \cap \tau^{-1}(\theta) \times \tau^{-1}(\theta) = \mathrm{Id}_Q \cap \tau^{-1}(\theta) \times \tau^{-1}(\theta).$$

We define a function $\mu_{\mathcal{A},1} : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathsf{B}(\mathsf{Q}_{\mathcal{A}})$ by: for every $s \in \mathbb{H} * \{t, \bar{t}\}^*, t \in \mathcal{T}$

$$\mu_{\mathcal{A},1}(t, s) = \mu_{\mathcal{A}}(s) \cap (\tau_{\mathcal{A}}^{-1}(\tau \mathrm{i}(t)) \times \tau_{\mathcal{A}}^{-1}(\tau \mathrm{e}(t))).$$

We establish in §3.4.1 that this map $\mu_{\mathcal{A},1}$ has a multiplicative property. For every $g \in \mathbb{G}$ we set:

$$\mu_{\mathcal{A},\mathbb{G}}(g) = \mu_{\mathcal{A},1}((1, H, \|\|s\|\|, 1, 1), s), \tag{12}$$

where s is any reduced t-sequence representing g. Since $\mathcal{A}$ is $\sim$-saturated, the value of $\mu_{\mathcal{A},1}(s)$ does not depend of the chosen representative s.

2.4. *Equations and inequations over a monoid*

2.4.1. *Equations*

Let $\mathbb{M} = (M, \cdot, 1_{\mathbb{M}})$ be some monoid and let

$$\mathcal{C} \in \mathcal{P}(\mathcal{P}(M)).$$

A *system of equations* $\mathcal{S}$, over $\mathbb{M}$, with variables in a set $\mathcal{U}$, is a family $((u_i, u'_i))_{i \in I}$ of elements of $\mathcal{U}^* \times \mathcal{U}^*$. This system is said to be *quadratic* if, for every $i \in I$,

$$|u_i| = 1, \ \ |u'_i| = 2.$$

A $\mathcal{C}$-*constraint* over $\mathcal{U}$ is a map $\mathsf{C} : \mathcal{U} \to \mathcal{C}$. A $\mathbb{M}$-solution of the system $\mathcal{S}$ with $\mathcal{C}$-constraint C is any monoid homomorphism

$$\sigma_{\mathbb{M}} : \mathcal{U}^* \to \mathbb{M}$$

fulfilling both conditions:

$$\forall i \in I, \sigma_{\mathbb{M}}(u_i) = \sigma_{\mathbb{M}}(u'_i) \tag{13}$$

$$\forall U \in \mathcal{U}, \sigma_{\mathbb{M}}(U) \in \mathsf{C}(U). \tag{14}$$

2.4.2. *Inequations*

A *system of equations and inequations* over $\mathbb{M}$ is a family $((u_i, c_i, u'_i))_{i \in I}$ of elements of $\mathcal{U}^* \times \{=, \neq\} \times \mathcal{U}^*$, where $\mathcal{U}$ is a set of variables.
A $\mathbb{M}$-solution of the system $((u_i, c_i, u'_i))_{i \in I}$ with constraint C is any monoid homomorphism

$$\sigma_{\mathbb{M}} : \mathcal{U}^* \to \mathbb{M}$$

fulfilling both conditions:

$$\forall i \in I, \sigma_{\mathbb{M}}(u_i) c_i \sigma_{\mathbb{M}}(u'_i) \tag{15}$$

and (14) above.

Special constraints In the case where, $\mathcal{C} = \{\{m\} \mid m \in M\} \cup \{M\}$, any system of equations with constraints in $\mathcal{C}$ is called a system of equations with *constants* (since the variables $U \in \mathcal{U}$ such that $\mathsf{C}(U)$ is a singleton are seen as constants in $\mathbb{M}$ while the variables $U \in \mathcal{U}$ such that $\mathsf{C}(U) = M$ are seen as variables without any constraint).
In the case where, $\mathcal{C} = \mathcal{B}(Rat(\mathbb{M}))$, (i.e. the boolean closure of the set of rational subsets of $\mathbb{M}$), any system of equations with constraints in $\mathcal{C}$ is called a system of equations with *rational constraints*. Partial Involution.

2.5. *Reductions among algorithmic problems*

Turing reduction from L_1 to L_2.
Turing reduction from L_1 to (L_2, L_3).
Informally: ...

3. AB-algebras

We define here the notion of AB-algebra, which turns out to be useful for handling equations with rational constraints in an HNN-extension.

3.1. *Types*

Let us consider the finite set of types $\mathcal{T}_6$ defined in [LS05]. We define a finite partial semi-group $\langle \mathcal{T}, \cdot \rangle$ as follows:

$$\mathcal{T} = \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6$$

where $\mathbb{B} = \langle \{0, 1\}, \cdot \rangle$ is the monoid of booleans; the partial product is defined by: for every $(p, b, q), (p', b', q') \in \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6$, if $q = p'$ then

$$(p, b, q) \cdot (p', b', q') = (p, b + b', q'),$$

otherwise the product is undefined. As we noticed in §2.1, $\langle \mathcal{P}(\mathcal{T}), \cdot \rangle$ is thus a semi-group.
We define an involutory map $\mathbb{I}_{\mathcal{R}} : \mathcal{T}_6 \to \mathcal{T}_6$ by:

$$(A, T) \mapsto (A, H), (A, H) \mapsto (A, T), (B, T) \mapsto (B, H), (B, H) \mapsto (B, T), (1, H) \mapsto (1, 1), (1, 1) \mapsto (1, H).$$

We then define an involution $\mathbb{I}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ by:

$$\mathbb{I}_{\mathcal{T}}(p, b, q) = (\mathbb{I}_{\mathcal{R}}(p), b, \mathbb{I}_{\mathcal{R}}(q)).$$

One can check that $\mathbb{I}_{\mathcal{T}}$ is an involutory anti-automorphism of $\mathcal{T}$. This involution induces an involutory semi-group anti-automorphism of $\langle \mathcal{P}(\mathcal{T}), \cdot \rangle$ that will be denoted by $\mathbb{I}_{\mathcal{T}}$ too. We associate to every element t of $\mathcal{T}$ an "initial type" $\tau\mathrm{i}(t) \in \mathcal{T}_6$, an "end type" $\tau\mathrm{e}(t) \in \mathcal{T}_6$, an "initial group" $\mathrm{Gi}(t) \in \{\{1\}, A, B\}$ and an "end group" $\mathrm{Ge}(t) \in \{\{1\}, A, B\}$:

$$\tau\mathrm{i}(p, b, q) = p, \quad \mathrm{Gi}(p, b, q) = p_1(p), \quad \tau\mathrm{e}(p, b, q) = q, \quad \mathrm{Ge}(p, b, q) = p_1(q).$$

Each element of $\mathcal{T}$ is called a *path-type* while the elements of $\mathcal{T}_6$ are called *vertex-types*. This terminology refers to the graph $\mathcal{R}$ exhibited on figure 1. (It is a variant of the t-automaton $\mathcal{R}_6$ defined in [LS05], which recognizes the set $\mathrm{Red}(\mathbb{H}, t)$). Let us call *atomic types* the path-types corresponding to the edges of $\mathcal{R}$ i.e. :

$$(A, T, 1, B, H), (B, T, 1, A, H) \tag{16}$$

$$(A, H, 0, B, T), (B, H, 0, A, T), \ (A, H, 0, A, T), (B, H, 0, B, T), \tag{17}$$

$$(1, H, 0, A, T), (1, H, 0, B, T), \quad (B, H, 0, 1, 1), (A, H, 0, 1, 1), \quad (1, H, 0, 1, 1) \tag{18}$$

$$(1, H, 0, 1, H), (A, T, 0, A, T), \ (B, T, 0, B, T), (B, H, 0, B, H), \ (A, H, 0, A, H), (1, 1, 0, 1, 1). \tag{19}$$

This set is closed under $\mathbb{I}_{\mathcal{T}}$. It is denoted by $\mathcal{TA}$. The partial submonoid generated by this set of atomic path-types will be denoted by $\mathcal{TR}$. It is closed under the involution $\mathbb{I}_{\mathcal{T}}$. The only path-types used in this work are those from $\mathcal{TR}$. We call T-types all

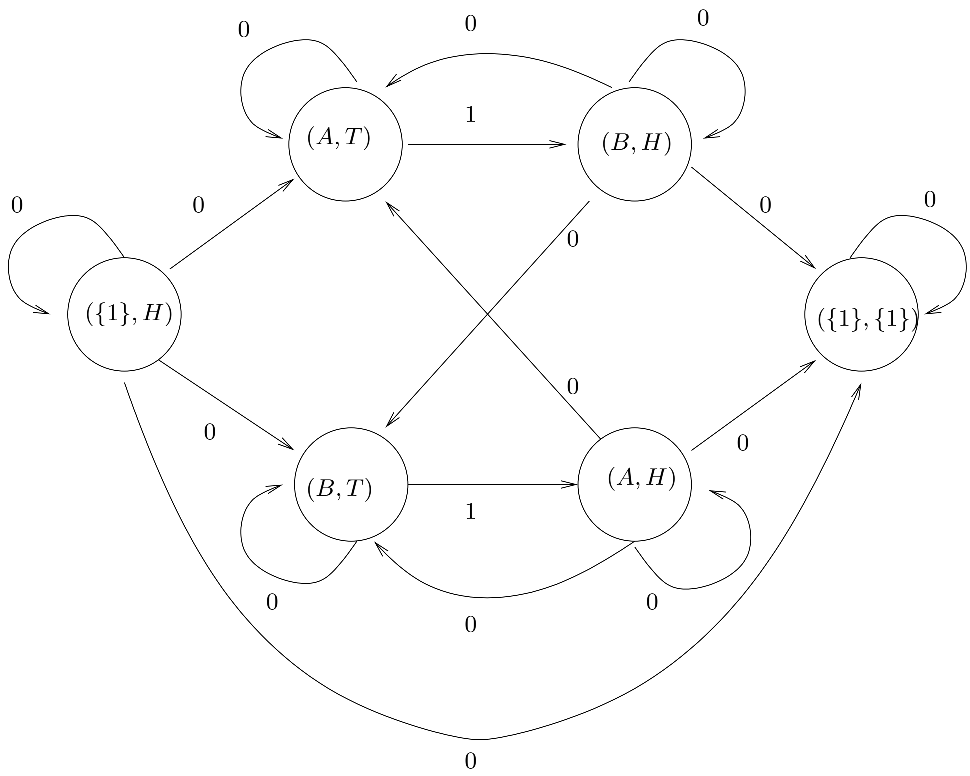

Fig. 1. graph $\mathcal{R}$

the atomic types listed in (16), H-types all the atomic types listed in (17)(18), $A \cup B$-types all the atomic types listed in (19). Note that some types are non-atomic: for example $(A, T, 1, A, H) = (A, T, 1, B, H) \cdot (B, H, 0, B, T) \cdot (B, T, 1, A, H)$ is an element of $\mathcal{TR} - \mathcal{TA}$.

3.2. *AB-algebra axioms*

Let A, B be two groups (what we have in mind are the two subgroups A, B of $\mathbb{H}$ leading to the HNN-extension $\mathbb{G}$ defined by (1)) and Q be some finite set (we have in mind the set of states of some t-automaton $\mathcal{A}$ over $\mathbb{H} * \{t, \bar{t}\}^*$). Let us denote by $\mathsf{B}(\mathsf{Q})$ the monoid $(\mathcal{P}(\mathsf{Q} \times \mathsf{Q}))$ of binary relations over Q and by $\mathsf{B}^2(\mathsf{Q})$ the direct product of the monoid $\mathsf{B}(\mathsf{Q})$ by itself. Given $m \in \mathsf{B}(\mathsf{Q}), m^{-1}$ is the binary relation $m^{-1} = \{(p, q) \in \mathsf{Q} \times \mathsf{Q} \mid (q, p) \in m\}$. We consider the involutory monoid anti-isomorphism $\mathbb{I}_{\mathsf{Q}} : \mathsf{B}^2(\mathsf{Q}) \to \mathsf{B}^2(\mathsf{Q})$ defined by

$$\forall m, m' \in \mathsf{B}(\mathsf{Q}), \;\; \mathbb{I}_{\mathsf{Q}}(m, m') = (m'^{-1}, m^{-1}).$$

We call AB-algebra a structure of the form

$$\langle \mathbb{M}, \cdot, 1_{\mathbb{M}}, \mathbb{I}, \iota_A, \iota_B, \gamma, \mu, \delta \rangle \tag{20}$$

where $\iota_A : A \to \mathbb{M}, \iota_B : B \to \mathbb{M}$ are total maps, $\mathbb{I} : \mathbb{M} \to \mathbb{M}$ is a partial map, $\gamma : \mathbb{M} \to \mathcal{P}(\mathcal{T})$ is a total map, $\mu : \mathcal{T} \times \mathbb{M} \to \mathsf{B}^2(\mathsf{Q})$ is a total map, $\delta : \mathcal{T} \times \mathbb{M} \to \mathrm{PGI}(A, B)$ is a total map ;
fulfilling the eleven axioms (21-31) below.
monoid:

$$(\mathbb{M}, \cdot, 1_{\mathbb{M}}) \text{ is a monoid },\tag{21}$$

embeddings:

$$\iota_A, \iota_B \text{ are injective monoid homomorphisms },\tag{22}$$

involution I:

$$\iota_A(A) \cup \iota_B(B) \subseteq \mathrm{dom}(\mathbb{I}) \subseteq \mathbb{M} - \gamma^{-1}(\{\emptyset\})\tag{23}$$

for every $m, m' \in \mathbb{M}$,

$$[\gamma(m) \cdot \gamma(m') \neq \emptyset] \Rightarrow [m \cdot m' \in \mathrm{dom}(\mathbb{I}) \Leftrightarrow (m \in \mathrm{dom}(\mathbb{I}) \wedge m' \in \mathrm{dom}(\mathbb{I}))]\tag{24}$$

$$\mathbb{I} : (\mathrm{dom}(\mathbb{I}), \cdot, 1_{\mathbb{M}}) \to (\mathrm{dom}(\mathbb{I}), \cdot, 1_{\mathbb{M}}) \text{ is a monoid anti-isomorphism, } \mathbb{I} \circ \mathbb{I} = \mathbb{I},\tag{25}$$

almost homomorphisms:
for every $m, m' \in \mathbb{M}$,

$$\gamma(m \cdot m') \supseteq \gamma(m) \cdot \gamma(m'),\tag{26}$$

for every $m, m' \in \mathbb{M}, t \in \gamma(m), t' \in \gamma(m')$, such that $(t, t') \in D(\cdot)$,

$$\mu(t \cdot t', m \cdot m') = \mu(t, m) \cdot \mu(t', m'),\tag{27}$$

$$\mathrm{dom}(\delta(t, m)) \subseteq \mathrm{Gi}(t), \mathrm{im}(\delta(t, m)) \subseteq \mathrm{Ge}(t)\tag{28}$$

$$\delta(t \cdot t', m \cdot m') = \delta(t, m) \circ \delta(t', m').\tag{29}$$

commutation with I:
for every $a \in A, b \in B, m \in \mathrm{dom}(\mathbb{I}), t \in \gamma(m)$,

$$\mathbb{I}(\iota_A(a)) = \iota_A(a^{-1});\;\; \mathbb{I}(\iota_B(b)) = \iota_B(b^{-1})\tag{30}$$

$$\gamma(\mathbb{I}(m)) = \mathbb{I}_{\mathcal{T}}(\gamma(m));\;\; \mu(\mathbb{I}_{\mathcal{T}}(t), \mathbb{I}(m)) = \mathbb{I}_{\mathsf{Q}}(\mu(t, m));\;\; \delta(\mathbb{I}_{\mathcal{T}}(t), \mathbb{I}(m)) = \delta(t, m)^{-1}.\tag{31}$$

Axiom (25) includes the assumption that $\mathrm{dom}(\mathbb{I})$ is a submonoid. From now on, we denote by $\hat{\mathbb{M}}$ this submonoid.

3.3. *AB-homomorphisms*

Let
$\mathcal{M}_1 = \langle \mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}, \iota_{A,1}, \iota_{B,1}, \mathbb{I}_1, \gamma_1, \mu_1, \delta_1 \rangle, \mathcal{M}_2 = \langle \mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}, \iota_{A,2}, \iota_{B,2}, \mathbb{I}_2, \gamma_2, \mu_2, \delta_2 \rangle$ be two AB-algebras with the same underlying groups A, B and set Q. We call AB-homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ any map $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ fulfilling the seven properties (32-38) below:
m-homomorphism:

$$\psi : (\mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}) \to (\mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}) \text{ is a monoid homomorphism} \tag{32}$$

ι-preservation:

$$\forall a \in A, \forall b \in B, \psi(\iota_{A,1}(a)) = \iota_{A,2}(a), \;\; \psi(\iota_{B,1}(b)) = \iota_{B,2}(b) \tag{33}$$

I-preservation:

$$\forall m \in \mathbb{M}_1 - \gamma_1^{-1}(\{\emptyset\}), \;\; m \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi(m) \in \mathrm{dom}(\mathbb{I}_2) \tag{34}$$

$$\forall m \in \hat{\mathbb{M}}_1, \;\; \mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m)) \tag{35}$$

γ-compatibility:

$$\forall m \in \mathbb{M}_1, \gamma_2(\psi(m)) \supseteq \gamma_1(m) \tag{36}$$

μ-preservation:

$$\forall m \in \mathbb{M}_1, \forall t \in \gamma_1(m), \mu_2(t, \psi(m)) = \mu_1(t, m), \tag{37}$$

δ-preservation:

$$\forall m \in \mathbb{M}_1, \forall t \in \gamma_1(m), \delta_2(t, \psi(m)) = \delta_1(t, m). \tag{38}$$

$\mathcal{M}_1$ is said to be a *sub-AB-algebra* of $\mathcal{M}_2$ if $\mathbb{M}_1 \subseteq \mathbb{M}_2$ and the inclusion map $\iota : \mathbb{M}_1 \to \mathbb{M}_2$ is an AB-homomorphism.

3.4. *The AB-algebra* $\mathbb{H}_t$

3.4.1. $\mathbb{H} * \{t, \bar{t}\}^*$

`subscript t everywhere` Given an HNN-extension (1) and a partitionned, $\sim$-saturated finite t-automaton $\mathcal{A}$, we define an AB-algebra with underlying monoid $\mathbb{H} * \{t, \bar{t}\}^*$ and set of states $\mathsf{Q}_{\mathcal{A}}$.

$$\langle \mathbb{H} * \{t, \bar{t}\}^*, \cdot, 1_{\mathbb{H}}, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma, \delta \rangle \tag{39}$$

as follows:

$$\iota_A, \iota_B$$

are the natural injections from A (resp. B) into $\mathbb{H} * \{t, \bar{t}\}^*$,

$$\mathrm{dom}(\mathbb{I}) = (I(\mathbb{H}) \cup \{t, \bar{t}\})^*$$

where $I(\mathbb{H})$ is the set of invertible elements of $\mathbb{H}$; $\mathbb{I}$ is the unique monoid anti-homomorphism $\mathrm{dom}(\mathbb{I}) \to \mathrm{dom}(\mathbb{I})$ such that

$$\forall h \in I(\mathbb{H}), \mathbb{I}(h) = h^{-1};\ \ \mathbb{I}(t) = \bar{t};\ \ \mathbb{I}(\bar{t}) = t.$$

The map γ is defined by: for every $s \in \mathbb{H} * \{t, \bar{t}\}^*$,

$$\gamma(s) = \{(p, b, q) \in \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6 \mid (p, q) \in \mu_{\mathcal{R}_6}(s) \wedge b = (\|s\| \neq 0)\} \tag{40}$$

where $(\|s\| \neq 0)$ is the boolean 1 if and only if it is true that $(\|s\| \neq 0)$. The map $\mu : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathsf{B}^2(\mathsf{Q})$ is defined by:

$$\begin{aligned} \mu(t, s) &= (\mu_1(t, s), (\mu_1(\mathbb{I}_{\mathcal{T}}(t), \mathbb{I}_t(s))^{-1})) && \text{if } s \in \mathrm{dom}(\mathbb{I}_t);\\ \mu(t, s) &= \qquad (\mu_1(t, s), \emptyset) && \text{if } s \notin \mathrm{dom}(\mathbb{I}_t). \end{aligned} \tag{41}$$

The map $\delta : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathrm{PGI}(A, B)$ is defined by:

$$\delta(t, s) = \{(g, g') \in \mathrm{Gi}(t) \times \mathrm{Ge}(t) \mid g \cdot s \sim s \cdot g'\}.$$

It is noteworthy that, for every $s \in \mathbb{H} * \{t, \bar{t}\}^*$

$$\gamma(s) \neq \emptyset \Leftrightarrow s \in \mathrm{Red}(\mathbb{H}, t). \tag{42}$$

Proposition 2. *The above structure $\langle \mathbb{H} * \{t, \bar{t}\}^*, \cdot, 1_{\mathbb{H}}, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma, \delta \rangle$ is an AB-algebra.*

In order to prove this proposition we show several lemmas.

Lemma 3. *For every $s, s' \in \mathbb{H} * \{t, \bar{t}\}^*, \gamma(s \cdot s') \supseteq \gamma(s) \cdot \gamma(s')$.*

Sketch of proof: Let Γ be the homomorphism from the partial semi-group of paths in $\mathcal{R}$ into the partial semi-group $\mathcal{T}$ which associates to every edge of $\mathcal{R}$ the same edge, viewed as an atomic path-type. Let $t \in \mathcal{T}, s \in \mathrm{Red}(\mathbb{H}, t)$. By $\mathrm{Path}(t, s)$ we denote the unique path in the graph $\mathcal{R}$ such that $\Gamma(\mathrm{Path}(t, s)) = t$ and $\Lambda(\mathrm{Path}(t, s)) = s$ (where Λ is the labelling map). By definition, a path-type t belongs to $\gamma(s)$ iff $\mathrm{Path}(t, s)$ exists. Suppose that $s, s' \in \mathrm{Red}(\mathbb{H}, t), t \in \gamma(s), t' \in \gamma(s')$, and $(t, t') \in D(\cdot)$. Then the product $\mathrm{Path}(t, s) \cdot \mathrm{Path}(t', s')$ is defined too, and

$$\Gamma(\mathrm{Path}(t, s) \cdot \mathrm{Path}(t', s')) = t \cdot t',\ \ \Lambda(\mathrm{Path}(t, s) \cdot \mathrm{Path}(t', s')) = s \cdot s',$$

showing that

$$\gamma(s \cdot s') \supseteq \gamma(s) \cdot \gamma(s').$$

$\square$

In order to check multiplicativity of the mapping μ, we claim the following facts

Claim 4.

1- $\mu_0(t) = \mu_1((A, T, 1, B, H), t)$
2- $\mu_0(\bar{t}) = \mu_1((B, T, 1, A, H), \bar{t})$
3- $\mu_0(h) = \cup_{\theta \in \gamma(h)} \mu_1(\theta, h)$
4- $\mu_1((A, T, 1, B, H), (\prod_{i=1}^{n-1} t^{\alpha_i} h_i) t^{\alpha_n}) = \prod_{i=1}^{n-1} \mu_0(t^{\alpha_i}) \mu_0(h_i) \mu_0(t^{\alpha_n})$.

Sketch of proof: Points (1)(2)(3) follow directly from the definition of μ. Point (4) follows from the property of $\mathcal{A}$:

$$\mu_1((\theta,0,\theta),1) = \mathrm{Id}_Q \cap \tau^{-1}(\theta) \times \tau^{-1}(\theta).$$

$\square$

Lemma 5. *Let $h,k \in H, \theta,\theta',\theta'' \in \mathcal{T}_6$, such that $(\theta,0,\theta') \in \gamma(h), (\theta',0,\theta'') \in \gamma(k)$ and $(\theta,0,\theta'') \in \mathcal{T}$. Then $\mu_1((\theta,0,\theta')\cdot(\theta',0,\theta''),h\cdot k) = \mu_1((\theta,0,\theta'),h)\cdot \mu_1((\theta',0,\theta''),k)$.*

Sketch of proof: If $\theta' \notin \{\theta,\theta''\}$, then the product $(\theta,0,\theta'') \notin \mathcal{T}$.
Let us suppose that $\theta' = \theta$. Hence h belongs to the subgroup $p_1(\theta)$, and thus, loops `precise ref` of [LS05] show that $\mu_1((\theta,0,\theta''),h\cdot k) = \mu_1((\theta,0,\theta),h)\cdot \mu_1((\theta,0,\theta''),k)$.
The case where $\theta' = \theta''$ can be treated symmetrically. $\square$

Lemma 6.
Let $n \geq 1$, θ,θ' be vertex-types and $\alpha_1,\ldots,\alpha_n \in \{-1,+1\}, h_0,h_1,\ldots,h_n \in \mathbb{H}$. Let $s = (\prod_{i=1}^{n-1} t^{\alpha_i} h_i)t^{\alpha_n}$. Then

$$\mu_1((\theta,1,\theta'),h_0 s h_n) = \mu_1((\theta,0,A(\alpha_1),T),h_0)\cdot\mu_1((A(\alpha_1),T,1,B(\alpha_n),H),s)\cdot\mu_1((B(\alpha_n),H,0,\theta'),h_n).$$

Sketch of proof: Follows immediately from the definitions of $\mu_{\mathcal{A}}$ and μ_1. $\square$

Lemma 7. *For every path-types t,t' and every t-sequences $s,s' \in \mathbb{H} * \{t,\bar{t}\}^*$, if $t \in \gamma(s)$, $t' \in \gamma(s')$ and $t\cdot t'$ is defined, then $\mu_1(t\cdot t', s\cdot s') = \mu_1(t,s)\cdot\mu_1(t',s')$.*

Sketch of proof: Suppose that $t = (\theta,1,\theta'), t' = (\theta',1,\theta'')$, $s = h\check{s}k, s' = h'\check{s}'k'$ where $\check{s} = (\prod_{i=1}^{n-1} t^{\alpha_i}h_i)t^{\alpha_n}$, $\check{s}' = (\prod_{j=1}^{m-1} t^{\beta_j}h'_j)t^{\beta_m}$.
Case 1: $\theta' = (B(\alpha_n),H)$.
Let us determine

$$\mu_1((\theta,1,\theta''),s\cdot s'). \tag{43}$$

By lemma 6 it is equal to

$$\mu_1((\theta,0,A(\alpha_1),T),h_0)\cdot\mu_1((A(\alpha_1),T,1,B(\beta_m),H),h),\check{s}\cdot kh\cdot\check{s}')\cdot\mu_1((B(\beta_m),H,0,\theta''),k'). \tag{44}$$

By claim 4, point (4), this can be rewritten:

$$\mu_1((\theta,0,A(\alpha_1),T),h_0)\cdot\prod_{i=1}^{n-1}\mu_0(t^{\alpha_i})\mu_0(h_i)\mu_0(t^{\alpha_n})\cdot\mu_0(kh)\cdot\prod_{j=1}^{m-1}\mu_0(t^{\beta_j})\mu_0(h'_j)\mu_0(t^{\beta_m})\mu_1((B(\beta_m),H,0,\theta''),k'). \tag{45}$$

but since the image of $\mu_0(t^{\alpha_n})$ is included in $\tau^{-1}(B(\alpha_n), H)$ and the domain of $\mu_0(t^{\beta_1})$ is included in $\tau^{-1}(A(\beta_1), T)$, the factor $\mu_0(kh)$ in formula (45) can be replaced by

$$\mu_1((B(\alpha_n), H, 0, A(\beta_1), T), kh),$$

which, by lemma 5 can be replaced by

$$\mu_1((B(\alpha_n), H, 0, B(\alpha_n), H), k) \cdot \mu_1((B(\alpha_n), H, 0, A(\beta_1), T), h).$$

After this replacement in formula (45) we obtain:

$$\mu_1((\theta, 0, A(\alpha_1), T), h_0) \cdot \prod_{i=1}^{n-1} \mu_0(t^{\alpha_i})\mu_0(h_i)\mu_0(t^{\alpha_n}) \cdot \ \mu_1(B(\alpha_n), H, 0, B(\alpha_n), H), k)$$
$$\cdot \mu_1((B(\alpha_n), H, 0, A(\beta_1), T), h) \cdot \prod_{j=1}^{m-1} \mu_0(t^{\beta_j})\mu_0(h'_j)\mu_0(t^{\beta_m}) \cdot \ \mu_1((B(\beta_m), H, 0, \theta''), k').$$

Using lemma 6 backwards and twice, we obtain

$$\mu_1((\theta, 0, B(\alpha_n), H), s) \cdot \mu_1((B(\alpha_n), H, 0, \theta''), s') \tag{46}$$

which is exactly

$$\mu_1((\theta, 1, \theta'), s) \cdot \mu_1((\theta', 1, \theta''), s') \tag{47}$$

We have established that (43) and (47) have the same value, as required.
Case 2: $\theta' = (A(\beta_1), T)$.
Symetric arguments can be applied.
Since every other value of θ' makes impossible that both $(\theta, 1, \theta') \in \gamma(s)$ and $(\theta', 1, \theta'') \in \gamma(s')$, we have treated all the possible cases.
It remains to treat the case where $s \in H$ or $s' \in H$:
Case 3: $s \in H, s' \notin H$.
By lemma 5,

$$\mu_1((\theta, 0, A(\beta_1), T)), s \cdot h') = \mu_1((\theta, 0, \theta'), s) \cdot \mu_1((\theta', 0, A(\beta_1), T), h').$$

Applying lemma 6 we get

$$\begin{aligned} \mu_1((\theta, 0, \theta''), s \cdot s') &= \mu_1((\theta, 0, A(\beta_1), T), sh') \cdot \mu_1((A(\beta_1), T, 0, \theta''), \check{s}'k') \\ &= \mu_1((\theta, 0, \theta'), s) \cdot \mu_1((\theta', 0, \theta''), h'\check{s}'k') \end{aligned}$$

as required.
Case 4: $s \notin H, s' \in H$.
can be handled as case 3.
Case 5: $s \in H, s' \in H$.
This case is treated by lemma 5. □
From the multiplicativity of μ_1 we can immediately deduce the multiplicativity of μ. All the other checks are easy.

Positive AB-structure Another AB-structure over $\mathbb{H} * \{t, \bar{t}\}^*$ can be defined by choosing, in place of the map γ defined in (**??**), the map map γ_+ defined by: for every $s \in \mathbb{H} * \{t, \bar{t}\}^*$,

$$\gamma(s) = \{(p, b, q) \in \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6 \mid (p, q) \in \mu_{\mathcal{G}_6}(s) \wedge b = (\|s\| \neq 0)\} \tag{48}$$

Assertion 42 is now replaced by

$$\forall s \in \mathbb{H} * \{t, \bar{t}\}^*, \gamma_+(s) \neq \emptyset. \tag{49}$$

We call the resulting structure

$$\langle \mathbb{H} * \{t, \bar{t}\}^*, \cdot, 1_{\mathbb{H}}, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma_+, \delta \rangle \tag{50}$$

the *positive* AB-algebra structure over $\mathbb{H} * \{t, \bar{t}\}^*$. This variant will be used in §9 where we deal with *positive* rational constraints.

3.4.2. $\mathbb{H}_t$

One can check that the monoid-congruence $\sim$ is compatible with $\mathbb{I}, \iota_A, \iota_B, \gamma, \mu, \delta$ in the sense that: for every $s, s' \in \mathbb{H} * \{t, \bar{t}\}^*, a \in A, b \in B, t \in \gamma(s)$, if $s \sim s'$ then,

$$\mathbb{I}(s) \sim \mathbb{I}(s'), \quad s = \iota_A(a) \Leftrightarrow s' = \iota_A(a), \quad s = \iota_B(b) \Leftrightarrow s' = \iota_B(b),$$

$$\gamma(s) = \gamma(s'), \quad \mu(t, s) = \mu(t, s'), \quad \delta(t, s) = \delta(t, s').$$

Let us denote by $\mathbb{H}_t$ the quotient set $\mathbb{H} * \{t, \bar{t}\}^* / \sim$. We can naturally endow $\mathbb{H}_t$ with a structure of AB-algebra:

$$\mathcal{H}_t := \langle \mathbb{H}_t, \cdot, 1_{\mathbb{H}}, \iota_{A,\sim}, \iota_{B,\sim}, \mathbb{I}_\sim, \mu_\sim, \gamma_\sim, \delta_\sim \rangle \tag{51}$$

where all the required maps are just obtained from the corresponding map in the AB-structure of $\mathbb{H} * \{t, \bar{t}\}^*$, by composition by $\pi_\sim : \mathbb{H} * \{t, \bar{t}\}^* \to \mathbb{H} * \{t, \bar{t}\}^* / \sim$. In addition

$$s \sim s' \Rightarrow \|s\| = \|s'\|$$

so that the notion of norm remains well-defined in the quotient $\mathbb{H}_t$.

Positive AB-structure Similarly, we can define the structure of AB-algebra:

$$\mathcal{H}_{t+} := \langle \mathbb{H}_t, \cdot, 1_{\mathbb{H}}, \iota_{A,\sim}, \iota_{B,\sim}, \mathbb{I}_\sim, \mu_\sim, \gamma_{+\sim}, \delta_\sim \rangle \tag{52}$$

where γ_+ is defined via the automaton $\mathcal{G}_6$ instead of $\mathcal{R}_6$, see (48).

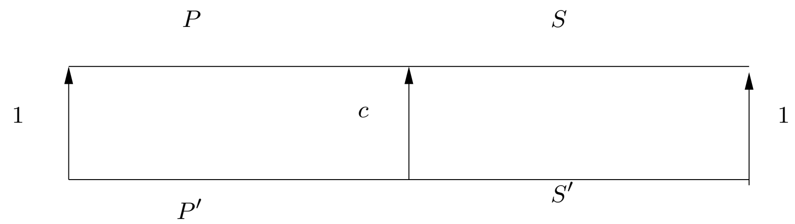

Fig. 2. Lemma 8

3.4.3. *Algebraic properties*

Lemma 8. *Let $P, P', S, S' \in \mathbb{H}_t$. Suppose that $PS = P'S'$,P, P' have an atomic H-type and $\gamma(P) \cdot \gamma(S) = \gamma(P') \cdot \gamma(S') \neq \emptyset$. Then, there exists $c \in \gamma_3(P)$ such that*

$$P = P'c, \quad cS = S'.$$

Lemma 9. *Let $P, P', S, S' \in \mathbb{H}_t$. Suppose that*

$$PS = P'S', P, P' \text{ have a T-type } \text{ and } \gamma(P) \cdot \gamma(S) = \gamma(P') \cdot \gamma(S') \neq \emptyset.$$

One of the following cases must occur:
1- $\|P\| = \|P'\|$ and there exists $c \in \gamma_3(P)$ such that

$$P = P'c, \quad cS = S'$$

2- $\|P\| < \|P'\|$ and there exist $c \in \gamma_3(P)$,P'_1,P'_2,$P'_3 \in \mathbb{H}_t$,P'_1, P'_3 have a T-type,P'_2 has a H-type ,$\gamma(P'_3) \cdot \gamma(S') \neq \emptyset$ and

$$P = P'_1 c, \quad P' = P'_1 P'_2 P'_3, \quad cS = P'_2 P'_3 S'.$$

3- $\|P\| > \|P'\|$ and there exists $c \in \gamma_3(P)$,P_1,P_2,$P_3 \in \mathbb{H}_t$,P_1, P_3 have a T-type ,P_2 has a H-type, $\gamma(P_3) \cdot \gamma(S) \neq \emptyset$ and

$$P = P_1 P_2 P_3, \quad P_1 = P'c, \quad cP_2 P_3 S = S'.$$

3.5. *The AB-algebra* $\mathbb{W}$

3.5.1. $\mathcal{W}^* * A * B$

Let $\mathcal{S}$ be a system of equations over $\mathbb{H}_t$ with involution and rational constraints. The rational constraints are expressed via the map μ_t defined by (41) in §3.4.1. We define an alphabet of "generic" symbols $\mathcal{W}$ with the underlying idea of representing inside each symbol the values of the functions $\gamma_t, \mu_t, \delta_t$ for the "concrete" value (i.e.

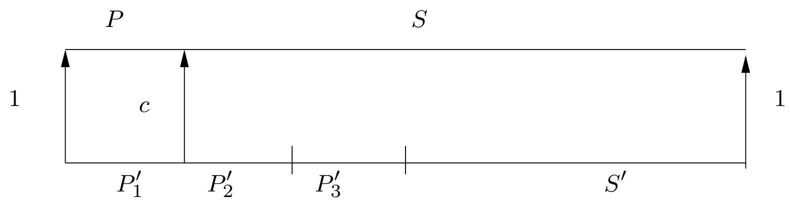

Fig. 3. Lemma 9

in $\mathbb{H} * \{t, \bar{t}\}^*$) of that variable that leads to a solution of the system of equations. Let $\mathcal{V}_0$ be some starting set. We then define

$$\Omega := \mathcal{V}_0 \times \{-1, 0, 1\} \times \mathcal{TA} \times \mathsf{B}^2(\mathsf{Q}) \times \mathrm{PGI}(A, B). \tag{53}$$

$$\mathcal{W} := \{(V, \epsilon, t, m, \varphi) \in \Omega \mid \varphi \in \mathrm{PIs}(\mathrm{Gi}(t), \mathrm{Ge}(t)), \forall (c, d) \in \varphi, \mu_{\mathcal{A}}(c) \cdot m = m \cdot \mu_{\mathcal{A}}(d)\}. \tag{54}$$

Let

$$\check{\mathcal{W}} = \{W \in \mathcal{W} \mid p_2(W) = 0\}, \;\; \hat{\mathcal{W}} = \{W \in \mathcal{W} \mid p_2(W) \neq 0\}$$

Let us consider the free product $\mathcal{W}^* * A * B$. We denote by $\iota_A : A \to \mathcal{W}^* * A * B$ the natural embedding of A into $\mathcal{W}^* * A * B$ (and use similarly the notation ι_B). Note that $\iota_A(A) \cap \iota_B(B) = \{1\}$. We define an AB-algebra with underlying monoid $\mathcal{W}^* * A * B$ and set of states Q

$$\langle \mathcal{W}^* * A * B, \cdot, 1, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma, \delta \rangle \tag{55}$$

as follows:

$$\mathrm{dom}(\mathbb{I}) = \hat{\mathcal{W}}^* * A * B$$

(the submonoid generated by $\hat{\mathcal{W}} \cup \iota_A(A) \cup \iota_B(B)$) $\mathbb{I}$ is the unique monoid anti-homomorphism $\hat{\mathcal{W}}^* * A * B \to \hat{\mathcal{W}}^* * A * B$ such that:

$$\forall a \in A, \mathbb{I}(\iota_A(a)) = \iota_A(a^{-1}); \;\; \forall b \in B, \mathbb{I}(\iota_B(b)) = \iota_B(b^{-1})$$

$$\mathbb{I}(V, \epsilon, t, m, \varphi) = (V, -\epsilon, \mathbb{I}_{\mathcal{T}}(t), \mathbb{I}_{\mathsf{Q}}(m), \varphi^{-1}).$$

$$\gamma : \mathcal{W}^* * A * B \to \mathcal{P}(\mathcal{T})$$

is defined by:

$$\gamma(V, \epsilon, t, m, \varphi) = \{t\},$$

for every $a \in A - \{1\}, b \in B - \{1\}$

$$\gamma(\iota_A(a)) = \{(A, T, 0, A, T), (A, H, 0, A, H)\}, \;\; \gamma(\iota_B(b)) = \{(B, T, 0, B, T), (B, H, 0, B, H)\},$$

$$\gamma(1) = \{(1, H, 0, 1, H), (1, 1, 0, 1, 1), (A, T, 0, A, T), (A, H, 0, A, H), (B, T, 0, B, T), (B, H, 0, B, H)\}$$

and finally, for every $g_1, \ldots, g_i, \ldots, g_n \in \mathcal{W} \cup \iota_A(A) \cup \iota_B(B)$,

$$\gamma(\prod_{i=1}^{n} g_i) = \prod_{i=1}^{n} \gamma(g_i) \tag{56}$$

μ is defined by

$$\mu(t, \iota_A(a)) = \mu_t(t, a), \;\; \mu(t, \iota_B(b)) = \mu_t(t, b),$$

for every $t' \in \mathcal{T}, t' \neq t$,

$$\mu(t, (V, \epsilon, t, m, \varphi)) = m, \;\; \mu(t', (V, \epsilon, t, m, \varphi)) = \emptyset.$$

The map $\delta : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathrm{PGI}(A, B)$ is defined by: for every $a \in \iota_A(A), b \in \iota_B(B), t \in \mathcal{T}$,

$$\delta(t, \iota_A(a)) = \{(\iota_A(c), \iota_A(d)) \mid (c, d) \in \mathrm{Gi}(t) \times \mathrm{Ge}(t), ca = ad\}, \text{ if } t \in \gamma(\iota_A(a))$$

$$\delta(t, \iota_A(a)) = \{(1, 1)\}, \text{ if } t \notin \gamma(\iota_A(a))$$

$$\delta(t, \iota_B(b)) = \{(\iota_B(c), \iota_B(d)) \mid (c, d) \in \mathrm{Gi}(t) \times \mathrm{Ge}(t), ca = ad\}, \text{ if } t \in \gamma(\iota_B(b))$$

$$\delta(t, \iota_B(b)) = \{(1, 1)\}, \text{ if } t \notin \gamma(\iota_B(b)).$$

For every $t' \in \mathcal{T}, t' \neq t$,

$$\delta(t, (V, \epsilon, t, m, \varphi)) = \varphi, \;\; \delta(t', (V, \epsilon, t, m, \varphi)) = \{(1, 1)\}$$

Since for every $W \in \mathcal{W}$, $\gamma(W)$ is simply a singleton of the form $\{t\}$, we also use the (abusive) notation

$$\tau\mathrm{i}(W), \tau\mathrm{e}(W), \mathrm{Gi}(W), \mathrm{Ge}(W), \mu(W), \delta(W)$$

for what should be denoted, in full rigor, by

$$\tau\mathrm{i}(t), \tau\mathrm{e}(t), \mathrm{Gi}(t), \mathrm{Ge}(t), \mu(t, W), \delta(t, W).$$

lengths For every $w \in (\iota_A(A) \cup \iota_B(B) \cup \mathcal{W})^*$, we set

$$\|w\| = |w|_{\mathcal{W}}.$$

One can check that

$$\|w \cdot w'\| = \|w\| + \|w'\|; \quad \|w\| = 0 \Leftrightarrow w \in A * B.$$

$$\chi_{AB}(w) := 1 \text{ if } w \in \iota_A(A) \cup \iota_B(B); \chi_{AB}(w) := 0 \text{ otherwise }.$$

$$\chi_H(w) := 1 \text{ if } \mathrm{Card}(\gamma(w)) = 1 \wedge p_2(\gamma(w)) \in \{H\}; \chi_H(w) := 0 \text{ otherwise }.$$

Given $P, S, P', S' \in \mathbb{W}$ and $\psi_t \in \mathrm{Hom}_{AB}(\mathbb{W}, \mathbb{H}_t)$ we define

$$\Delta(P, S, P', S', \psi_t) = 1 - \frac{1}{2}(\chi_{AB}(P) + \chi_{AB}(P')) + \chi_H(S) + \chi_H(S') + 2\|\psi_t(S)\| + 2\|\psi_t(S')\| \tag{57}$$

3.5.2. $\mathbb{W}$

Let us consider the monoid congruence $\equiv$ over $\mathcal{W}^* * A * B$ generated by the set of pairs

$$cW \equiv Wd, \tag{58}$$

for all $W \in \mathcal{W}, (c,d) \in \delta(W)$. We define the monoid $\langle \mathbb{W}, \cdot, 1_{\mathbb{W}} \rangle$ as the quotient-monoid $\mathcal{W}^* * A * B/\equiv$. One can check that the monoid-congruence $\equiv$ is compatible with $\mathbb{I}, \iota_A, \iota_B, \gamma, \mu, \delta$ in the sense that: for every $u, u' \in \mathcal{W}^* * A * B, a \in A, b \in B, t \in \gamma(u)$, if $u \equiv u'$ then,

$$\mathbb{I}(u) \equiv \mathbb{I}(u'), \;\; u = \iota_A(a) \Leftrightarrow u' = \iota_A(a), \;\; u = \iota_B(b) \Leftrightarrow u' = \iota_B(b), \tag{59}$$

$$\gamma(u) \equiv \gamma(u'), \;\; \mu(t,u) = \mu(t,u'), \;\; \delta(t,u) = \delta(t,u'). \tag{60}$$

We can naturally endow $\mathbb{W}$ with a structure of AB-algebra:

$$\langle \mathbb{W}, \cdot, 1_{\mathbb{W}}, \iota_{A,\equiv}, \iota_{B,\equiv}, \mathbb{I}_{\equiv}, \mu_{\equiv}, \gamma_{\equiv}, \delta_{\equiv} \rangle \tag{61}$$

where all the required maps are just obtained from the corresponding map in the AB-structure of $\mathcal{W}^* * A * B$, by composition by $\pi_{\equiv} : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B/\equiv$. In addition

$$u \equiv u' \Rightarrow \|u\| = \|u'\|$$

so that the notion of norm remains well-defined in the quotient $\mathbb{W}$.

3.5.3. $\mathbb{W}_t, \mathbb{W}_{\mathbb{H}}$

Let us consider the set $\mathcal{W}_t$ consisting of all the letters $W \in \mathcal{W}$ fulfilling

$$\exists s \in \mathbb{H} * \{t, \bar{t}\}^*, W \in \mathrm{dom}(\mathbb{I}_{\mathbb{W}}) \Leftrightarrow s \in \mathrm{dom}(\mathbb{I}_t), \gamma(W) \subseteq \gamma(s), \quad \text{and}$$

$$\forall t \in \gamma(W), \mu(t,W) = \mu(t,s), \delta(t,W) = \delta(t,s).$$

We define the subset

$$\mathcal{W}_{\mathbb{H}} := \{W \in \mathcal{W}_t \mid \gamma(W) \text{ is a H-type }\}.$$

We then define the quotients

$$\mathbb{W}_t := \mathcal{W}_t^* * A * B/\equiv, \;\; \mathbb{W}_{\mathbb{H}} := \mathcal{W}_{\mathbb{H}}^* * A * B/\equiv .$$

One can easily check that the successive inclusions $\mathbb{W}_{\mathbb{H}} \to \mathbb{W}_t \to \mathbb{W}$ are AB-homomorphisms.

3.5.4. *Involutions*

Apart from the natural involution $\mathbb{I} : \hat{\mathbb{W}} \to \hat{\mathbb{W}}$ there might exist other involutory monoid anti-isomorphisms $\mathbb{I}' : (\hat{\mathbb{W}}, \cdot, 1_{\mathbb{W}}) \to (\hat{\mathbb{W}}, \cdot, 1_{\mathbb{W}})$. Let us consider those $\mathbb{I}'$ defined by a partition $\hat{\mathcal{W}} = \hat{\mathcal{W}}_0 \cup \mathcal{W}_1 \cup \overline{\mathcal{W}}_1, \mathcal{W}_1 = \{W_1, \cdots, W_p\}, \overline{\mathcal{W}}_1 = \{\bar{W}_1, \cdots, \bar{W}_p\}$, a tuple $(a_1, b_1, \cdots, a_k, b_k, \cdots, a_p, b_p)$, with $(a_k, b_k) \in (\mathrm{Gi}(W_k), \mathrm{Ge}(W_k))$ and formulas of the form

$$\mathbb{I}'(W) = \mathbb{I}(W), \quad \text{for all } W \in \hat{\mathcal{W}}_0$$

$$\mathbb{I}'(W_k) = a_k W_k b_k; \mathbb{I}'(\bar{W}_k) = a_k^{-1} \bar{W}_k b_k^{-1} \quad \text{for all } k \in [1, p]. \tag{63}$$

The two following conditions on the tuple $(a_1, b_1, \cdots, a_k, b_k, \cdots, a_p, b_p)$ are necessary and sufficient for this involutory monoid anti-isomorphism to exist:

$$(b_k^{-1} a_k, a_k b_k^{-1}) \in \delta(W_k) \text{ for all } k \in [1, p], \tag{64}$$

$$\delta(\bar{W}_k) = \delta(a_k) \circ \delta(W_k) \circ \delta(b_k) \text{ for all } k \in [1, p]. \tag{65}$$

`here a proof of NS`
The structure

$$\langle \mathbb{W}, \iota_A, \iota_B, \mathbb{I}', \gamma, \mu, \delta \rangle$$

is still an AB-algebra iff the following additional condition is satisfied: for every $k \in [1, p]$

$$\gamma(W_k) = \mathbb{I}_{\mathcal{T}}(\gamma(W_k)), \quad \mu(a_k W_k b_k) = \mathbb{I}_{\mathsf{Q}}(\mu(W_k)). \tag{66}$$

We denote by $\mathcal{I}$ the set of all partial involution $\mathbb{I}'$ of the form (63) satisfying conditions (64-66)

3.5.5. *Homomorphisms*

Let us show here some properties of AB-homomorphisms from $\mathcal{W}^* * A * B$ or $\mathbb{W}$ to other AB-algebras. Let us denote by $\mathcal{G}_{\mathbb{W}}$ the set of generators:

$$\mathcal{G}_{\mathbb{W}} = \mathcal{W} \cup \iota_A(A) \cup \iota_B(B).$$

Lemma 10. *Let $\mathcal{M}_2 = \langle \mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}, \iota_{A,2}, \iota_{B,2}, \mathbb{I}_2, \gamma_2, \mu_2, \delta_2 \rangle$ be some AB-algebra. Let $\psi : \mathbb{W} \to \mathbb{M}_2$ be some monoid-homomorphism. This map ψ is an AB-homomorphism if and only if,*
1- $\iota_A \circ \psi = \iota_{A,2}, \quad \iota_B \circ \psi = \iota_{B,2}$
and for every $g \in \mathcal{G}_{\mathbb{W}}, t \in \gamma(g)$:
2- $g \in \mathrm{dom}(\mathbb{I}) \Leftrightarrow \psi(g) \in \mathrm{dom}(\mathbb{I}_2)$
2'- $\mathbb{I}_2(\psi(g)) = \psi(\mathbb{I}(g))$
3- $\gamma_2(\psi(g)) \supseteq \gamma(g)$
4- $\mu_2(t, \psi(g)) = \mu(t, g)$
5- $\delta_2(t, \psi(g)) = \delta(t, g)$.

Proof: Suppose that ψ is an AB-homomorphism. By definition it must fulfilll conditions (33-38). But for all $g \in \mathcal{G}_{\mathbb{W}}$, $\gamma(g) \neq \emptyset$. Hence condition (34) implies condition (2) of the lemma. The other five conditions translate immediately into (1)(2')(3)(4)(5).
Conversely, let suppose that ψ fulfills conditions (1-5) of the lemma.
By (1), condition (33) is fulfilled.
Extending (2) to $\mathbb{W}$
Let $w \in \mathbb{W}$ with $\gamma(w) \neq \emptyset$. It must have a decomposition

$$w = g_1 g_2 \cdots g_n$$

with $1 \leq n, \forall i \in [1,n], g_i \in \mathcal{G}_{\mathbb{W}}$.
Let us suppose that

$$w \in \mathrm{dom}(\mathbb{I}). \tag{67}$$

By definition of $\gamma_{\mathbb{W}}$, $\gamma_{\mathbb{W}}(w) = \prod_{i=1}^{n} \gamma(g_i)$, hence, (67) and axiom (23) imply

$$\prod_{i=1}^{n} \gamma(g_i) \neq \emptyset \tag{68}$$

and, in particular

$$\forall i \in [1,n], \gamma(g_i) \neq \emptyset \tag{69}$$

Applying axiom (24) of AB-algebras, (67) and (68) give:

$$\forall i \in [1,n], \quad g_i \in \mathrm{dom}(\mathbb{I}). \tag{70}$$

By condition (2) of the lemma, (69) and (70) entail that

$$\forall i \in [1,n], \quad \psi(g_i) \in \mathrm{dom}(\mathbb{I}_2). \tag{71}$$

By condition (3) of the lemma and (68), we know that

$$\prod_{i=1}^{n} \gamma(\psi(g_i)) \neq \emptyset. \tag{72}$$

Using again axiom (24) and (71) we obtain that

$$\prod_{i=1}^{n} \psi(g_i) \in \mathrm{dom}(\mathbb{I}_2). \tag{73}$$

But ψ is a monoid homomorphism, hence this implies that

$$\psi(w) \in \mathrm{dom}(\mathbb{I}_2). \tag{74}$$

We have proved that, under the hypothesis that $\gamma(w) \neq \emptyset$, (67) implies (74).
Let us establish the converse. Let us assume that

$$\gamma(w) \neq \emptyset \tag{75}$$

and (74). As ψ is a monoid-homomorphism, we obtain (73). From (75) we get (68) and, as above (72). From (73) and (72), by axiom (24) of AB-algebras we can deduce (71). By condition (2) of the lemma, (71) implies (70). From (70) and (68), by axiom (24) we obtain (67), as required.

Extending (2') to $\mathbb{W}$

By the point above, we know that $\mathrm{im}(\mathbb{I} \circ \psi) \subseteq \mathrm{dom}(\mathbb{I}_2)$. Let us consider the map

$$\theta = \mathbb{I} \circ \psi \circ \mathbb{I}_2 : \hat{\mathbb{W}} \to \hat{\mathbb{M}}_2.$$

Condition (2') shows that,

$$\forall g \in \mathcal{G}_{\mathbb{W}}, \theta(g) = \psi(g).$$

As θ, ψ are monoid-homomorphism and $\mathcal{G}_{\mathbb{W}}$ is a set of monoid generators of $\mathbb{W}$, it follows that $\theta = \psi$, thus $\theta \circ \mathbb{I}_2 = \psi \circ \mathbb{I}_2$ i.e.

$$\mathbb{I} \circ \psi = \psi \circ \mathbb{I}_2,$$

as required.

Extending (3) to $\mathbb{W}$

Let us consider the pairs $(\mathbb{W}, =)$,$(\mathbb{M}_2, =)$,$(\mathcal{P}(\mathcal{T}), \subseteq)$. Each of them is an *ordered* monoid i.e., the second element of each pair is an ordering relation which is compatible with right(resp. left) product. Let us consider the sequence of two maps:

$$\mathbb{W} \xrightarrow{\psi} \mathbb{M}_2 \xrightarrow{\gamma_2} \mathcal{P}(\mathcal{T})$$

These two maps are *overmorphic* in the sense that:

$$\psi(w \cdot w') = \psi(w) \cdot \psi(w'), \;\; \gamma_2(m \cdot m') \supseteq \gamma_2(m) \cdot \gamma_2(m').$$

(see axiom (26) for the second inequality). It follows that their composition $\psi \circ \gamma_2$ is overmorphic too, i.e. for every $w, w' \in \mathbb{W}$

$$\gamma_2(\psi(w \cdot w')) \supseteq \gamma_2(\psi(w)) \cdot \gamma_2(\psi(w')) \tag{76}$$

From this property (76), one can show, by induction over the integer n that, for every $g_1, \cdots, g_i, \cdots, g_n$

$$\gamma_2(\psi(\prod_{i=1}^{n} g_i)) \supseteq \prod_{i=1}^{n} \gamma_2(\psi(g_i)),$$

which, by condition (3) of the lemma , and the definition of $\gamma_{\mathbb{W}}$ implies that

$$\gamma_2(\psi(\prod_{i=1}^{n} g_i)) \supseteq \gamma_{\mathbb{W}}(\prod_{i=1}^{n} g_i),$$

as required.

Extending (4) to $\mathbb{W}$

Let $w \in \mathbb{W}$ such that

$$\gamma_{\mathbb{W}}(w) \neq \emptyset. \tag{77}$$

This w must have a decomposition

$$w = \prod_{i=1}^{n} g_i \tag{78}$$

where $g_i \in \mathcal{G}_{\mathbb{W}}$. As we saw in the extension of (3) to $\mathbb{W}$, hypothesis (77) entails that

$$\prod_{i=1}^{n} \gamma_{\mathbb{W}}(g_i) \neq \emptyset, \ \ \prod_{i=1}^{n} \gamma_2(\psi(g_i)) \neq \emptyset. \tag{79}$$

Let $t \in \gamma_{\mathbb{W}}(w)$: by definition of $\gamma_{\mathbb{W}}$ it must have the form

$$t = \prod_{i=1}^{n} t_i, \ \ \forall i \in [1,n], t_i \in \gamma_{\mathbb{W}}(g_i). \tag{80}$$

We thus have

$$\begin{aligned} \mu_2(t, \psi(w)) &= \mu_2(\textstyle\prod_{i=1} t_i, \prod_{i=1}^{n} \psi(g_i)) & (\psi \text{ is a monoid hom. }) \\ &= \textstyle\prod_{i=1}^{n} \mu_2(\psi(t_i, g_i)) & (\text{ axiom (27)}) \\ &= \textstyle\prod_{i=1}^{n} \mu_{\mathbb{W}}(t_i, g_i) & (\text{ by condition (4)}) \\ &= \mu_{\mathbb{W}}(t, w) & (\text{ axiom (27)}), \end{aligned} \tag{81}$$

as required.

Extending (5) to $\mathbb{W}$

We start again with some $w \in \mathbb{W}$ fulfilling (77), hence (79) and consider some $t \in \gamma_{\mathbb{W}}(w)$, hence of the form (80). We thus have

$$\begin{aligned} \delta_2(t, \psi(w)) &= \delta_2(\textstyle\prod_{i=1} t_i, \prod_{i=1}^{n} \psi(g_i)) & (\psi \text{ is a monoid hom. }) \\ &= \textstyle\prod_{i=1}^{n} \delta_2(t_i, \psi(g_i)) & (\text{ axiom (29)}) \\ &= \textstyle\prod_{i=1}^{n} \delta(t_i, g_i) & (\text{ by condition (5)}) \\ &= \delta_{\mathbb{W}}(t, w) & (\text{ axiom (29)}). \end{aligned} \tag{82}$$

□

Lemma 11. *Let $\psi : \mathbb{W} \to \mathbb{H}_t$ be some AB-homomorphism. Let $P, S, P', S' \in \mathbb{W}$ such that $\psi(PS) = \psi(P'S')$ and $\gamma(PS) = \gamma(P'S') \neq \emptyset$. Then, $\psi(P) = \psi(P') \Leftrightarrow \psi(S) = \psi(S')$.*

`a proof; we do not need H itself cancellative; used in factorization lemma`

Lemma 12. *Let $P, S, P', S' \in \mathbb{W}$ such $\gamma(P) = \gamma(P')$ and $\gamma(PS) = \gamma(P'S') \neq \emptyset$. Then one of the following occurs*
1- there exist $t, t' \in \mathcal{T}_6$, $\gamma(S), \gamma(S') \in \{\{(t, 0, t')\}, \{(t, 1, t')\}\}$.
2- $P, S, P', S' \in A$
3- $P, S, P', S' \in B$

Proof: Let P, S, P', S' fulfill the hypothesis of the lemma.
Case 1: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| \geq 1, \|S'\| \geq 1$.
Since $\gamma(PS) = \gamma(P'S') \neq \emptyset$ we must have $\tau\mathrm{i}(S) = \tau\mathrm{e}(P), \tau_i(S') = \tau\mathrm{e}(P')$. Similarly

we obtain that $\tau\text{e}(S) = \tau\text{e}(S')$, so that conslusion 1 holds.
Case 2: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| = \|S'\| = 0$.
In this case, $S, S' \in A \cup B$. Moreover, $\gamma(P) = \gamma(P') \Rightarrow \text{Ge}(P) = \text{Ge}(P') \Rightarrow S, S' \in \text{Ge}(P) \Rightarrow \gamma(S) = \gamma(S')$.
Case 3: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| \geq 1, \|S'\| = 0$.
As $\|S\| \geq 1, \text{Card}(\gamma(S)) = 1$ and $\tau\text{e}(S) \neq \tau\text{i}(S) = \tau\text{e}(P)$ so that

$$\tau\text{e}(S) \neq \tau\text{e}(P) \tag{83}$$

Since $\|S'\| = 0$, for every $t \in \gamma(S'), \tau\text{i}(t) = \tau\text{e}(t)$, hence

$$\tau\text{e}(P'S') = \tau\text{e}(P') \tag{84}$$

We also have $\gamma(PS) = \gamma(P'S') \neq \emptyset$ which implies that $\tau\text{e}(S) = \tau\text{e}(PS) = \tau\text{e}(P'S')$ hence, taking into account (84) we get

$$\tau\text{e}(S) = \tau\text{e}(P') \tag{85}$$

But equations (83)(85) entail that $\tau\text{e}(P) \neq \tau\text{e}(P')$ contradicting the hypothesis that $\gamma(P) = \gamma(P')$. This case is thus impossible.
Case 4: $\|P\| = \|P'\| = 0, \|S\| \geq 1, \|S'\| \geq 1$.
In this case $P, P' \in A \cup B$. The fact that $\gamma(P) = \gamma(P')$ implies that

$$(P \in A - \{1\} \wedge P' \in A - \{1\}) \vee (P \in B - \{1\} \wedge P' \in B - \{1\}) \vee (P = P' = 1).$$

Here $\gamma(S) = \gamma(PS) = \gamma(P'S') = \gamma(S')$. Hence conclusion 1 holds.
Case 5: $\|P\| = \|P'\| = 0, \|S\| \geq 1, \|S'\| = 0$.
We should then have $\text{Card}(\gamma(PS)) = 1$ while $\text{Card}(\gamma(P'S')) \in \{0, 2, 6\}$. This is contradicts the hypothesis that $\gamma(PS) = \gamma(P'S')$. This case is thus impossible.
Case 6: $\|P\| = \|P'\| = 0, \|S\| = \|S'\| = 0$.
Then $P, P', S, S' \in A \cup B$. But $\gamma(PS) = \gamma(P'S')$ implies that conclusion 2 or conlusion 3 holds. □

Lemma 13. *Let $\psi : \mathbb{W} \to \mathbb{M}$ be some AB-homomorphism into some AB-algebra $\mathbb{M}$. Let $P, S, P', S' \in \mathbb{W}$ such that $\gamma(P) = \gamma(P'), \psi(P) = \psi(P'), \psi(PS) = \psi(P'S')$ and $\gamma(PS) = \gamma(P'S') \neq \emptyset$. Then $\gamma(S) = \gamma(S')$.*

Proof: One of conclusions 1,2,3 of lemma 12 must hold. If conclusion 1 holds,$\|\psi(P)\| = \|\psi(P')\|, \|\psi(PS)\| = \|\psi(P'S')\|$ imply that $\|\psi(S)\| = \|\psi(S')\|$. The boolean component of $\gamma(S), \gamma(S')$ are thus equal and finally, $\gamma(S) = \gamma(S')$.
If conclusion 2 (resp. 3) holds, since ψ restricted to the group A (resp. B) is a group isomorphism onto its image, $S = S'$. □

Lemma 14. *Let $\psi : \mathbb{W} \to \mathbb{H}_t$ be some AB-homomorphism. Let $Q \in \mathbb{W}, P', S' \in \mathbb{H}_t$ such that $\psi(Q) = P'S'$, $\gamma(P') \cdot \gamma(S') \neq \emptyset$. and $\gamma(P')$ is a H-type. Then, there exist $P, S \in \mathbb{W}$ such that:*

$$Q = P \cdot S, \;\; \psi(P) = P', \;\; \psi(S) = S'.$$

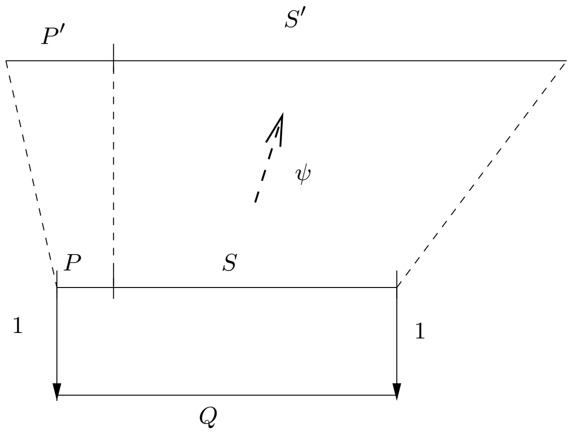

Fig. 4. lemma 14

Lemma 15 (factorization). *Let $\tilde{\psi} : \mathcal{W}^* * A * B \to \mathbb{W}$ be some AB-homomorphism. Then, there exists a unique AB-homomorphism $\psi : \mathbb{W} \to \mathbb{W}$ such that $\pi_\equiv \circ \psi = \tilde{\psi}$.*

Proof: Since $\tilde{\psi}$ is an AB-homomorphism, for every $W \in \mathcal{W}$, $\delta(W) = \delta(\tilde{\psi}(W))$.Hence $\ker(\pi_\equiv) \subseteq \ker(\tilde{\psi})$, which ensures the existence of $\psi \in \mathrm{Hom}(\mathbb{W}, \mathbb{W})$ such that $\pi_\equiv \circ \psi = \tilde{\psi}$. By hypothesis, $\tilde{\psi}$ preserves $\iota_A, \iota_B, \mathbb{I}, \gamma, \mu, \delta$, over the elements of $\mathcal{G}_\mathbb{W}$. As $\pi_\equiv$ does also preserves all these maps, it follows that ψ fulfills conditions (1-5) of lemma 10, hence it is an AB-homomorphism. $\Box$

Lemma 16 (quotient). *Let $\tilde{\sigma} : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ be some AB-homomorphism. Then, there exists a unique AB-homomorphism $\sigma : \mathbb{W} \to \mathbb{W}$ such that $\pi_\equiv \circ \sigma = \tilde{\sigma} \circ \pi_\equiv$.*

Proof: Applying lemma 15 to the AB-homomorphism $\tilde{\psi} = \tilde{\sigma} \circ \pi_\equiv$ we obtain this lemma. $\Box$

Lemma 17 (lifting). *Let $\check{\psi} : \mathcal{W}^* * A * B \to \mathbb{W}$ be some AB-homomorphism. Then, there exists an AB-homomorphism $\tilde{\psi} : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ such that $\check{\psi} = \tilde{\psi} \circ \pi_\equiv$.*

Proof: Let us consider some map $\tilde{\psi} : \mathcal{G}_\mathbb{W} \to \mathcal{W}^* * A * B$ fulfilling, for every $a \in A, b \in B, W \in \mathcal{W}$:

$$\tilde{\psi}(\iota_A(a)) = \iota_A(a), \;\; \tilde{\psi}(\iota_B(b)) = \iota_B(b)$$

$$\tilde{\psi}(W) \in \pi_\equiv^{-1}(\check{\psi}(W)).$$

Since $\pi_\equiv$ preserves $\iota_A, \iota_B, \mathbb{I}, \gamma, \mu, \delta$ and $\check{\psi}$ fulfills conditions (1-5) of lemma 10, the map $\tilde{\psi}$ does also fulfill conditions (1-5). By the universal property of the free product,it can be extended into a monoid homomorphism from $\mathcal{W}^* * A * B$ to $\mathcal{W}^* * A * B$. By lemma 10, it is also an AB-homomorphism from $\mathcal{W}^* * A * B$ to $\mathcal{W}^* * A * B$. Moreover, for every $g \in \mathcal{G}_{\mathbb{W}}, \pi_\equiv(\tilde{\psi}(g)) = \check{\psi}(g)$, hence $\tilde{\psi} \circ \pi_\equiv = \check{\psi}$, as required. □

Lemma 18 (inverse image). *Let $\sigma : \mathbb{W} \to \mathbb{W}$ be some AB-homomorphism. Then, there exists an AB-homomorphism $\tilde{\sigma} : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ such that $\pi_\equiv \circ \sigma = \tilde{\sigma} \circ \pi_\equiv$.*

Proof: Applying lemma 17 to the AB-homomorphism $\check{\psi} = \pi_\equiv \circ \sigma$, we obtain this lemma. □

$$
\begin{tikzcd}
\mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"'] \arrow[dr, "\tilde{\psi}"] & \\
\mathbb{W} \arrow[r, dashed, "\psi"] & \mathbb{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{W}^* * A * B \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi_\equiv"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"'] \\
\mathbb{W} \arrow[r, dashed, "\sigma"] & \mathbb{W}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{W}^* * A * B \arrow[r, dashed, "\tilde{\psi}"] \arrow[dr, "\check{\psi}"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"'] \\
& \mathbb{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{W}^* * A * B \arrow[r, dashed, "\tilde{\sigma}"] \arrow[d, "\pi_\equiv"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"'] \\
\mathbb{W} \arrow[r, "\sigma"] & \mathbb{W}
\end{tikzcd}
$$

Fig. 5. AB-homomorphisms

4. Equations with rational constraints over $\mathbb{G}$

Let us consider a *system of equations* $\mathcal{S}$ over $\mathbb{G}$:

$$((u_i, u'_i))_{i \in I}$$

where I is a finite set, $\mathcal{U}$ is a finite set of variables and $u_i \in \mathcal{U}^*, u'_i \in \mathcal{U}^*$; let us consider also a rational constraint $\mathsf{C} : \mathcal{U} \to \mathcal{B}(Rat(\mathbb{G}))$.

Quadratic normal form Such a system can be effectively transformed into a finite system $\mathcal{S}'$, with set of variables $\mathcal{U}'$ and with rational constraint $\mathsf{C}' : \mathcal{U}' \to \mathcal{B}(Rat(\mathbb{G}))$ such that
(1) $\mathcal{S}'$ is quadratic
(2) $\mathcal{U} \subseteq \mathcal{U}'$
(3) the solutions of $(\mathcal{S}, \mathsf{C})$ are exactly the restrictions to $\mathcal{U}^*$ of the solutions of $(\mathcal{S}', \mathsf{C}')$.
Such a transformation consists of applying iteratively the following elementary transformations T_k, for $1 \leq k \leq 4$:
T1 Suppose that $|u'_i| \leq 1$.
Then set $\mathcal{U}' := \mathcal{U} \cup \{U'\}$, where U' is a new variable, and set

$$v_j := u_j \text{ for all } j \in I, \;\; v'_j := u'_j \text{ for all } j \in I - \{i\}, \;\; v'_i := u'_i U',$$

$$\mathsf{C}'(U) = \mathsf{C}(U) \text{ for all } U \in \mathcal{U}, \;\; \mathsf{C}'(U') = \{\varepsilon\}.$$

T2 Suppose that $|u_i| = 0$.
Then set $\mathcal{U}' := \mathcal{U} \cup \{U'\}$, where U' is a new variable, and set

$$v_j := u_j \text{ for all } j \in I - \{i\}, \;\; v'_j := u'_j \text{ for all } j \in I, \;\; v_i := U',$$

$$\mathsf{C}'(U) = \mathsf{C}(U) \text{ for all } U \in \mathcal{U}, \;\; \mathsf{C}'(U') = \{\varepsilon\}.$$

T3 Suppose that $|u_i| \geq 2$.
Then set $\mathcal{U}' := \mathcal{U} \cup \{U'\}$, where U' is a new variable, $I' := I \cup \{\bar{\imath}\}$ where $\bar{\imath} \notin I$

$$v_j := u_j \text{ for all } j \in I - \{i\}, \;\; v'_j := u'_j \text{ for all } j \in I - \{i\},$$

$$v_i := U', \;\; v'_i := u_i, \;\; v_{\bar{\imath}} := U', \;\; v'_{\bar{\imath}} := u'_i,$$

$$\mathsf{C}'(U) = \mathsf{C}(U) \text{ for all } U \in \mathcal{U}, \;\; \mathsf{C}'(U') = \mathbb{G}.$$

T4 Suppose that $|u'_i| \geq 3 : u'_i = u''_i U$ for some $U \in \mathcal{U}$
Then set $\mathcal{U}' := \mathcal{U} \cup \{U'\}$, where U' is a new variable, $I' := I \cup \{\bar{\imath}\}$ where $\bar{\imath} \notin I$

$$v_j := u_j \text{ for all } j \in I - \{i\}, \;\; v'_j := u'_j \text{ for all } j \in I - \{i\},$$

$$v_i := u_i, \;\; v'_i := U'U, \;\; v_{\bar{\imath}} := U', \;\; v'_{\bar{\imath}} := u''_i.$$

$$\mathsf{C}'(U) = \mathsf{C}(U) \text{ for all } U \in \mathcal{U}, \;\; \mathsf{C}'(U') = \mathbb{G}.$$

Rational constraint Let $\mathsf{C} : \mathcal{U} \to \mathcal{B}(Rat(\mathbb{G}))$. By the results of [LS05], one can construct some normalized finite t-automaton $\mathcal{A}$ (see §2.3) over the labelling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ such that: for every $U \in \mathcal{U}$, there exists some subset $\mathsf{B}(U) \subseteq \mathsf{B}(\mathsf{Q})$ recognizing $\mathsf{C}(U)$ in the sense that

$$\mathsf{C}(U) = \mu^{-1}_{\mathcal{A},\mathbb{G}}(\mathsf{B}(U)).$$

Let us denote by

$$M(\mathsf{C}) := \{\mu : \mathcal{U} \to \mathsf{B}(\mathsf{Q}) \mid \forall U \in \mathcal{U}, \mu(U) \in \mathsf{B}(U)\}.$$

We can thus express the initial system of equations with constraint as follows:

$$\bigvee_{\mu_{\mathcal{U}} \in M(\mathsf{C})} (\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}}). \tag{86}$$

A solution of the system $(\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}})$ is now any monoid-homomorphism $\sigma_{\mathbb{G}} : \mathcal{U}^* \to \mathbb{G}$ fulfilling both conditions:

$$\forall i \in I, \sigma_{\mathbb{G}}(u_i) = \sigma_{\mathbb{G}}(u'_i) \tag{87}$$

$$\forall U \in \mathcal{U}, \mu_{\mathcal{A},\mathbb{G}}(\sigma_{\mathbb{G}}(U)) = \mu_{\mathcal{U}}(U). \tag{88}$$

The above discussion proves the following

Proposition 19. *The satisfiability problem for systems of equations with rational constraints over $\mathbb{G}$ is Turing-reducible to the satisfiability problem for systems of equations with rational constraints over $\mathbb{G}$ of the form*

$$\mathcal{S} = ((u_i, u'_i)_{i \in I}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}}) \tag{89}$$

where every equation (u_i, u'_i) is quadratic, $\mu_{\mathcal{A},\mathbb{G}}$ is the map associated with a normalized finite t-automaton $\mathcal{A}$ and $\mu_{\mathcal{U}}$ is a map $\mathcal{U} \to \mathsf{B}(\mathsf{Q}_{\mathcal{A}})$.

Any system $\mathcal{S}$ of the form (89) described in the proposition is said to be in *normal form.*

`must be adapted to inequations/constants/special constraints`.

5. Equations over $\mathbb{H}_t$

5.1. *t-equations*

A system of t-equations is a family of ordered pairs

$$\mathcal{S} = (w_i, w'_i)_{i \in I} \tag{90}$$

where $w_i, w'_i \in \mathbb{W}_t, \gamma(w_i) = \gamma(w'_i) \neq \emptyset$. A *solution* of $\mathcal{S}$ is any AB-homomorphism $\psi_t : \mathbb{W}_t \to \mathbb{H}_t$ such that, for every $i \in I$

$$\psi_t(w_i) = \psi_t(w'_i). \tag{91}$$

Notice that here, the rational constraints are replaced by the even more restrictive conditions that define the notion of AB-homomorphism: beside preservation of μ the map ψ_t must also preserve $\mathbb{I}, \gamma$ and δ.

5.2. *From* $\mathbb{G}$*-equations to t-equations*

Let us start with a system of equations with rational constraints , over $\mathbb{G}$, which is in normal form (see proposition 19):

$$((u_i, u'_i)_{i \in I}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}}) \tag{92}$$

We suppose $I = [1, n]$, we denote by $U_{i,1}$(resp. $U_{i,2}, U_{i,3}$) the unique letter of u_i (resp. the first, second letter of u'_i). therefore, the equations of $\mathcal{S}$ take the form

$$E_i : \quad (U_{i,1}, U_{i,2}U_{i,3}) \text{ for all } 1 \leq i \leq n \tag{93}$$

We define here a reduction of the satisfiability problem for such systems to the satisfiability problem for systems of t-equations. The leading idea is simply that, since $\pi_G : \mathrm{Red}(\mathbb{H}, t)/\sim \to \mathbb{G}$ is a bijection, every solution in $\mathbb{G}$ corresponds to a map into $\mathbb{H}_t$. Nevertheless the product in $\mathbb{G}$ corresponds to a somewhat complicated operation over $\mathrm{Red}(\mathbb{H}, t)/\sim$ that we must deal with. Let us consider the alphabets $\mathcal{V}_0 := I \times [1, 3] \times [1, 9]$ and the alphabet $\mathcal{W}$ constructed from this $\mathcal{V}_0$ in §3.5. We consider all the vectors $(W_{i,j,k})$ where $1 \leq i \leq n, 1 \leq j \leq 3, 1 \leq k \leq 9$ of elements of $\mathcal{W} \cup \{1\}$ and all triple $(e_{i,1,2}, e_{i,2,3}, e_{i,3,1}) \in (A \cup B)^3$ such that:

$$
\begin{aligned}
p_1(W_{i,j,k}) = &\quad (i,j,k) \in \mathcal{V}_0 && \text{for } W_{i,j,k} \neq 1 && (94)\\
\gamma(\prod_{k=1}^{9} W_{i,j,k}) = &\quad (1, H, b, 1, 1) && \text{for some } b \in \{0,1\} && (95)\\
\mu_1(\prod_{k=1}^{9} W_{i,j,k}) = &\quad \mu_{\mathcal{U}}(U_{i,j}) && && (96)\\
W_{i,2,k}, W_{i,3,10-k} \in &\quad \hat{\mathcal{W}} \cup \{1\} && \text{for } 6 \leq k \leq 9 && (97)\\
\gamma(\prod_{k=1}^{4} W_{i,1,k}) = &\quad \gamma(\textstyle\prod_{k=1}^{4} W_{i,2,k}) && && (98)\\
\gamma(\prod_{k=6}^{9} W_{i,2,k}) = &\quad \gamma(\textstyle\prod_{k=6}^{9} \overline{W}_{i,3,10-k}) && && (99)\\
\gamma(\prod_{k=6}^{9} W_{i,1,k}) = &\quad \gamma(\textstyle\prod_{k=6}^{9} W_{i,3,k}) && && (100)\\
W_{i,j,5} \in \mathcal{W} \wedge \gamma(W_{i,j,5}) \text{ is a } &\quad \text{H-type} && && (101)\\
e_{i,1,2} \in &\ \mathrm{Gi}(W_{i,1,5}) = \mathrm{Gi}(W_{i,2,5}) && && (102)\\
e_{i,2,3} \in &\ \mathrm{Ge}(W_{i,2,5}) = \mathrm{Gi}(W_{i,3,5}) && && (103)\\
e_{i,3,1} \in &\ \mathrm{Ge}(W_{i,3,5}) = \mathrm{Ge}(W_{i,1,5}). && && (104)
\end{aligned}
$$

A vector $(\vec{W}, \vec{e})$ fulfilling all the properties (94-104) is called an *admissible* vector. For every admissible vector $(\vec{W}, \vec{e})$ we define the following equations:

$$
\begin{aligned}
(\prod_{k=1}^{9} W_{i,j,k}, &\quad \textstyle\prod_{k=1}^{9} W_{i',j',k}) && \text{if } U_{i,j} = U_{i',j'} && (105)\\
(W_{i,1,1}W_{i,1,2}W_{i,1,3}W_{i,1,4}e_{i,1,2}, &\quad W_{i,2,1}W_{i,2,2}W_{i,2,3}W_{i,2,4}) && && (106)\\
(W_{i,2,6}W_{i,2,7}W_{i,2,8}W_{i,2,9}, &\ e_{i,2,3}\overline{W}_{i,3,4}\overline{W}_{i,3,3}\overline{W}_{i,3,2}\overline{W}_{i,3,1}) && && (107)\\
(W_{i,1,5}W_{i,1,6}W_{i,1,7}W_{i,1,8}, &\ e_{i,1,3}W_{i,3,6}W_{i,3,7}W_{i,3,8}W_{i,3,9}) && && (108)\\
(W_{i,1,5}, &\quad e_{i,1,2}W_{i,2,5}e_{i,2,3}W_{i,3,5}e_{i,3,1}) && && (109)
\end{aligned}
$$

for all $1 \leq i, i' \leq n, 1 \leq j, j' \leq 3$. Equations (106-109) correspond to a decomposition of the planar diagram associated with the relation $U_{i,1} \approx U_{i,2}U_{i,3}$ into four pieces, as indicated on figure 6. Equation (105) expresses the fact that some variables from $\mathcal{U}$ are common to several equations of $\mathcal{S}$.

We denote by $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ the sequence of equations (105-108) and by $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ the sequence of equations (109). For every $(i,j) \in [1,n] \times [1,3]$ we denote by $\overline{\mathrm{i},\mathrm{j}}$ the smallest pair such that $U_{i,j} = U_{\overline{\mathrm{i},\mathrm{j}}}$. By $\sigma_{\vec{W},\vec{e}} : \mathcal{U}^* \to \mathbb{W}$ we denote the unique monoid-homomorphism such that,

$$\sigma_{\vec{W},\vec{e}}(U_{i,j}) = \prod_{k=1}^{9} W_{\overline{\mathrm{i},\mathrm{j}},k}.$$

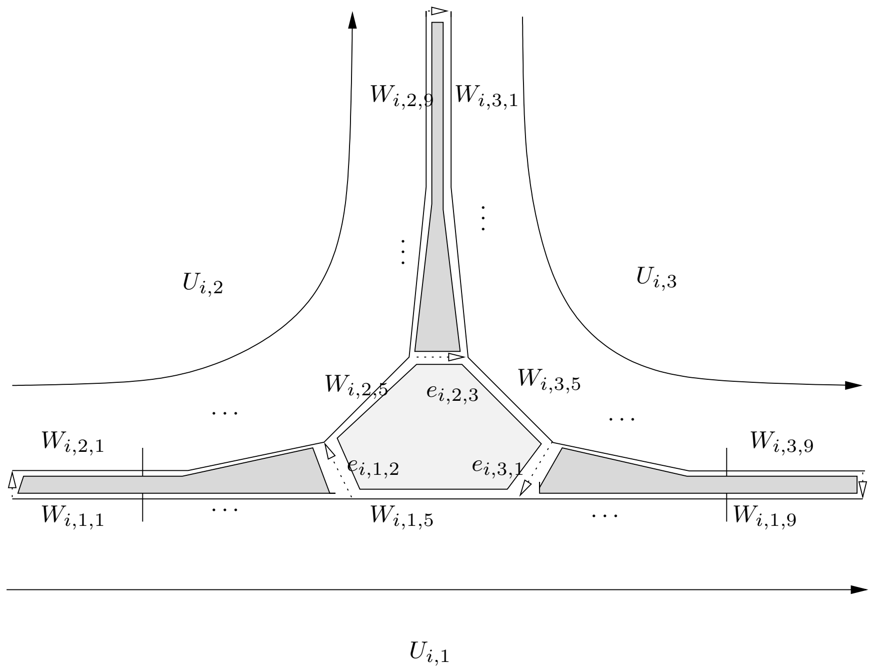

Fig. 6. Equation cut into 4 pieces

Notice that, by the conditions imposed through the notion of "admissible vector", the equations of $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ are really t-equations, while some of the righthand-sides of the equations of $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ might have an empty image by γ.

Lemma 20. *Let $\mathcal{S} = ((E_i)_{1\leq i\leq n}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}})$ be a system of equations over $\mathbb{G}$, with rational constraint. Let us suppose that $\mathcal{S}$ is in normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{S}$ if and only if, there exists an admissible choice $(\vec{W}, \vec{e})$ of variables of $\mathcal{W}_t$ and elements of $A \cup B$ and an AB-homomorphism

$$\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$$

solving both systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$, such that

$$\sigma = \sigma_{\vec{W},\vec{e}} \circ \sigma_t \circ \bar{\pi}_{\mathbb{G}}.$$

(We denote by $\bar{\pi}_{\mathbb{G}} : \mathbb{H}_t \to \mathbb{G}$ the canonical projection; see figure 7). We prove this lemma in the following two subsections.

$$\begin{tikzcd} \mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\sigma_t"] & \mathbb{W}_t \arrow[d] \\ \mathbb{G} & \mathbb{H}_t \arrow[l, "\bar{\pi}_{\mathbb{G}}"] \end{tikzcd}$$

Fig. 7. lemma 20

5.2.1. *From $\mathbb{G}$-solutions to t-solutions*

Let $\sigma : \mathcal{U}^* \to \mathbb{G}$ be a monoid homomorphism solving the system $\mathcal{S}$. Let us fix some equation from $\mathcal{S}$, i.e. some integer $1 \leq i \leq n$. Let us construct the vectors $(W_{i,*,*}), (e_{i,*,*})$ corresponding to this equation. Let us choose, for every $j \in [1, 3]$, some $s_{i,j} \in \mathrm{Red}(\mathbb{H}, t)$ such that:

$$\sigma(U_{i,j}) = \pi_G(s_{i,j}).$$

Let us consider decompositions of the form (5) for $s_{i,2}, s_{i,3}$:

$$s_{i,2} = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell, \tag{110}$$

$$s_{i,3} = k_0 t^{\beta_1} k_1 \cdots t^{\beta_\rho} k_\rho \cdots t^{\beta_m} k_m. \tag{111}$$

We know that $s_{i,1} \approx s_{i,2}s_{i,3}$. Either there exist some integers $\lambda \in [1, \ell], \rho \in [1, m]$ such that

$$\alpha_\lambda + \beta_\rho = 0, \tag{112}$$

$$t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell \cdot k_0 t^{\beta_1} k_1 \cdots t^{\beta_\rho} \approx e_{i,2,3} \in A(\beta_\rho), \tag{113}$$

$$h_0 \cdots t^{\alpha_{\lambda-1}}(h_{\lambda-1} e_{i,2,3} k_\rho) t^{\beta_{\rho+1}} \cdots t^{\beta_m} k_m \in \mathrm{Red}(\mathbb{H}, t), \tag{114}$$

or ($\alpha_\ell + \beta_1 \neq 0$)or ($\alpha_\ell + \beta_1 = 0$ and $h_\ell \cdot k_0 \notin A(\beta_1)$) or $\ell = 0$ or $m = 0$. We include in the above notation the following "degenerated" cases:

• [Left-degenerated case]: $\lambda = 1$; then $h_0 \cdots t^{\alpha_{\lambda-1}}$ must be understood as 1
• [Right-degenerated case]: $\rho = m$; then $t^{\beta_{\rho+1}} \cdots t^{\beta_m} k_m$ must be understood as 1,
• [Middle-degenerated case]: $\alpha_\ell + \beta_1 \neq 0$ or ($\alpha_\ell + \beta_1 = 0$ and $h_\ell \cdot k_0 \notin A(\beta_1)$);
we then consider that $\lambda = \ell + 1, \rho = 0, e_{i,2,3} = 1$. Equality (112) disappears, (113) becomes the trivial equation $1 = 1$ while (114) remains valid.
• [LM-degenerated case]: $\ell = 0$;
We then consider that $\lambda = \rho = 0, e_{i,2,3} = 1$ and equation (113) becomes the trivial equation $1 = 1$. The lefthand-side of assertion (114) consists just of $s_{i,3}$.
• [MR-degenerated case]: $m = 0$;
Same notation as for LM. The l.h.s. of (114) consists just of $s_{i,2}$.
Notice that these cases are not disjoint; in particular, when $\ell = m = 0$ the three

kinds of degeneracy occur simultaneously. Each kind of degeneracy can be visualized on figure 8 as one of the three triangular pieces consisting of a trivial relation. Let us consider the following factors of the reduced sequences $s_{i,2}, s_{i,3}$:

$$P_{i,2} = h_0 \cdots t^{\alpha_{\lambda-1}}, \;\; M_{i,2} = h_{\lambda-1}, \;\; S_{i,2} = t^{\alpha_\lambda} \cdots t^{\alpha_\ell} h_\ell,$$

$$P_{i,3} = k_0 \cdots t^{\beta_\rho}, \;\; M_{i,3} = k_\rho, \;\; S_{i,3} = t^{\beta_{\rho+1}} \cdots t^{\beta_m} k_m.$$

Since $s_{i,1} \approx s_{i,2}s_{i,3} \approx P_{i,2}(M_{i,2}e_{i,2,3}M_{i,3})S_{i,3}$ and $s_{i,1}, P_{i,2}(M_{i,2}e_{i,2,3}M_{i,3})S_{i,3}$ are reduced sequences, by lemma 1 we get:

$$s_{i,1} \sim P_{i,2}(M_{i,2}e_{i,2,3}M_{i,3})S_{i,3}.$$

Case 1[standard case] $\lambda \in [2, \ell], \rho \in [1, m-1]$.
There must exist $P_{i,1}, S_{i,1} \in \mathrm{Red}(\mathbb{H}, t), M_{i,1} \in \mathbb{H}$ and connecting elements $e_{i,1,2} \in B(\alpha_{\lambda-1}), e_{i,2,3} \in A(\alpha_\lambda)$ such that:

$$P_{i,1}e_{i,1,2} \sim P_{i,2}, \tag{115}$$

$$e_{i,1,2}M_{i,2}e_{i,2,3}M_{i,3}e_{i,3,1} =_{\mathbb{H}} M_{i,1} \tag{116}$$

$$e_{i,3,1}S_{i,1} \sim S_{i,3}, \tag{117}$$

while relation (113) can be rewritten as:

$$S_{i,2} \sim e_{i,2,3}\mathbb{I}_t(P_{i,3}), \tag{118}$$

see figure 8. Let $\pi_T : \mathbb{H} * \{t, \bar{t}\}^* \to \{t, \bar{t}\}^*$ be the natural projection. By (115), (resp.(117),(118))we know that

$$\pi_T(P_{i,1}) = \pi_T(P_{i,2}), \;\; \pi_T(S_{i,1}) = \pi_T(S_{i,3}), \;\; \pi_T(S_{i,2}) = \pi_T(\mathbb{I}_t(P_{i,3})).$$

Hence the t-automaton $\mathcal{R}_6$ has computations of the following forms:

$$(1, H) \overset{P_{i,1}}{\to} q_{i,1} \overset{M_{i,1}}{\to} r_{i,1} \overset{S_{i,1}}{\to} (1, 1) \tag{119}$$

$$(1, H) \overset{P_{i,2}}{\to} q_{i,1} \overset{M_{i,2}}{\to} r_{i,2} \overset{S_{i,2}}{\to} (1, 1) \tag{120}$$

$$(1, H) \overset{P_{i,3}}{\to} \mathbb{I}_{\mathcal{R}}(r_{i,2}) \overset{M_{i,3}}{\to} r_{i,1} \overset{S_{i,3}}{\to} (1, 1). \tag{121}$$

Since $q_{i,1} \neq (1,1)$, an easy inspection of the automaton $\mathcal{R}_6$ shows that the computation $(1, H) \overset{P_{i,1}}{\to} q_{i,1}$ can be factored into four subcomputations

$$(1, H) = q_{i,1,0} \overset{P_{i,1,1}}{\to} q_{i,1,1} \overset{P_{i,1,2}}{\to} q_{i,1,2} \overset{P_{i,1,3}}{\to} q_{i,1,3} \overset{P_{i,1,4}}{\to} q_{i,1,4} = q_{i,1} \tag{122}$$

such that each $(q_{i,1,k}, |||P_{i,1,k}|||, q_{i,1,k+1})$ is an edge of $\mathcal{R}$. Any decomposition having this property w.r.t. its projection on $\mathcal{R}$ will be called $\mathcal{R}$-compatible. Similarly the computations $r_{i,1} \overset{S_{i,1}}{\to} (1,1)$ (resp. $r_{i,2} \overset{S_{i,2}}{\to} (1,1)$) have $\mathcal{R}$-compatible decompositions:

$$q_{i,1,5} \overset{P_{i,1,6}}{\to} q_{i,1,6} \overset{P_{i,1,7}}{\to} q_{i,1,7} \overset{P_{i,1,8}}{\to} q_{i,1,8} \overset{P_{i,1,9}}{\to} q_{i,1,9} = (1, 1), \tag{123}$$

$$q_{i,2,5} \overset{P_{i,2,6}}{\to} q_{i,2,6} \overset{P_{i,2,7}}{\to} q_{i,2,7} \overset{P_{i,2,8}}{\to} q_{i,2,8} \overset{P_{i,2,9}}{\to} q_{i,2,9} = (1, 1). \tag{124}$$

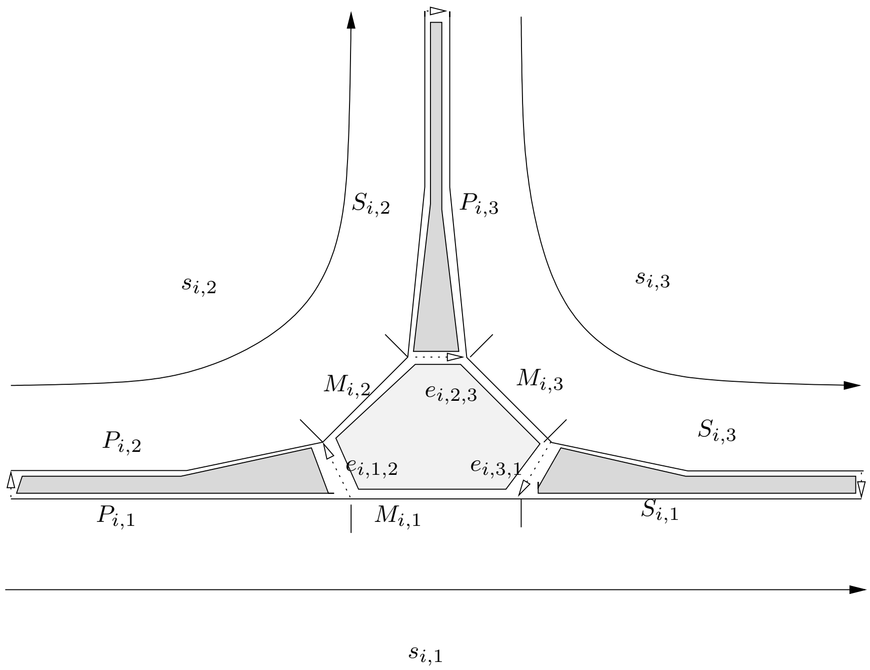

Fig. 8. Cutting the solution into three factors

Combining decomposition (122) with equation (115), (123) with (117), (124) with (118), we get the three $\mathcal{R}$-compatible decompositions:

$$(1,H) = q_{i,1,0} \stackrel{P_{i,2,1}}{\rightarrow} q_{i,1,1} \stackrel{P_{i,2,2}}{\rightarrow} q_{i,1,2} \stackrel{P_{i,2,3}}{\rightarrow} q_{i,1,3} \stackrel{P_{i,2,4}}{\rightarrow} q_{i,1,4} = q_{i,1} \tag{125}$$

$$r_{i,1} = q_{i,1,5} \stackrel{P_{i,3,6}}{\rightarrow} q_{i,1,6} \stackrel{P_{i,3,7}}{\rightarrow} q_{i,1,7} \stackrel{P_{i,3,8}}{\rightarrow} q_{i,1,8} \stackrel{P_{i,3,9}}{\rightarrow} q_{i,1,9} = (1,1), \tag{126}$$

$$\begin{aligned}(1,H) = \mathbb{I}(q_{i,2,9}) = q_{i,3,0} \stackrel{P_{i,3,1}}{\rightarrow} \mathbb{I}(q_{i,2,8}) = q_{i,3,1} \stackrel{P_{i,3,2}}{\rightarrow} \\ \mathbb{I}(q_{i,2,7}) = q_{i,3,2} \stackrel{P_{i,3,3}}{\rightarrow} \mathbb{I}(q_{i,2,6}) = q_{i,3,3} \stackrel{P_{i,3,4}}{\rightarrow} \mathbb{I}(r_{i,2}) = q_{i,3,4}.\end{aligned} \tag{127}$$

Finally, we define $P_{i,j,5} := M_{i,j}$. We summarize on figure 9 the above decompositions and relations. Let us extract from these the vector $(\vec{W}, \vec{e})$ and the AB-homomorphism σ_t:
- we choose for $W_{i,j,k}$ a letter from $\mathcal{W}$ with $\gamma(W_{i,j,k}) = (q_{i,j,k-1}, |||P_{i,j,k}|||, q_{i,j,k})$ and which can be mapped by some AB-homomorphism on the t-sequence $P_{i,j,k}$.
- we define $\sigma_t(W_{i,j,k}) := [P_{i,j,k}]_\sim$; the choice of the letters $W_{i,j,k}$ together with lemma 10 imply that any extension of σ_t on the alphabet $\mathcal{W}_t$ (respecting conditions 1-5 of lemma 10), will posess a unique extension as an AB-homomorphism from $\mathbb{W}_t$ to $\mathbb{H}_t$.
One can check that $(\vec{W}, \vec{e})$ is an admissible vector, that σ_t solves the systems of

equations $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ and that

$$\sigma = \sigma_{\vec{W},\vec{e}} \circ \sigma_t \circ \bar{\pi}_{\mathbb{G}}.$$

We indicate now how these arguments must be adapted to the degenerated cases.

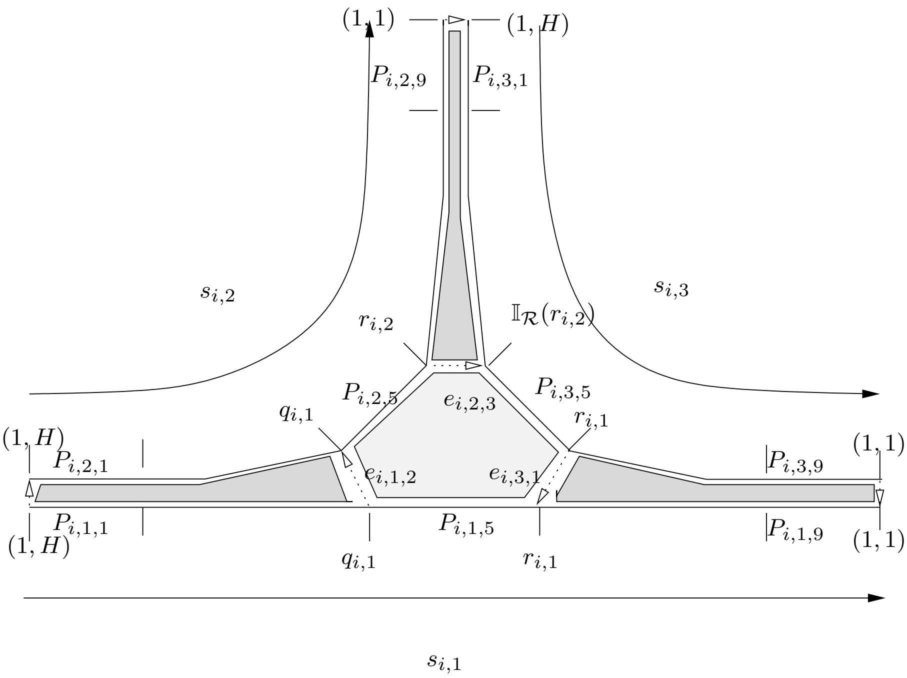

Fig. 9. Cutting the solution into nine factors

Case 2[L]:$\lambda = 1$.
We take: $P_{i,2} = P_{i,1} = 1$, $e_{i,1,2} = 1$ and, accordingly

$$P_{i,2,k} = P_{i,1,k} = 1, \;\; W_{i,2,k} = W_{i,1,k} = 1$$

for $1 \leq k \leq 4$.
Case 3[R]:$\rho = m$.
We take: $S_{i,3} = S_{i,1} = 1$, $e_{i,3,1} = 1$ and, accordingly

$$S_{i,3,k} = S_{i,1,k} = 1, \;\; W_{i,3,k} = W_{i,1,k} = 1$$

for $6 \leq k \leq 9$.
Case 4[M]:$\alpha_\ell + \beta_1 \neq 0$ or $(\alpha_\ell + \beta_1 = 0$ and $h_\ell \cdot k_0 \notin A(\beta_1))$.
We take: $S_{i,2} = P_{i,3} = 1$, $e_{i,2,3} = 1$ and, accordingly

$$S_{i,2,k} = P_{i,3,10-k} = 1, \;\; W_{i,2,k} = W_{i,3,10-k} = 1$$

for $6 \leq k \leq 9$.
Case 5[LM]:$\ell = 0$.
We choose all the special values chosen in Case 2 and Case 4 i.e. $P_{i,2} = P_{i,1} = S_{i,2} = P_{i,3} = 1$, $e_{i,1,2} = e_{i,2,3} = 1$ and the resulting choices for $W_{i,*,*}$.
Case 6[MR]:$m = 0$.
We choose all the special values chosen in Case 3 and Case 4.

5.2.2. *From t-solutions to* $\mathbb{G}$*-solutions*

Let $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ be an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$.
1- Let us show that, for every $i \in [1, n]$,

$$\sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,1})) \approx \sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,2}U_{i,3})). \tag{128}$$

Let us fix such an integer i. The conjunction of equivalences (106-109), implies that

$$\sigma_t(\prod_{k=1}^{9} W_{i,1,k}) \approx \sigma_t(\prod_{k=1}^{9} W_{i,2,k} \prod_{k=1}^{9} W_{i,3,k}).$$

(figure 6 gives a decomposition of the Van-Kampen diagram corresponding to the above equivalence into four diagrams corresponding to (106-109)). Using equation (105) we get:

$$\sigma_t(\prod_{k=1}^{9} W_{\overline{1},\overline{1},k}) \approx \sigma_t(\prod_{k=1}^{9} W_{\overline{1},\overline{2},k} \prod_{k=1}^{9} W_{\overline{1},\overline{3},k}).$$

Taking into account the definition of $\sigma_{\vec{W},\vec{e}}$ and the fact that σ_t is a monoid-homomorphism, we get a proof of (128).
2- Let us show that, for every $i \in [1, n]$

$$\mu_{\mathcal{A},\mathbb{G}}(\bar{\pi_{\mathbb{G}}}(\sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,j})))) = \mu_{\mathcal{U}}(U_{i,j}).$$

By definition (12) this means that the value of

$$\mu_{\mathcal{A},1}((1, H, b, 1, 1), \sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,j}))), \tag{129}$$

where $b = \||\sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,j}))\||$ is equal to $\mu_{\mathcal{U}}(U_{i,j})$.
We first remark that

$$\begin{aligned} \sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,j})) &= \sigma_t(\textstyle\prod_{k=1}^{9} U_{\overline{1},\overline{\jmath},k}) \text{ by definition of } \sigma_{\vec{W},\vec{e}} \\ &= \sigma_t(\textstyle\prod_{k=1}^{9} U_{i,j,k}) \text{ by condition (105)} \end{aligned} \tag{130}$$

As σ_t is an AB-homomorphism

$$\mu_{\mathcal{A},1}((1, H, b, 1, 1), \sigma_t(\prod_{k=1}^{9} U_{i,j,k})) = \mu_{\mathcal{A},1}((1, H, b, 1, 1), \prod_{k=1}^{9} U_{i,j,k}). \tag{131}$$

Using now the conditions defining the notion of admissible vector we get:

$$\begin{aligned} \mu_{\mathcal{A},1}((1,H,b,1,1),\prod_{k=1}^{9} U_{i,j,k}) &= \mu_{\mathcal{A},1}(\textstyle\prod_{k=1}^{9} U_{i,j,k}) && \text{by (95)} \\ &= \mu_{\mathcal{U}}(U_{i,j}) && \text{by (96)} \qquad (132) \end{aligned}$$

Putting together (130)(131)(132), we obtain that the value of term (129) is exactly $\mu_{\mathcal{U}}(U_{i,j})$, as required.

6. Equations over $\mathbb{W}$

We suppose here that a system of equations with involution and rational constraints over $\mathbb{G}$ is fixed. Thus the AB-algebras $\mathbb{W}$ and $\mathbb{H}_t$ are completely defined (from the variable alphabet of the system and the t-automaton expressing the constraint).Given an involution $\mathbb{I}'$ fulfilling conditions (64)(65)(66), we abbreviate as $(\mathbb{W}, \mathbb{I}')$ the AB-algebra obtained from the AB-algebra $\mathbb{W}$ by replacing the standard involution $\mathbb{I}$ by $\mathbb{I}'$ (see (3.5.4)).

6.1. $\mathbb{W}$-*equations*

A system of $\mathbb{W}$-equations is a family of ordered pairs together with an involution:

$$\mathcal{S} = ((w_i, w_i')_{i \in I}, \mathbb{I}') \tag{133}$$

where $w_i, w_i' \in \mathbb{W}_t, \gamma(w_i) = \gamma(w_i') \neq \emptyset, \mathbb{I}' \in \mathcal{I}$.
A *solution* of $\mathcal{S}$ is any AB-homomorphism $\sigma_{\mathbb{W}} : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I}')$ such that, for every $i \in I$

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w_i'). \tag{134}$$

6.2. *From t-equations to* $\mathbb{W}$-*equations*

6.2.1. *From t-solutions to* $\mathbb{W}$-*solutions*

Lemma 21 (factorisation of t-solutions). *Let $\mathcal{S} = ((w_i, w_i'))_{1 \le i \le n}$ be a system of t-equations of the form (90). Let us suppose that $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism solving the system $\mathcal{S}$. Then there exists an involution $\mathbb{I}' \in \mathcal{I}$ and AB -homomorphisms*

$$\sigma_{\mathbb{W}} : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I}'), \;\; \psi_t : (\mathbb{W}_t, \mathbb{I}') \to (\mathbb{H}_t, \mathbb{I}_t)$$

such that, $\sigma_t = \sigma_{\mathbb{W}} \circ \psi_t$ and

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w_i') \text{ for all } 1 \le i \le n.$$

In other words: every solution in $\mathbb{H}_t$ of a system of t-equations factorizes through a solution in $\mathbb{W}_t$ of the same system of equations, with an involution in $\mathcal{I}$.
This factorization lemma is obtained via the more technical lemma 22 below. For every $w \in \mathbb{W}$ let us note

$$\mathrm{A}(w) := \mathrm{Card}\{W \in \check{\mathcal{W}} \mid |w|_W \neq 0\} + \frac{1}{2}\mathrm{Card}\{W \in \hat{\mathcal{W}} \mid |w|_{W,\bar{W}} \neq 0\}.$$

Lemma 22. *Let K_0 be an integer such that $K_0 < \mathrm{Card}(\mathcal{V}_0)$ and $((w_i, w_i'))_{1 \le i \le n+m}$ be a sequence of pairs $(w_i, w_i') \in \mathbb{W} \times \mathbb{W}$ such that $\gamma(w_i) = \gamma(w_i') \neq \emptyset$. Let us suppose that $\lambda_i : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I})$ (for $1 \le i \le n+m$) and $\theta_t : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{H}_t, \mathbb{I}_t)$ are AB-homomorphisms such that*

$$\theta_t(\lambda_i(w_i)) = \theta_t(\lambda_i(w_i')) \text{ for all } 1 \le i \le n+m,$$

$$\begin{tikzcd}
\mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[d, "\sigma"'] & (\mathbb{W}_t, \mathbb{I}) \arrow[d, "\sigma_t"'] \arrow[dr, "\sigma_{\mathbb{W}}"] & \\
\mathbb{G} & (\mathbb{H}_t, \mathbb{I}_t) \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & (\mathbb{W}_t, \mathbb{I}') \arrow[l, "\psi_t"]
\end{tikzcd}$$

Fig. 10. lemma 21

$$\lambda_i = \lambda_j \textit{ for all } 1 \leq i \leq n, 1 \leq j \leq n,$$

$$\mathrm{A}(\prod_{i=1}^{n+m} \lambda_i(w_i w_i')) \leq K_0.$$

Then there exists an involution $\mathbb{I}' \in \mathcal{I}$ *and AB homomorphisms*

$$\lambda_i' : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I}'), \;\; \theta_t' : (\mathbb{W}_t, \mathbb{I}') \to (\mathbb{H}_t, \mathbb{I}_t)$$

such that,

$$\lambda_i \circ \theta_t = \lambda_i' \circ \theta_t', \tag{135}$$

$$\lambda_i'(w_i) = \lambda_i'(w_i') \textit{ for all } 1 \leq i \leq n+m, \tag{136}$$

$$\lambda_i' = \lambda_j' \textit{ for all } 1 \leq i \leq n, 1 \leq j \leq n. \tag{137}$$

Proof: Let $\mathcal{S} = ((w_i, w_i'))_{1 \leq i \leq n+m}$ be a a sequence of pairs $(w_i, w_i') \in \mathbb{W}_t \times \mathbb{W}_t$ such that $\gamma(w_i) = \gamma(w_i')$ and let $\vec{\lambda} = (\lambda_i)_{1 \leq i \leq n+m}$ be a sequence of AB-homomorphisms from $\mathbb{W}_t$ to $\mathbb{W}_t$.

Distinguishing pair For every $i \in [1, n+m]$ we define $\equiv_i$ as the least monoid congruence over $\mathbb{W}$ containing $\{(\lambda_j(w_j), \lambda_j(w_j')) \mid i+1 \leq j \leq n+m\}$. For every $i \in [1, n+m]$ let us consider some decompositions

$$\lambda_i(w_i) = P_i \cdot S_i, \;\; \lambda_i(w_i') = P_i' \cdot S_i' \tag{138}$$

such that $P_i \equiv_i P_i'$, and this choice of the decomposition (138) minimizes the integer

$$\Delta(P_i, S_i, P_i', S_i', \theta_t).$$

(this integer was defined by equation (57)). Such a (P_i, P_i') is called a *distinguishing pair* for $(\lambda_i(w_i), \lambda_i(w_i'))$ and we denote by

$$\Delta_i(\mathcal{S}, \vec{\lambda}, \theta_t)$$

the corresponding value of $\Delta(P_i, S_i, P_i', S_i', \theta_t)$.

Size We call *size* of the triple $(\mathcal{S}, \vec{\lambda}, \theta_t)$ the multiset of natural integers:

$$\|(\mathcal{S}, \lambda, \theta_t)\| = \{\{\Delta_1(\mathcal{S}, \lambda, \theta_t)), \ldots, \Delta_i(\mathcal{S}, \lambda, \theta_t), \ldots, \Delta_{n+m}(\mathcal{S}, \lambda, \theta_t)\}\}. \tag{139}$$

For every $w \in \mathbb{W}$ we use the notation

$$\begin{aligned}
\mathrm{Alph}(w) &:= \{W \in \check{\mathcal{W}} \mid |w|_W \neq 0\} \cup \{W \in \hat{\mathcal{W}} \mid |w|_{W,\bar{W}} \neq 0\}, \\
\mathrm{w}(\mathcal{S}, \vec{\lambda}) &:= \prod_{i=1}^{n+m} \lambda_i(w_i w_i'), \\
\mathrm{Alph}(\mathcal{S}, \vec{\lambda}) &:= \mathrm{Alph}(\mathrm{w}(\mathcal{S}, \vec{\lambda})), \\
\mathrm{A}(\mathcal{S}, \vec{\lambda}) &:= \mathrm{A}(\mathrm{w}(\mathcal{S}, \vec{\lambda})).
\end{aligned}$$

Induction Let us prove lemma 22 by induction over $\|(\mathcal{S}, \vec{\lambda}, \theta_t)\|$, with respect to the partial ordering over multisets of integers induced by the natural ordering over $\mathbb{N}$ (it is known that this ordering is well-founded). Let $(\mathcal{S}, \vec{\lambda}, \theta_t)$ fulfill the hypothesis of the lemma.
Case 1: In this case we suppose that, for every $i \in [1, n+m]$, one of the two following situations occurs:

$$\lambda_i(w_i) \equiv_i \lambda_i(w_i') \tag{140}$$

$$\lambda_i(w_i) = e_i W f_i, \ \ \lambda_i(w_i') = c_i^{-1} \bar{W} d_i^{-1} \tag{141}$$

for some $W \in \hat{\mathcal{W}}, (e_i, f_i), (d_i, c_i) \in (\mathrm{Gi}(W), \mathrm{Ge}(W))$.
Let us consider the partition

$$\hat{\mathcal{W}} = \hat{\mathcal{W}}_0 \cup \mathcal{W}_1 \cup \bar{\mathcal{W}}_1, \mathcal{W}_1 = \{W_1, \cdots, W_p\}$$

where $\mathcal{W}_1 \cup \bar{\mathcal{W}}_1$ is exactly the set of variables occuring in the equations of type (141). We can modify the system $\mathcal{S}$ in such a way that in every equation of type (141), $e_i = f_i = 1$, with preservation of the hypothesis of the lemma (with the same morphisms) and also of the norm of $(\mathcal{S}, \vec{\lambda}, \theta_t)$. We can also suppose that $\mathcal{W}_1$ is exactly the set of lefthand-sides of the equations (141). For every $k \in [1, p]$, let $(W_k, c_{i(k)}^{-1} \bar{W}_k d_{i(k)}^{-1})$ be the equation of type (141) with smallest index, $i(k) \in [1, n+m]$, such that $\lambda_{i(k)}(w_{i(k)}) = W_k$. We thus have $\lambda_{i(k)}(w'_{i(k)}) = c_{i(k)}^{-1} \bar{W}_k d_{i(k)}^{-1}$. Let us notice that, since θ_t is an AB-homomorphism and $\theta_t(W_k) = \theta_t(\lambda_{i(k)}(w_{i(k)})) = \theta_t(\lambda_{i(k)}(w'_{i(k)})) = \theta_t(c_{i(k)}^{-1} \bar{W}_k d_{i(k)}^{-1})$ we know that

$$\theta_t(\bar{W}_k) = \theta_t(c_{i(k)} W_k d_{i(k)}). \tag{142}$$

As θ_t preserves δ, for every $e, e' \in (\gamma_1(W_k), \gamma_3(W_k))$,

$$e\bar{W}_k = \bar{W}_k e' \Leftrightarrow e c_{i(k)} W_k d_{i(k)} = c_{i(k)} W_k d_{i(k)} e'. \tag{143}$$

It follows that there exists a unique monoid homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$ fulfilling:

$$\lambda'(e) = \quad e \quad \text{for all } e \in \iota_A(A) \cup \iota_B(B) \tag{144}$$
$$\lambda'(W) = \quad W \quad \text{for all } W \in \mathcal{W} - \mathcal{W}_1 \tag{145}$$
$$\lambda'(W_k) = \quad W_k \quad \text{for all } 1 \leq k \leq p \tag{146}$$
$$\lambda'(\bar{W}_k) = c_{i(k)} W_k d_{i(k)} \quad \text{for all } 1 \leq k \leq p. \tag{147}$$

The involutory anti-isomorphism $\mathbb{I}_t$ defined on $\mathbb{H}_t$, maps $\theta_t(W_k)$ to $\theta_t(c_{i(k)} W_k d_{i(k)})$ and $\theta_t(\bar{W}_k)$ to $\theta_t(c_k^{-1} W_k d_k^{-1})$.
Thus the tuple $(c_{i(1)}, d_{i(1)}, \ldots, c_{i(k)}, d_{i(k)}, \ldots, c_{i(p)}, d_{i(p)})$ fulfills conditions (64)(65). Hence, by the result of §3.5.4, there exists a unique involutory monoid anti-isomorphism $\mathbb{I}' : \hat{\mathbb{W}} \to \hat{\mathbb{W}}$ such that

$$\mathbb{I}'(e) = \quad e^{-1} \quad \text{for all } e \in \iota_A(A) \cup \iota_B(B) \tag{148}$$
$$\mathbb{I}'(W) = \quad \mathbb{I}(W) \quad \text{for all } W \in \hat{\mathcal{W}}_0 \tag{149}$$
$$\mathbb{I}'(W_k) = c_{i(k)} W_k d_{i(k)} \quad \text{for all } 1 \leq k \leq p \tag{150}$$
$$\mathbb{I}'(\bar{W}_k) = c_{i(k)}^{-1} \bar{W}_k d_{i(k)}^{-1} \quad \text{for all } 1 \leq k \leq p. \tag{151}$$

It is clear that λ' preserves ι_A, ι_B. The fact that

$$\mathbb{I} \circ \lambda' = \lambda' \circ \mathbb{I}' \tag{152}$$

can be checked over the generators of $\mathbb{W}$. The only non-trivial verification is for $W = \bar{W}_k$:

$$\lambda'(\mathbb{I}(\bar{W}_k)) = \lambda'(W_k) = W_k$$

$$\mathbb{I}'(\lambda'(\bar{W}_k)) = \mathbb{I}'(c_{i(k)} W_k d_{i(k)}) = d_{i(k)}^{-1} c_{i(k)} W_k d_{i(k)} c_{i(k)}^{-1},$$

and by condition (64), the righthand-side of this last equality is W_k. Thus (152) is established. The fact that λ' preserves μ, γ, δ is ensured by:
-the hypothesis that θ_t does so,
-the fact that all the monoid generators $W \in \mathcal{W}$ are mapped by γ into a singleton
-the fact that, for $e \in \iota_A(A) \cup \iota_B(B)$, $\gamma_t(e) = \gamma_{\mathbb{W}}(e)$, and for all $t \in \gamma(e)$, $\mu_t(t, e) = \mu_{\mathbb{W}}(t, e), \delta_t(t, e) = \delta_{\mathbb{W}}(t, e)$.
We have thus established that

$$\lambda' : (\mathbb{W}, \mathbb{I}) \to (\mathbb{W}, \mathbb{I}') \text{ is an AB-homomorphism.}$$

Let us define

$$\lambda_i' = \lambda_i \circ \lambda', \theta_t' = \theta_t.$$

As every λ_i' is a composite of two AB-homomorphisms, it is an AB-homomorphism from $(\mathbb{W}_t, \mathbb{I})$ to $(\mathbb{W}_t, \mathbb{I}')$. By hypothesis, $\theta_t : \mathbb{W}_t \to \mathbb{H}_t$ is a monoid-homomorphism which preserves $\iota, \mu, \gamma, \delta$. It is clear that $\theta_t \circ \mathbb{I}$ is equal to $\mathbb{I}' \circ \theta_t$ over $\iota_A(A) \cup \iota_B(B) \cup \hat{\mathcal{W}}_0$. Moreover

$$\mathbb{I}(\theta_t(W_k)) = \theta_t(\mathbb{I}(W_k)) = \theta_t(\bar{W}_k), \qquad \text{while}$$

$$\theta_t(\mathbb{I}'(W_k)) = \theta_t(c_{i(k)}^{-1} W_k d_{i(k)}^{-1}).$$

But the hypothesis that $\theta_t(\lambda_{i(k)}(w_{i(k)})) = \theta_t(\lambda_{i(k)}(w'_{i(k)}))$ ensures that $\theta_t(\bar{W}_k) = \theta_t(c_{i(k)}^{-1} W_k d_{i(k)}^{-1})$. We have established that

$$\theta_t : (\mathbb{W}_t, \mathbb{I}') \to (\mathbb{H}_t, \mathbb{I}_t) \text{ is an AB-homomorphism.}$$

Let us check now that (λ'_i, θ'_t) fulfill conclusions (135),(136),(137) of the lemma.
Let us show that

$$\theta_t = \lambda' \circ \theta_t. \tag{153}$$

The only non-trivial verification is for generators of the form $\bar{W}_k$: from (142) we get that $\theta_t(\bar{W}_k) = \theta_t(\lambda'(\bar{W}_k))$, which establishes (153). This equality (153) and the fact that $\lambda'_i = \lambda_i \circ \lambda'$ prove (135).
Let $1 \leq i \leq n+m$. Suppose that (w_i, w'_i) is one of the pairs of the form (141) (recall we reduced to the case where $e_i = f_i = 1$): there exists some $k \in [1, p]$ such that

$$\lambda_i(w_i) = W_k, \;\; \lambda_i(w'_i) = c_i^{-1} \bar{W}_k d_i^{-1}$$

Therefore

$$\lambda'(\lambda_i(w_i)) = W_k, \;\; \lambda'(\lambda_i(w'_i)) = c_i^{-1} c_{i(k)} W_k d_{i(k)} d_i^{-1} \tag{155}$$

Applying $\lambda_i \circ \theta_t$ on (w_i, w'_i), on one hand, on $(w_{i(k)}, w'_{i(k)})$ on the other hand, we obtain

$$\theta_t(W_k) = c_i^{-1} \theta_t(\bar{W}_k) d_i^{-1} = c_{i(k)}^{-1} \theta_t(\bar{W}_k) d_{i(k)}^{-1}$$

hence

$$\theta_t(W_k) = c_i^{-1} c_{i(k)} \theta_t(W_k) d_{i(k)} d_i^{-1}.$$

This shows that $(c_i^{-1} c_{i(k)}, d_i d_{i(k)}^{-1}) \in \delta(\theta_t(W_k))$ which, as θ_t is an *AB*-homomorphism, implies that $(c_i^{-1} c_{i(k)}, d_i d_{i(k)}^{-1}) \in \delta(W_k)$, which finally proves that both righthand-sides of the two equalities in (155) are equal. We have thus established (136).
By hypothesis $\lambda_i = \lambda_j$ for all $1 \leq i, j \leq n$, which immediately implies (137).
Case 2: We suppose here that, there exists some $i \in [1, n+m]$, such that none of conditions (140)(141) does hold.
Let $i \in [1, n+m]$ be the minimal integer fulfilling $\neg(140) \wedge \neg(141)$.
Let (P_i, P'_i) be a distinguishing pair for $(\lambda_i(w_i), \lambda_i(w'_i))$. The hypothesis that $P_i \equiv_i P'_i$ implies that $\gamma(P_i) = \gamma(Pi), \theta_t(P_i) = \theta_t(P'_i)$. Lemma 11 applied on the *AB*-morphism θ_t and the elements P_i, S_i, P'_i, S'_i shows that

$$\theta_t(S_i) = \theta_t(S'_i). \tag{157}$$

Lemma 13 applied on $P_i, S_i, P'_i, S'_i \in \mathbb{W}$ asserts that $\gamma(S_i) = \gamma(S'_i)$. As θ_t restricted to $\iota_A(A)$ (resp. to $\iota_B(B)$) is injective, the value $\|S_i\| = 0$ would lead to $S_i = S'_i$, which contradicts the choice of i. Finally we must have:

$$\|S_i\| \geq 1, \;\; \|S'_i\| \geq 1, \;\; \gamma(S_i) = \gamma(S'_i).$$

Decomposing S_i, S'_i over the set of generators $\mathcal{G}_{\mathbb{W}}$ we must have

$$S_i = cW \cdot L_i; \;\; S'_i = c'W' \cdot L'_i \tag{158}$$

where

$$W, W' \in \mathcal{W}, \gamma(W) = \gamma(W') \in \mathcal{TA}, c, c' \in Gi(W).$$

Subcase 2.1:$W' = W$,$\gamma(W)$ is a H-type.
Equation (157), decomposition (158) and lemma 8 imply that there exists $d, d' \in$ Ge(W) such that

$$\theta_t(cWd) = \theta_t(c'Wd'); \;\; \theta_t(d^{-1}L_i) = \theta_t(d'^{-1}L'_i).$$

As θ_t is δ-preserving, this implies that $cWd = c'Wd'$. Taking $Q_i = P_icWd, T_i = d^{-1}L_i, Q'_i = P'_ic'Wd', T'_i = d'^{-1}L'_i$, we obtain a decomposition of $\lambda_i(w_i), \lambda_i(w'_i)$ such that $2\|\psi_t(T_i)\| + 2\|\psi_t(T'_i)\| \leq 2\|\psi_t(S_i)\| + 2\|\psi_t(S'_i)\|$ and $\chi_H(T_i) + \chi_H(T'_i) < \chi_H(S_i) + \chi_H(S'_i)$ (because T_i, T'_i cannot begin with a letter having a H-type). This is enough to entail

$$\Delta(Q_i, T_i, Q'_i, T'_i, \theta_t) < \Delta(P_i, S_i, P'_i, S'_i, \theta_t)$$

and $Q_i \equiv_i Q'_i$, violating the hypothesis of minimality in the choice of decomposition (138). This subcase is thus impossible.
Subcase 2.2: $W' = \bar{W}$, $\gamma(W)$ is a H-type.
Reasoning as in subcase 2.1, we obtain $d, d' \in$ Ge(W) such that

$$\theta_t(cWd) = \theta_t(c'\bar{W}d'); \;\; \theta_t(d^{-1}L_i) = \theta_t(d'^{-1}L'_i).$$

Let us define a new equation

$$w_{n+m+1} = cWd; \;\; w'_{n+m+1} = c'\bar{W}d',$$

and the AB-morphisms:

$$\lambda'_i = \lambda_i \text{ for all } 1 \leq i \leq n+m, \lambda'_{n+m+1} = \mathrm{Id}_{\mathbb{W}_t}, \theta'_t = \theta_t.$$

The new system $\mathcal{S}' = ((w_i, w'_i))_{1 \leq i \leq n+m+1}$ together with $\vec{\lambda}' = (\lambda_i)_{1 \leq i \leq n+m+1}$ and θ'_t fulfills the hypothesis of lemma 22.

$$P_icWd \equiv_i P'_ic'\bar{W}d'$$

$$\|\theta_t(\lambda_i(L_i))\| = \|\theta_t(\lambda_i(S_i))\| \tag{159}$$

$$2\chi_H(L_i) = 0 \leq 2\chi_H(S_i) - 2 \tag{160}$$

$$1 - \chi_{AB}(P_icWd) = 1 \leq (1 - \chi_{AB}(P_i)) + 1. \tag{161}$$

Adding up comparisons (159)(160)(161), we obtain:

$$\Delta(P_icWd, P'_ic'W'd', d^{-1}L_i, d'^{-1}L'_i, \theta_t) < \Delta(P_i, P'_i, S_i, S'_i, \theta_t)$$

hence, by definition of Δ_i,

$$\Delta_i(\mathcal{S}', \vec{\lambda}', \theta'_t) < \Delta_i(\mathcal{S}, \vec{\lambda}, \theta_t). \tag{163}$$

The integer i was supposed to do not fullfill (141), hence

$$1 \leq 1 - \chi_{AB}(P_i) + 4\|\theta_t(L_i)\|,$$

which implies

$$\Delta_{n+m+1}(\mathcal{S}', \lambda', \theta_t) < \Delta_i(\mathcal{S}, \lambda, \theta_t). \tag{165}$$

The above inequalities (163)(165) prove that

$$\{\{\Delta_i(\mathcal{S}', \vec{\lambda}', \theta'_t), \Delta_{n+m+1}(\mathcal{S}', \vec{\lambda}', \theta'_t)\}\} < \{\{\Delta_i(\mathcal{S}, \vec{\lambda}, \theta_t)\}\}.$$

Hence

$$\|(\mathcal{S}', \vec{\lambda}', \theta'_t)\| < \|(\mathcal{S}, \vec{\lambda}, \theta_t)\|.$$

By induction hypothesis, the conclusion of the lemma holds for $(\mathcal{S}', \vec{\lambda}', \theta'_t)$. This proves that it holds for $(\mathcal{S}, \vec{\lambda}, \theta_t)$ too.
Subcase 2.3: $W' \notin \{W, \bar{W}\}$,$\gamma(W)$ is a H-type.
As in subcase 2.1 we obtain that there exists $d, d' \in \mathrm{Ge}(W)$ such that

$$\theta_t(cWd) = \theta_t(c'W'd');\ \ \theta_t(d^{-1}L_i) = \theta_t(d'^{-1}L'_i). \tag{166}$$

Let us consider the monoid homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$ fulfilling:

$$\lambda'(e) = \quad e \quad \text{for all } e \in \iota_A(A) \cup \iota_B(B) \tag{167}$$
$$\lambda'(W) = c^{-1}c'W'd'd^{-1} \tag{168}$$
$$\lambda'(\bar{W}) = dd'^{-1}\bar{W}'c'^{-1}c \tag{169}$$

(this definition is written for the case where $W \in \hat{\mathcal{W}}$, in the case where $W \notin \hat{\mathcal{W}}$, last line of this definition must be cancelled). Such an homomorphism exists because (166) ensures that $(W, c^{-1}c'W'd'd^{-1})$ (resp. $(\bar{W}, dd'^{-1}\bar{W}'c'^{1}c)$ in case $W \in \hat{\mathcal{W}}$) have the same image by δ. Unicity of λ' is straightforward. Such an homomorphism λ' also preserves μ, γ, δ because θ_t does so. It preserves the partial involution because $dd'^{-1}\bar{W}'c'^{1}c = \mathbb{I}(c^{-1}c'W'd'd^{-1})$. Hence λ' is an AB-homomorphism. Let us define:

$$\lambda'_i = \lambda_i \circ \lambda' \text{ for all } 1 \leq i \leq n+m,\ \ \theta'_t = \theta_t.$$

Since $\mathrm{Alph}(\mathcal{S}, \vec{\lambda}') = \mathrm{Alph}(\mathcal{S}, \vec{\lambda}) - \{W, \bar{W}\}$, the inequality $\mathrm{A}(\mathcal{S}, \vec{\lambda}') \leq K_0$ still holds. In this sytem we now have:

$$P_i cWd \equiv_i P'_i c'W'd'$$

$$\|\theta_t(\lambda_i(d^{-1}L_i))\| = \|\theta_t(\lambda_i(S_i))\| \tag{170}$$

$$2\chi_H(d^{-1}L_i) = 0 \leq 2\chi_H(S_i) - 2 \tag{171}$$

The conjunction of comparisons (170) (171) imply

$$\Delta_i(\mathcal{S}', \lambda', \theta'_t) < \Delta_i(\mathcal{S}, \lambda, \theta_t). \tag{172}$$

and finally

$$\|(\mathcal{S}', \vec{\lambda}', \theta'_t)\| < \|(\mathcal{S}, \vec{\lambda}, \theta_t)\|.$$

The system $\mathcal{S}' = \mathcal{S}$ together with $\vec{\lambda}' = (\lambda'_i)_{1\leq i\leq n+m}$ and θ'_t fulfills the hypothesis of lemma 22 and has smaller size. By induction hypothesis the conclusion of the lemma holds for $(\mathcal{S}', \vec{\lambda}', \theta'_t)$. This proves that it holds for $(\mathcal{S}, \vec{\lambda}, \theta_t)$ too.
Subcase 2.4:$W' = W$,$\gamma(W)$ is a T-type.
Let us observe that $\|\theta_t(cWd)\| = \|\theta_t(c'Wd')\|$. In view of equation (157), decomposition (158) and the above equality, point (1) of lemma 9 applies: there exists $d, d' \in \gamma_3(W)$ such that

$$\theta_t(cWd) = \theta_t(c'Wd');\;\; \theta_t(d^{-1}L_i) = \theta_t(d'^{-1}L'_i).$$

We can then conclude as in subcase 2.1.
Subcase 2.5:$W' = \bar{W}$,$\gamma(W)$ is a T-type.
Then $\gamma(W) = \gamma(\bar{W}) = \mathbb{I}_{\mathcal{T}}(\gamma(W))$. But the two atomic T-types are exchanged by the involution $\mathbb{I}_{\mathcal{T}}$, which makes this subcase impossible.
Subcase 2.6:$W' \notin \{W, \bar{W}\}$,$\gamma(W)$ is a T-type.
By (157) $\theta_t(cWL_i) = \theta_t(c'W'L'_i)$. Using that $\gamma(W) = \gamma(W')$ is a T-type, we can apply lemma 9 on $P = \theta_t(cW), P' = \theta_t(cW'), S = \theta_t(L_i), S' = \theta_t(L'_i)$. We distinguish 3 subsubcases according to which point of lemma 9 occurs.
subsubcase 2.6.1: $\|\theta_t(cW)\| = \|\theta_t(cW')\|$.
This corresponds to point (1) of lemma 9: there exists some $d \in \gamma_3(W)$ such that,

$$\theta_t(cW) = \theta_t(cW')d,\;\; d\theta_t(L_i) = \theta_t(L'_i). \tag{173}$$

We can end this subsubcase as for case 2.3.
subsubcase 2.6.2: $\|\theta_t(cW)\| < \|\theta_t(cW')\|$.
This corresponds to point (2) of lemma 9: $\exists d \in \mathrm{Ge}(W), P'_1, P'_2, P'_3 \in \mathbb{H}_t$,$P'_1, P'_3$ have a T-type,P'_2 has a H-type $\gamma(P'_3) \cdot \gamma(\theta_t(L'_i)) \neq \emptyset$ and

$$\theta_t(cW) = P'_1 d,\;\; \theta_t(c'W') = P'_1 \cdot P'_2 \cdot P'_3,\;\; d\theta_t(L_i) = P'_2 P'_3 \theta_t(L'_i). \tag{174}$$

Applying lemma 14, we obtain some $P_2, S \in \mathbb{W}$ such that:

$$dL_i = P_2 \cdot S,\;\; \theta_t(P_2) = P'_2,\;\; \theta_t(S) = \theta_t(P'_3 L'_i). \tag{175}$$

first step
Let us assume, in this step, that $W' \in \hat{\mathbb{W}}$.
By axiom (23), $c' \in \hat{\mathbb{W}}$, by axiom (24) $c'W' \in \hat{\mathbb{W}}$. Axiom(34) on AB-homomorphisms implies that $P'_1 \cdot P'_2 \cdot P'_3 \in \mathrm{dom}\mathbb{I}_t$. Axiom (24) implies that all the P'_i $(1 \leq i \leq 3)$ belong to $\mathrm{dom}(\mathbb{I}_t)$ and axiom (34) implies that

$$W, P_2 \in \hat{\mathbb{W}}. \tag{176}$$

We saw above that $\gamma(P'_3)$ is a T-type and, by hypothesis, $\mathrm{A}(\mathcal{S}, \vec{\lambda}) \leq K_0 < \mathrm{Card}(\mathcal{V}_0)$. Hence, we can choose a letter $W_3 \in \mathcal{W}$ such that

$$W_3 \notin \mathrm{Alph}(\mathcal{S}, \vec{\lambda}),\;\; \gamma(W_3) = \gamma(P'_3),\;\; \mu(W_3) = \mu(P'_3),\;\; \delta(W_3) = \delta(P'_3).$$

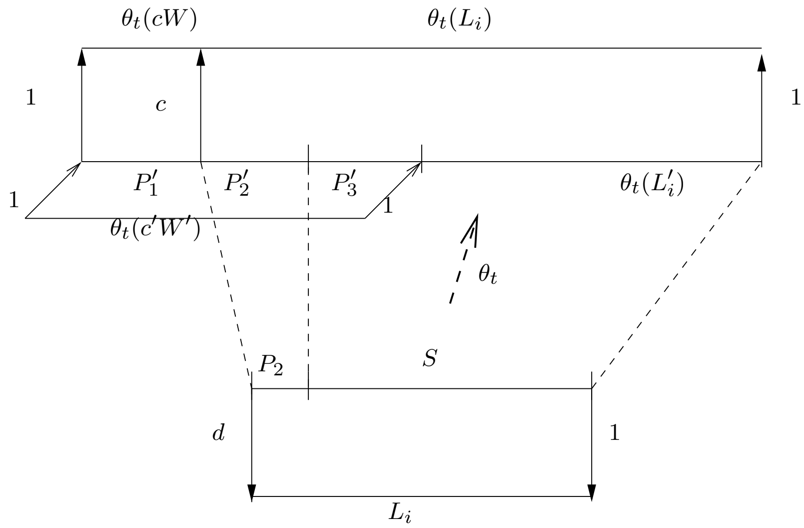

Fig. 11. subsubcase 2.6.2

We define a monoid-homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$ by

$$\begin{aligned}
\lambda'(e) = &\quad e && \text{for all } e \in \iota_A(A) \cup \iota_B(B) \\
\lambda'(W'') = &\quad W'' && \text{for all } W'' \in \mathcal{W} - \{W', \bar{W}'\} \\
\lambda'(W') = &\quad c'^{-1}cWd^{-1}P_2W_3 && (177) \\
\lambda'(\bar{W}') = &\ \mathbb{I}(c'^{-1}cWd^{-1}P_2W_3)\ . && (178)
\end{aligned}$$

and we define a new monoid-homomorphism $\theta' : \mathbb{W}_t \to \mathbb{H}_t$ by

$$\begin{aligned}
\theta'(\iota_A(a) = &\quad a && \text{for all } a \in A \\
\theta'(\iota_B(b) = &\quad b && \text{for all } b \in B \\
\theta'(W'') = &\quad W'' && \text{for all } W'' \in \mathcal{W} - \{W_3, \bar{W}_3\} \\
\theta'(W_3) = &\quad P'_3 && (179) \\
\theta'(\bar{W}_3) = &\ \mathbb{I}_t(P'_3). && (180)
\end{aligned}$$

As in the above cases we can check that λ', θ'_t are AB-homomorphisms. Let us check that, for every $W'' \in \mathrm{Alph}(\mathcal{S}, \vec{\lambda})$

$$\theta'_t(\lambda'(W'')) = \theta_t(W''). \tag{181}$$

Since $W_3 \notin \mathrm{Alph}(\mathcal{S}, \vec{\lambda}), \overline{W}_3 \notin \mathrm{Alph}(\mathcal{S}, \vec{\lambda})$, for every $W'' \in \mathrm{Alph}(\mathcal{S}, \vec{\lambda}) - \{W', \bar{W}'\}$, $\theta'_t(\lambda'_t(W'')) = \theta'_t(W'') = \theta_t(W'')$.

Moreover

$$\begin{aligned}
\theta'_t(\lambda'(W')) &= \theta'_t(c'^{-1}cWd^{-1}P_2W_3) && \text{by (179)}\\
&= c'^{-1}\theta_t(cW)d^{-1}\theta'_t(P_2)P'_3 && \text{by (179)}\\
&= c'^{-1}\theta_t(cW)d^{-1}\theta_t(P_2)P'_3 && (\text{ no occurrence of } W_3, \bar{W}_3 \text{ in } P_2)\\
&= c'^{-1}\theta_t(cW)d^{-1}P'_2P'_3 && \text{by (175), second equation}\\
&= c'^{-1}P'_1dd^{-1}P'_2P'_3 && \text{by (174), first equation}\\
&= \theta_t(W') && \text{by (174), second equation.}
\end{aligned}$$

Finally, since $\lambda', \theta'_t, \theta_t$ preserve the involutions $\mathbb{I}_{\mathbb{W}}, \mathbb{I}_t$, we also get

$$\theta'_t(\lambda'(\bar{W}')) = \theta_t(\bar{W}').$$

Let us define

$$\lambda'_i = \lambda_i \circ \lambda' \text{ for all } 1 \leq i \leq n+m.$$

Let us notice that $\gamma(W')$ is a T-type while $\gamma(P_2)$ is a H-type. Hence $\|\theta_t(P_2)\| = 0 < \|\theta_t(W')\|$ which proves that

$$W', \bar{W}' \notin \mathrm{Alph}(P_2)$$

Hence $\mathrm{Alph}(\mathcal{S}', \vec{\lambda}') = \mathrm{Alph}(\mathcal{S}, \vec{\lambda}) \cup \{W_3, \bar{W}_3\} - \{W', \bar{W}'\}$. We thus obtain

$$\mathrm{A}(\mathcal{S}', \vec{\lambda}') \leq K_0. \tag{182}$$

Equality (181) and inequality (182) ensure that the new triple $(\mathcal{S}, \vec{\lambda}', \theta')$ fulfills the hypothesis of lemma 22.
Let us evaluate now the size of this new triple.

$$\begin{aligned}
(\lambda'_i(w_i), \lambda'_i(w'_i)) &= (\lambda'(P_icWL_i), \lambda'(P'_ic'W'L'_i))\\
&= (\lambda'(P_i)cW\lambda'(d^{-1}P_2S), \lambda'(P'_i)c'(c'^{-1}cWd^{-1}P_2W_3)\lambda'(L'_i))\\
&= (\lambda'(P_i)cWd^{-1}P_2 \cdot \lambda'(S)), \lambda'(P'_i)cWd^{-1}P_2 \cdot W_3\lambda'(L'_i)) \quad \text{by (6.2.1)}
\end{aligned}$$

Let us set

$$Q_i = \lambda'(P_i)cWd^{-1}P_2,\;\; T_i = \lambda'(S),\;\; Q'_i = \lambda'(P'_i)cWd^{-1}P_2,\;\; T'_i = W_3\lambda'(L'_i)$$

$$\begin{aligned}
\Delta(Q_i, T_i, Q'_i, T'_i, \theta'_t) &\leq 3 + 4\|\theta'_t(W_3)\theta'_t(\lambda'(L'_i))\|\\
&= 3 + 4\|P'_3\| + 4\|\theta_t(L'_i)\| && (\text{ by (179), (181)})\\
&< 4(\|P'_3\| + \|P'_3\| + \|P'_3\|) + 4\|\theta_t(L'_i)\| && (\text{ because } \|P'_1\| \geq 1)\\
&\leq \Delta(P_i, S_i, P'_i, S'_i, \theta_t).
\end{aligned} \tag{183}$$

The hypothesis that P_i, P'_i are related by the monoid-congruence generated by the set of pairs $\{(\lambda_j(w_j), \lambda_j(w'_j)) \mid i+1 \leq j \leq n+m\}$ implies that

$\lambda'(P_i), \lambda'(P'_i)$ are related by the monoid-congruence generated by the set of pairs $\{(\lambda'(\lambda_j(w_j)), \lambda'(\lambda_j(w'_j))) \mid i+1 \leq j \leq n+m\}$. It follows that

$$\Delta_i(\mathcal{S}, \vec{\lambda}', \theta'_t) \leq \Delta(Q_i, T_i, Q'_i, T'_i, \theta'_t) < \Delta(P_i, S_i, P'_i, S'_i, \theta_t) = \Delta_i(\mathcal{S}, \vec{\lambda}, \theta_t)$$

and thus:

$$\|(\mathcal{S}', \vec{\lambda}', \theta')\| < \|(\mathcal{S}, \vec{\lambda}, \theta)\|.$$

By induction hypothesis, conclusions (135-137) are true for $(\mathcal{S}, \vec{\lambda}')$, which also implies that they hold for $(\mathcal{S}, \vec{\lambda})$.
second step
Let us handle the situation where W' does not belong to $\hat{\mathbb{W}}$.
We just cancel the last line of (178) and can conclude as in first step.
subsubcase 2.6.3: $\|\theta_t(cW)\| > \|\theta_t(c'W')\|$.
Symetric to subsubcase 2.6.2. □
`Check that the algebraic lemmas apply to` $\mathbb{W}_t$ `(as well as to` $\mathbb{W}$`).`

6.2.2. *From* $\mathbb{W}$*-solutions to t-solutions*

Easy direction.

7. Equations over $\mathbb{U}$

7.1. *The group* $\mathbb{U}$

Let us adjoin to $\check{W}$ an alphabet of inverses $\overline{\check{W}}$. The extended alphabet $\mathcal{W}' := \check{W} \cup \overline{\check{W}} \cup \hat{W}$ is now endowed with a total involution $W \mapsto \bar{W}$, which extends the partial involution $\mathbb{I}_{\mathbb{W}}$. The maps γ, μ, δ are extended to $\mathcal{W}'$ in such a way that the axiom (31) of AB-algebras is fulfilled by $\mathcal{W}'^* * A * B$ endowed with this involution. We define the group

$$\mathbb{U} := \langle A * B, \mathcal{W}'; \bar{W}eW = \delta(W)(e) \quad (e \in \mathrm{Gi}(W), W \in \mathcal{W})\rangle \tag{184}$$

i.e. it is an HNN-extension of the free product $A * B$ with, as stable letters, all the letters W from $\mathcal{W}'$ and as partial isomorphisms, the maps $\delta(W)$. We identify ι_A (resp. ι_B) with the natural embedding of A (resp. B) into $A * B$. We denote by $\equiv_{\mathbb{U}}$ the monoid-congruence over $\mathcal{W}'^* * A * B$ generated by the set of relations (184). We denote by

$$\pi_U : \mathcal{W}'^* * A * B \to \mathbb{U}$$

the homomorphism $z \mapsto [z]_{\equiv_{\mathbb{U}}}$. All the pairs of (58) also belong to $\equiv_{\mathbb{U}}$, hence $\equiv \subseteq \equiv_{\mathbb{U}}$. Thus, there exists a unique map $\bar{\pi}_U : \mathbb{W} \to \mathbb{U}$ such that

$$\pi_U|_{\mathcal{W}^* * A * B} = \pi_{\equiv} \circ \bar{\pi}_U.$$

An element $z \in \mathcal{W}'^* * A * B$ is said to be a *reduced sequence* iff it does not contain any factor of the form $\bar{W}eW$ with $W \in \mathcal{W}', e \in \mathrm{Gi}(W)$. We denote by $\mathrm{Red}(A * B, \mathcal{W}')$ the subset of $\mathcal{W}'^* * A * B$ consisting of all reduced sequences.

Lemma 23. *Let z, z' be some reduced sequences in $\mathcal{W}^* * A * B$. Then $z \equiv_{\mathbb{U}} z'$ if and only if $z \equiv z'$.*

This lemma is obtained via the analogue of lemma 1, but in the case of an HNN-extension with a set $\mathcal{W}'$ of stable letters (instead of just $\{t, \bar{t}\}$). Such an analogue can be obtained from lemma 1, by induction over the number of stable letters.

Lemma 24. *Let $z, z' \in \mathcal{W}^* * A * B$ such that $\gamma(z) = \gamma(z') \neq \emptyset$. Then $z \equiv_{\mathbb{U}} z'$ if and only if $z \equiv z'$.*

Every z such that $\gamma(z) \neq \emptyset$ is a reduced sequence. This lemma is thus a direct corollary of lemma 23.

7.2. *From* $\mathbb{W}$*-equations to* $\mathbb{U}$*-equations*

We use here the notion of system of equations with rational constraint over $\mathbb{U}$, as defined in §2.4.1 for any monoid. Let us consider a system of $\mathbb{W}$-equations of the form described in (133), together with a morphism:

$$\mathcal{S}_{\mathbb{W}} = ((w_i, w_i')_{i \in I}, \mathbb{I}'); \;\; \sigma_H \in \mathrm{Hom}_{AB}((\mathbb{W}_H, \mathbb{I}), (\mathbb{W}_H, \mathbb{I}')).$$

The involution $\mathbb{I}'$ is given by formulas (63). Let us notice that every letter $W_k \in \mathcal{W}_1$, i.e. on which $\mathbb{I}'$ has a "non-standard"value, fulfills (66), so that $\gamma(W_k)$ must be a H-type and $\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k)$.
Let $z_i, z'_i \in \mathcal{W}_t^* * A * B$ be some representatives, modulo $\equiv$, of w_i, w'_i. By lemma 18, we can also choose an AB-homomorphism $\tilde{\sigma}_H : \mathcal{W}_t^* * A * B \to \mathcal{W}_t^* * A * B$ such that $\pi_\equiv \circ \sigma_H = \tilde{\sigma}_H \circ \pi_\equiv$. For every $W \in \mathcal{W}_t$ we consider the following rational subsets of $\mathcal{W}_t^* * A * B$:

$$\begin{aligned}
R_{\mathbb{I},W} &:= \{z \in \mathcal{W}_t^* * A * B \mid \gamma(z) = \gamma(W) \wedge z \in \hat{\mathcal{W}}_t^* * A * B \Leftrightarrow W \in \hat{\mathcal{W}}_t\},\\
R_{\mu,W} &:= \qquad \{z \in \mathcal{W}_t^* * A * B \mid \gamma(z) = \gamma(W) \wedge \mu(z) = \mu(W)\},\\
R_{\delta,W} &:= \qquad \{z \in \mathcal{W}_t^* * A * B \mid \gamma(z) = \gamma(W) \wedge \delta(z) = \delta(W)\}.\\
R_{H,W} &:= \quad \{\tilde{\sigma}_H(W)\} \text{ if } W \in \mathcal{W}_H; \;\; R_{H,W} := \mathcal{W}_t^* * A * B \text{ if } W \notin \mathcal{W}_H
\end{aligned}$$

We define the rational constraint $\mathsf{C} : \mathcal{W}_t^* * A * B \to \mathrm{Rat}(\mathbb{U})$ by:

$$\forall W \in \mathcal{W}_t, \;\; \mathsf{C}(W) := \pi_\mathbb{U}(R_{\mathbb{I},W}) \cap \pi_\mathbb{U}(R_{\mu,W}) \cap \pi_\mathbb{U}(R_{\delta,W}) \cap \pi_\mathbb{U}(R_{H,W}),$$

$$\forall e \in \iota_A(A) \cup \iota_B(B), \;\; \mathsf{C}(e) := \pi_\mathbb{U}(\{e\}).$$

Let us define a system of equations over $\mathbb{U}$ with rational constraint:

$$\mathcal{S}_\mathbb{U}(\sigma_H) := ((z_i, z'_i)_{1 \le i \le n}, \mathsf{C}),$$

(note that $\mathcal{S}_\mathbb{U}(\sigma_H)$ does really depend on $\mathcal{S}_\mathbb{W}$ and σ_H, but not on the choice of $\tilde{\sigma}_H$).

Lemma 25. *The map* $\Phi : \mathrm{Hom}_{AB}((\mathbb{W}_t, \mathbb{I}), (\mathbb{W}_t, \mathbb{I}')) \to \mathrm{Hom}(\mathcal{W}_t^* * A * B, \mathbb{U})$, $\sigma_\mathbb{W} \mapsto \pi_\mathbb{W} \circ \sigma_\mathbb{W} \circ \bar{\pi}_\mathbb{U}$ *induces a bijection from the set of solutions of* $\mathcal{S}_\mathbb{W}$ *which extend* σ_H, *into the set of solutions of* $\mathcal{S}_\mathbb{U}(\sigma_H)$.

We prove this lemma in the subsequent three subsections; see figure 12 for the general context and figure 13 for the details of the proof.

7.2.1. *From* $\mathbb{W}$*-solutions to* $\mathbb{U}$*-solutions*

Let $\sigma_\mathbb{W}$ be a solution of $\mathcal{S}_\mathbb{W}$, extending σ_H.Let $\sigma_\mathbb{U} := \Phi(\sigma_\mathbb{W})$ i.e.

$$\sigma_\mathbb{U} := \pi_\equiv \circ \sigma_\mathbb{W} \circ \bar{\pi}_\mathbb{U}.$$

From the definitions of z_i, z'_i we get

$$\sigma_\mathbb{U}(z_i) = \bar{\pi}_\mathbb{U}(\sigma_\mathbb{W}(w_i) = \bar{\pi}_\mathbb{U}(\sigma_\mathbb{W}(w'_i) = \sigma_\mathbb{U}(z'_i), \tag{186}$$

thus $\sigma_\mathbb{U}$ is a solution of the system of equations $(z_i, z'_i)_{1 \le i \le n}$.
The map $\pi_\equiv \circ \sigma_\mathbb{W}$ is an AB-homomorphism, hence it maps $\mathbb{I}$ on $\mathbb{I}'$ and preserves γ, μ, δ. It follows that, for every $W \in \mathcal{W}_t$:

$$\sigma_\mathbb{W}(\pi_\equiv(W)) \in \pi_\equiv(R_{\mathbb{I},W}) \cap \pi_\equiv(R_{\mu,W}) \cap \pi_\equiv(R_{\delta,W}).$$

$$
\begin{tikzcd}
 & \mathcal{W}_t^* * A * B \arrow[d, "\pi_{\equiv}"'] \arrow[dddrrr, "\sigma_{\mathbb{U}}"] & & & \\
\mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[dd, "\sigma"'] & (\mathbb{W}_t, \mathbb{I}) \arrow[dd, "\sigma_t"'] \arrow[ddrr, "\sigma_{\mathbb{W}}"'] & \mathbb{W}_H \arrow[l] \arrow[dr, "\sigma_H"'] & & \\
 & & & \mathbb{W}_H \arrow[d] & \\
\mathbb{G} & (\mathbb{H}_t, \mathbb{I}_t) \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & & (\mathbb{W}_t, \mathbb{I}') \arrow[ll, "\psi_t"] \arrow[r, "\bar{\pi}_{\mathbb{U}}"'] & \mathbb{U}
\end{tikzcd}
$$

Fig. 12. lemma 25: the context

As $\sigma_{\mathbb{W}}$ extends σ_H we get, for every $W \in \mathcal{W}$:

$$\sigma_{\mathbb{W}}(\pi_{\equiv}(W)) \in \pi_{\equiv}(R_{H,W}).$$

Applying $\bar{\pi}_{\mathbb{U}}$ on both sides of the above membership relations we obtain that, for every $W \in \mathcal{W}_t$:

$$\sigma_{\mathbb{U}}(W) \in \mathsf{C}(W). \tag{187}$$

Moreover the three maps $\pi_{\equiv}, \sigma_{\mathbb{W}}, \bar{\pi}_{\mathbb{U}}$ are fixing every element of $A \cup B$, so that, for every $e \in A \cup B$:

$$\sigma_{\mathbb{U}}(e) \in \mathsf{C}(e). \tag{188}$$

By (186)(187)(188) $\sigma_{\mathbb{U}}$ is a solution of $\mathcal{S}_{\mathbb{U}}(\sigma_H)$.

7.2.2. *From $\mathbb{U}$-solutions to $\mathbb{W}$-solutions*

Let $\sigma_{\mathbb{U}}$ be a solution of $\mathcal{S}_{\mathbb{U}}(\sigma_H)$.
Since, for every $W \in \mathcal{W}_t$, $\sigma_{\mathbb{U}}(W) \in \mathsf{C}(W) \subseteq \pi_{\mathbb{U}}(R_{\mu,W})$, there is a choice map $\tilde{\sigma}_{\mathbb{U}} : \mathcal{W}_t \to \mathcal{W}_t^* * A * B$ fulfilling:

$$\forall W \in \mathcal{W}_t, \tilde{\sigma}_{\mathbb{U}}(W) \in R_{\mu,W}, \;\; \pi_{\mathbb{U}}(\tilde{\sigma}_{\mathbb{U}}(W)) = \sigma_{\mathbb{U}}(W). \tag{189}$$

Let us denote by $\tilde{\sigma}_{\mathbb{U}}$ the unique monoid homomorphism fixing every element of $A * B$ and extending the above choice map. Since $\sigma_{\mathbb{U}}(W) \in \mathsf{C}(W)$, there exist $z_{\mathbb{I},W} \in R_{\mathbb{I},W}, z_{\delta,W} \in R_{\delta,W}, z_{H,W} \in R_{H,W}$ fulfilling

$$\pi_{\mathbb{U}}(\tilde{\sigma}_{\mathbb{U}}(W)) = \pi_{\mathbb{U}}(z_{\mathbb{I},W}) = \pi_{\mathbb{U}}(z_{\delta,W}) = \pi_{\mathbb{U}}(z_{H,W}).$$

$$
\begin{tikzcd}
\mathcal{W}_t^* * A * B \arrow[r, "\tilde{\sigma}_{\mathbb{U}}"] \arrow[d, "\pi_{\equiv}"'] \arrow[rr, bend left, "\sigma_{\mathbb{U}}"] & \mathcal{W}_t^* * A * B \arrow[r, "\pi_{\mathbb{U}}"] \arrow[d, "\pi_{\equiv}"'] & \mathbb{U} \\
\mathbb{W}_t \arrow[r, "\sigma_{\mathbb{W}}"] & \mathbb{W}_t \arrow[ur, "\tilde{\pi}_{\mathbb{U}}"'] & \\
\mathbb{W}_H \arrow[u] \arrow[r, "\sigma_H"] & \mathbb{W}_H \arrow[u] & \\
\mathcal{W}_H^* * A * B \arrow[u, "\pi_{\equiv}"] \arrow[r, "\tilde{\sigma}_H"] & \mathcal{W}_H^* * A * B \arrow[u, "\pi_{\equiv}"] &
\end{tikzcd}
$$

Fig. 13. lemma 25:the proof

All these $z_{\mathbb{I},W}, z_{\delta,W}, z_{H,W}$ have a non-empty image by γ and are equivalent with $\tilde{\sigma}_{\mathbb{U}}(W)$ modulo $\equiv_{\mathbb{U}}$. By lemma 24:

$$\tilde{\sigma}_{\mathbb{U}}(W) \equiv z_{\mathbb{I},W} \equiv z_{\delta,W} \equiv z_{H,W}.$$

The equivalence $\equiv$ preserves the AB-structure of $\mathcal{W}^* * A * B$ (see (59-60)), hence, for every $W \in \mathcal{W}_t$,

$$\tilde{\sigma}_{\mathbb{U}}(W) \in \hat{\mathcal{W}}_t * A * B \Leftrightarrow W \in \hat{\mathcal{W}}_t$$

$$\delta(\tilde{\sigma}_{\mathbb{U}}(W)) = \delta(W)$$

$$\pi_{\equiv}(\tilde{\sigma}_{\mathbb{U}}(W)) = \pi_{\equiv}(\tilde{\sigma}_H(W)) \text{ if } W \in \mathcal{W}_H. \tag{190}$$

The map $\tilde{\sigma}_{\mathbb{U}} : \mathcal{W}_t \to \mathcal{W}_t^* * A * B$ defined by (189) can thus be extended into an AB-homomorphism $\tilde{\sigma}_{\mathbb{U}} : \mathcal{W}_t^* * A * B \to \mathcal{W}_t^* * A * B$. By lemma 16 it defines a unique AB-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t$ fulfilling:

$$\pi_{\equiv} \circ \sigma_{\mathbb{W}} = \tilde{\sigma}_{\mathbb{W}} \circ \pi_{\equiv}. \tag{191}$$

By (190)(191) $\sigma_{\mathbb{W}}$ extends σ_H.
By hypothesis $\tilde{\sigma}_{\mathbb{U}} \circ \pi_{\mathbb{U}}$ is a solution of $\mathcal{S}_{\mathbb{U}}(\sigma_H)$ hence:

$$\pi_{\mathbb{U}}(\tilde{\sigma}_{\mathbb{U}}(z_i)) = \pi_{\mathbb{U}}(\tilde{\sigma}_{\mathbb{U}}(z_i'))$$

i.e. $\tilde{\sigma}_{\mathbb{U}}(z_i) \equiv_{\mathbb{U}} \tilde{\sigma}_{\mathbb{U}}(z'_i)$. We know that $\gamma(z_i) = \gamma(z'_i) \neq \emptyset$ and that $\tilde{\sigma}_{\mathbb{U}}$ preserves γ. Using lemma 24 we conclude that $\tilde{\sigma}_{\mathbb{U}}(z_i) \equiv \tilde{\sigma}_{\mathbb{U}}(z'_i)$, hence

$$\sigma_{\mathbb{U}}(w_i) = \sigma_{\mathbb{U}}(w'_i).$$

Using (191), $\pi_{\equiv} \circ \sigma_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}} = \tilde{\sigma}_{\mathbb{W}} \circ \pi_{\equiv} \circ \bar{\pi}_{\mathbb{U}} = \tilde{\sigma}_{\mathbb{W}} \circ \pi_{\mathbb{U}} = \sigma_{\mathbb{U}}$. Finally, $\sigma_{\mathbb{W}}$ is a solution of $\mathcal{S}_{\mathbb{W}}$ which extends σ_H and

$$\sigma_{\mathbb{U}} = \Phi(\sigma_{\mathbb{W}}).$$

7.2.3. *Bijection* Φ

Subsubsection 7.2.2 established that Φ is surjective. Let us check it is injective. Suppose that $\sigma_{\mathbb{W}}, \sigma'_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t)$ fulfill:

$$\pi_{\equiv} \circ \sigma_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}} = \pi_{\equiv} \circ \sigma'_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}}$$

As $\pi_{\equiv}$ is surjective we get

$$\sigma_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}} = \sigma'_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}}.$$

By lemma 24, $\bar{\pi}_{\mathbb{U}}$ is injective over $\{z \in \mathbb{W}_t \mid \gamma(z) \neq \emptyset\}$, hence, for every $g \in \mathcal{W}_t \cup \iota_A(A) \cup \iota_B(B)$,

$$\sigma_{\mathbb{W}}(g) = \sigma'_{\mathbb{W}}(g),$$

which implies

$$\sigma = \sigma'.$$

By the above three subsubsections, lemma 25 is proved.

8. Transfer of solvability

8.1. *The structure of* $\mathbb{U}$

Since $\mathbb{U}$ is built from two finite groups A, B by a finite number of operations which are either a free product or an HNN-extension, $\mathbb{U}$ is a virtually free group.
Let us notice that, for the particular virtually free groups $\mathbb{K}$ of the form:

$$\mathbb{K} = A \propto F(\mathcal{V}) \tag{192}$$

i.e. a semi-direct product of a finite group A by a free group with finite rank, the decidability of the satisfiability problem for equations with rational constraints in $\mathbb{K}$ is Turing-reducible to the same problem in the free group $F(\mathcal{V})$. Hence, by the main theorem of [DHG05], this problem is decidable. By successive sequences of Tietze transformations, we show that $\mathbb{U}$ can be constructed from a group $\mathbb{K}$ of the form (192) by a finite number of HNN-extensions, with associated subgroups of *strictly smaller* cardinality than A.

First transformation Let $t \in \mathcal{W}$ such that $\delta(t)$ is a full isomorphism $A \to B$, for example we can choose t such that $\delta(t) = \varphi$. Let us apply the Tietze transformation:

$$b --> \bar{t}\varphi^{-1}(b)t \text{ for all } b \in B$$

we obtain a presentation with A as base group and relations of the form:

$$\begin{aligned} \bar{W}aW = &\quad \delta(W)(a); && \text{for } \mathrm{Gi}(W) = \mathrm{Ge}(W) = A \\ \bar{W}aW = &\ \bar{t}\varphi^{-1}(\delta(W)(a))t; && \text{for } \mathrm{Gi}(W) = A, \mathrm{Ge}(W) = B, W \neq t. \end{aligned}$$

Second transformation Taking as new set of generators:

$$A \cup \{t, \bar{t}\} \cup \{W \in \mathcal{W}, \mathrm{Gi}(W) = \mathrm{Ge}(W) = A, W \neq t\}$$

$$\cup \{t\bar{W}, \mathrm{Gi}(W) = B, \mathrm{Ge}(W) = A\} \cup \{W\bar{t}, \mathrm{Gi}(W) = A, \mathrm{Ge}(W) = B\}$$

we obtain a set of relations of the form:

$$\bar{V}aV = \delta'(V)(a) \text{ for } V \in \mathcal{V},$$

with $\delta' : \mathcal{V} \to \mathrm{PIs}(A, A)$.

Decomposition Let $\mathcal{V}_A := \{V \in \mathcal{V}, \mathrm{dom}(\delta'(V)) = \mathrm{im}(\delta'(V) = A\}$ and let

$$\mathbb{K} := \langle A; \bar{V}aV = \delta'(V)(a) \text{ for } V \in \mathcal{V}_A \rangle.$$

We claim that $\mathbb{K}$ is of the form (192) and $\mathbb{U}$ is obtained from $\mathbb{K}$ by a finite number of HNN-extension operations, with associated subgroups of cardinality $< |A|$.

8.2. *Inductive transfer*

We prove now a general transfer theorem for systems of equations with rational constraints. We first treat the case of groups since it is technically simpler.

8.2.1. *Inductive transfer for groups*

Proposition 26. *Let $\mathbb{H}$ be a group and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the pair of problems (Q_1, Q_2), where*
1- Q_1 is the SAT-problem for systems of equations with rational constraints in $\mathbb{U}$
2- Q_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$

Proof: Let us consider a system $\mathcal{S}_0$ of equations with rational constraints in $\mathbb{G}$. By proposition 89 it can be reduced to a system $\mathcal{S}_{\mathbb{G}}$ in quadratic normal form. By lemma 20, solving $\mathcal{S}_{\mathbb{G}}$ is reduced to problem P1 and the auxiliary problem AP1:
P1 Compute the alphabet $\mathcal{W}_t$.
AP1 Solve the disjunction of all the pairs of systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e}), \mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ by a common $\sigma_t \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{H}_t)$.

For every value of $\vec{W}, \vec{e}$, by lemma 21 AP1 reduces to:
AP2 Find an involution $\mathbb{I}' \in \mathcal{I}$ and a solution $\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}((\mathbb{W}_t, \mathbb{I}), (\mathbb{W}_t, \mathbb{I}'))$ to the systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$, and find a $\psi_t \in \mathrm{Hom}_{AB}((\mathbb{W}_t, \mathbb{I}'), (\mathbb{H}_t, \mathbb{I}_t))$ such that $\sigma_{\mathbb{W}} \circ \psi_t$ solves $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$.
By lemma 25, and a (finite) enumeration of all the $\mathbb{I}' \in \mathcal{I}$, and $\sigma_H \in \mathrm{Hom}_{AB}(\mathbb{W}_H, \mathbb{W}_H)$, AP2 reduces to
P2
P2.1 Solve the system $\mathcal{S}_{\mathbb{U}}(\sigma_H)$,
P2.2 Find a $\psi_{H,t} \in \mathrm{Hom}_{AB}((\mathbb{W}_H, \mathbb{I}'), (\mathbb{H}_t, \mathbb{I}_t))$, solving $\sigma_H(\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e}))$. Let us examine now the remaining problems P1,P2.1,P2.2.
Problem P1 consists, for every letter $W \in \mathcal{W}$, in deciding whether there exists an AB-homomorphism from the sub-AB-algebra $\langle W \rangle$ into $\mathbb{H}_t$. Let us suppose $\mathbb{I}'$ is given by the formulas (63) of §3.5.4. For every $W \in \mathcal{W}_H$ we define the sets

$$\mathsf{C}_I(W) := \mathbb{H}; \quad \text{if } W \in \hat{\mathcal{W}}_0,$$

$$\mathsf{C}_I(W_k) := \{h \in \mathbb{H} \mid a_k h b_k = h^{-1}\},; \;\; \mathsf{C}_I(\bar{W}_k) := \{h \in \mathbb{H} \mid a_k^{-1} h b_k^{-1} = h^{-1}\},$$

$$\mathsf{C}_\gamma(W) := \{h \in \mathbb{H} \mid \gamma(W) \subseteq \gamma(h)\}; \;\; \mathsf{C}_\mu(W) := \{h \in \mathbb{H} \mid \mu_1(W) = \mu_{\mathcal{A},1}(h)\},$$

$$\mathsf{C}_\delta(W) := \{h \in \mathbb{H} \mid \forall t \in \gamma(W), \delta(t, W) = \delta(t, h)\},$$

$$\mathsf{C}(W) := \mathsf{C}_I(W) \cap \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W).$$

Let us denote, for any monoid $\mathbb{M}$, by EQR($\mathbb{M}$) the set of all subsets of $\mathbb{M}$ which can be defined by a system of equations with rational constraints. We observe that C_I take values in EQR($\mathbb{H}$), C_γ take values in $\mathcal{B}(\{\{1\}, A, B, \mathbb{H}\})$ and C_μ take values in $\mathcal{B}(\mathrm{RAT}(\mathbb{H}))$. The map C_δ take values which are intersections of subsets of the form

$$S_H(c, d) := \{h \in \mathbb{H} \mid cs \sim sd\} \tag{193}$$

for $c, d \in A \cup B$ and of subsets of the form $\mathbb{H} - S_H(c,d)$. It is clear that $S_H(c,d) \in \mathrm{EQR}(\mathbb{H})$. We can notice that, $ch \neq hd \Leftrightarrow \exists h' \in \mathbb{H}, h^{-1}chd^{-1} = h' \wedge h' \notin \{1\}$ i.e. $\mathbb{H} - S_H(c,d)$ belongs to EQR($\mathbb{H}$). Therefore, each of the four subsets $\mathsf{C}_I(W), \mathsf{C}_\gamma(W), \mathsf{C}_\mu(W), \mathsf{C}_\delta(W)$ belongs to EQR($\mathbb{H}$), so that $\mathsf{C}(W)$ also belongs to EQR($\mathbb{H}$). Deciding whether $\mathsf{C}(W) = \emptyset$ is thus an instance of Q_2.
Let $W \in \mathcal{W} - \mathcal{W}_H$. We set now

$$\mathsf{C}_I(W) := \mathbb{H} * \{t, \bar{t}\}^*,$$

$$\mathsf{C}_\gamma(W) := \{s \in \mathbb{H}*\{t,\bar{t}\}^* \mid \gamma(W) \subseteq \gamma(s)\}; \;\; \mathsf{C}_\mu(W) := \{s \in \mathbb{H}*\{t,\bar{t}\}^* \mid \mu_1(W) = \mu_{\mathcal{A}}(s)\}$$

$$\mathsf{C}_\delta(W) := \{s \in \mathbb{H} * \{t,\bar{t}\}^* \mid \forall t \in \gamma(W), \delta(t,W) = \delta(t,s)\}$$

and finally

$$\mathsf{C}(W) := \mathsf{C}_I(W) \cap \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W).$$

The subset $\mathsf{C}_\gamma(W)$ is recognized by the t-automaton $\mathcal{R}_6$, with suitably chosen initial and terminal states (along equation (40)), $\mathsf{C}_\mu(W)$ is a finite union of subsets recognized by the t-automaton $\mathcal{A}$ with modified initial and terminal states. The subset $\mathsf{C}_\delta(W)$ is an intersection of subsets of the form:

$$S(c,d) := \{s \in \mathbb{H} * \{t,\bar{t}\}^* \mid cs \sim sd\} \tag{194}$$

for some $c, d \in A \cup B$, and of subsets of the form $\mathbb{H} * \{t,\bar{t}\}^* - S(c,d)$. Each set $S(c,d)$ is recognized by some t-automaton with set of labels in

$$\mathcal{F} = \{S_H(c,d) \mid c,d \in A \cup B\} \subseteq EQR(\mathbb{H}).$$

Each set $\mathbb{H} * \{t,\bar{t}\}^* - S(c,d)$ is recognized by some t-automaton with set of labels in

$$\mathcal{F} = \{S_H(c,d) \mid c,d \in A \cup B\} \cup \{\mathbb{H} - S_H(c,d) \mid c,d \in A \cup B\} \subseteq EQR(\mathbb{H}).$$

It follows that $\mathsf{C}_\delta(W)$ is recognized by some t-automaton with labels in $EQR(\mathbb{H})$. Finally, the subset $\mathsf{C}(W)$ is recognized by some t-automaton $\mathcal{D}$ with labels in $EQR(\mathbb{H})$. The emptiness problem for $\mathrm{L}(\mathcal{D})$ reduces to the emptiness problem for elements of $EQR(\mathbb{H})$ (given by a system of equations with rational constraints), which are themselves instances of Q_2.
The problem P2.1, first line, is an instance of $Q1$, while P2.2 is an instance of $Q2$.
□
By induction over the size of the associated subgroups, we obtain

Theorem 27. *Let $\mathbb{H}$ be a group and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$.*

8.2.2. *Inductive transfer for cancellative monoids*

We will prove the extension of proposition 26 to cancellative monoids:

Proposition 28. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the pair of problems (Q_1, Q_2), where*
1- Q_1 is the SAT-problem for systems of equations with rational constraints in $\mathbb{U}$
2- Q_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$

What becomes difficult here is the computation of $\mathcal{W}_t$: it requires to determine, for a given symbol $W \in \mathcal{W}$ whether there exists a t-sequence s such that: $s \in \mathbb{H}$ (if $\gamma(s)$ is a H-type), s is not invertible (if $\gamma(W)$ specifies that W does not belong to the domain of $\mathbb{I}_{\mathbb{W}}$), and $cs \neq sd$ for some $c, d \in A \cup B$ (if the value of $\delta(W)$ imposes this). The non-invertibility condition is expressed via an universally quantified formula $(\forall h', h \cdot h' \neq 1)$ and the non-commutation condition is a disequation. Since no hypothesis ensures that the satisfiability of such formulas over $\mathbb{H}$ is decidable, we give up the hope to compute $\mathcal{W}_t$. Instead, we enumerate all the subalphabets $\mathcal{W}' \subseteq \mathcal{W}$ (closed under $\mathbb{I}_{\mathbb{W}}$) and apply the above method in the sub-AB-algebra $\mathbb{W}'$ generated by $\mathcal{W}'$ instead of the sub-AB-algebra $\mathbb{W}_t$. But the above difficulty arises in the computation of $\psi_t : \mathbb{W}' \to \mathbb{H}_t$. We have to compute, for every $W' \in \mathcal{W}'$, an image $\psi_t(W')$ having the same behaviour w.r.t. $\mathbb{I}, \gamma, \mu, \delta$. We avoid here this difficulty by computing a *weak* AB-homomorphism $\psi_t : \mathbb{W}' \to \mathbb{H}_t$.
We call *weak AB*-homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ any map $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ fulfilling the seven properties (195-201) below:

$$\psi : (\mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}) \to (\mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}) \text{ is a monoid homomorphism} \tag{195}$$

$$\forall a \in A, \forall b \in B, \psi(\iota_{A,1}(a)) = \iota_{A,2}(a), \quad \psi(\iota_{B,1}(b)) = \iota_{B,2}(b) \tag{196}$$

$$\forall m \in \mathbb{M}_1 - \gamma_1^{-1}(\{\emptyset\}), \quad m \in \mathrm{dom}(\mathbb{I}_1) \Rightarrow \psi(m) \in \mathrm{dom}(\mathbb{I}_2) \tag{197}$$

$$\forall m \in \hat{\mathbb{M}}_1, \quad \mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m)) \tag{198}$$

$$\forall m \in \mathbb{M}_1, \gamma_2(\psi(m)) \supseteq \gamma_1(m) \tag{199}$$

$$\forall m \in \mathbb{M}_1, \forall t \in \gamma_1(m), p_1(\mu_2(t, \psi(m))) = p_1(\mu_1(t, m)), \tag{200}$$

$$\forall m \in \mathbb{M}_1, \forall t \in \gamma_1(m), \delta_2(t, \psi(m)) \supseteq \delta_1(t, m). \tag{201}$$

where the map $p_1 : \mathsf{B}^2(\mathsf{Q}) \to \mathsf{B}(\mathsf{Q})$ denotes the first projection. We have thus replaced the axioms (34),(37),(38) by the weaker axioms (197),(200),(201). It remains true that the above list of axioms can be checked on the generators only, i.e. the following analogue of lemma 10 is true.

Lemma 29. *Let $\mathcal{M}_2 = \langle \mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}, \iota_{A,2}, \iota_{B,2}, \mathbb{I}_2, \gamma_2, \mu_2, \delta_2 \rangle$ be some AB-algebra. Let $\mathbb{W}'$ be the sub-AB-algebra generated by some subalphabet $\mathcal{W}' \subseteq \mathcal{W}$, which is closed*

under $\mathbb{I}_{\mathbb{W}}$. *Let* $\psi : \mathbb{W}' \to \mathbb{M}_2$ *be some monoid-homomorphism. This map* ψ *is a weak* AB*-homomorphism if and only if,*
1- $\iota_A \circ \psi = \iota_{A,2}, \ \ \iota_B \circ \psi = \iota_{B,2}$
and for every $g \in \mathcal{W}' \cup A \cup B, t \in \gamma(g)$*:*
2- $g \in \mathrm{dom}(\mathbb{I}) \Rightarrow \psi(g) \in \mathrm{dom}(\mathbb{I}_2)$
2'- $\mathbb{I}_2(\psi(g)) = \psi(\mathbb{I}(g))$
3- $\gamma_2(\psi(g)) \supseteq \gamma(g)$
4- $p_1(\mu_2(t, \psi(g))) = p_1(\mu(t, g))$
5- $\delta_2(t, \psi(g)) \supseteq \delta(t, g)$.

Given two AB-algebras $\mathcal{M}_1, \mathcal{M}_2$, we denote by $WHom_{AB}(\mathcal{M}_1, \mathcal{M}_2)$ the set of all weak AB-homomorphisms from $\mathcal{M}_1$ to $\mathcal{M}_2$. We can then express the solutions of the original equation over $\mathbb{G}$ via some $AB-$ (resp. weak $AB-$) homomorphisms.

Lemma 30. *Let* $\mathcal{S} = ((E_i)_{1 \le i \le n}, \mu_{\mathcal{A}}, \mu_{\mathcal{U}})$ *be a system of equations over* $\mathbb{G}$*, with rational constraint. Let us suppose that* $\mathcal{S}$ *is in normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{S}$ *if and only if, there exists an admissible choice* $(\vec{W}, \vec{e})$ *of variables of* $\mathcal{W}$ *(resp. elements of* $A \cup B$*), an alphabet* $\mathcal{W}' \subseteq \mathcal{W}$ *posessing all these variables and closed under* $\mathbb{I}_{\mathbb{W}}$*, an involution* $\mathbb{I}' \in \mathcal{I}$*, an AB-homomorphism* $\sigma_{\mathbb{W}} : (\langle \mathcal{W}' \rangle, \mathbb{I}), \to (\langle \mathcal{W}' \rangle, \mathbb{I}')$ *and a weak AB-homomorphism* $\psi_t : \mathbb{W}_t \to \mathbb{H}_t$ *such that:*
1- $\sigma_{\mathbb{W}}$ *is a solution of* $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$*,*
2- $\sigma_{\mathbb{W}} \circ \psi_t$ *is a solution of* $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$*,*
3- $\sigma = \sigma_{\vec{W}, \vec{e}} \circ \sigma_{\mathbb{W}} \circ \psi_t \circ \bar{\pi}_{\mathbb{G}}$.

Proof: Follows from previous lemmas. `Make more precise`. □
Algorithm scheme
1- `Enumerate` the subalphabets $\mathcal{W}'$ which are closed under the involutin $\mathbb{I}_{\mathbb{W}}$.
2- `Enumerate` the involutions $\mathbb{I}' \in \mathcal{I}$ and the admissible vectors $\vec{W}, \vec{e}$.
3- `For` every value of $\mathcal{W}', \mathbb{I}', \vec{W}, \vec{e}$:
4- $\mathbb{W}' := \langle \mathcal{W}' \rangle$.
5- `Find` a solution $\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}((\mathbb{W}', \mathbb{I}), (\mathbb{W}', \mathbb{I}'))$ to the system $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ by the following procedure
5.1- `Enumerate` all the $\sigma_H \in \mathrm{Hom}_{AB}((\mathbb{W}'_H, \mathbb{I}), (\mathbb{W}'_H, \mathbb{I}'))$,
5.2- `Find` a solution $\sigma_{\mathbb{U}}$ for the system $\mathcal{S}_{\mathbb{U}}(\sigma_H)$,
5.3- $\sigma_{\mathbb{W}} := \Phi^{-1}(\sigma_{\mathbb{U}})$
6- `Find` a $\psi_t \in \mathrm{WHom}_{AB}((\mathbb{W}', \mathbb{I}'), (\mathbb{H}_t, \mathbb{I}_t))$ such that $\sigma_{\mathbb{W}} \circ \psi_t$ solves $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$.
7- `Endfor`
8- `If` some pair $\sigma_{\mathbb{W}}, \psi_t$ is found `then` $\mathcal{S}$ is satisfiable `else` $\mathcal{S}$ is unsatisfiable.
(We summarize on figure 14 the different maps to be found; doted arrows correspond to weak AB-homomorphisms). Let us precise how one can achieve every line of this scheme.

$$
\begin{tikzcd}
 & \mathcal{W}'^{*} * A * B \arrow[d, "\pi_{\equiv}"'] \arrow[dddrrr, "\sigma_{\mathbb{U}}"] & & & \\
\mathcal{U}^{*} \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[ddd, "\sigma"'] & (\mathbb{W}', \mathbb{I}) \arrow[ddd, dotted, "\sigma_t"'] \arrow[dddrr, "\sigma_{\mathbb{W}}"] & \mathbb{W}'_H \arrow[l] \arrow[dr, "\sigma_H"'] & & \\
 & & & \mathbb{W}'_H \arrow[dd] & \\
 & & & & \\
\mathbb{G} & (\mathbb{H}_t, \mathbb{I}_t) \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & & (\mathbb{W}', \mathbb{I}') \arrow[ll, dotted, "\psi_t"] \arrow[r, "\bar{\pi}_{\mathbb{U}}"'] & \mathbb{U}
\end{tikzcd}
$$

Fig. 14. the algorithm: monoid case

Line 5.2 is an instance of Q_1. Let us show that line 6 can be achieved by a Turing-reduction to Q_2.

By lemma 29, line 6 amounts to find some tuple $(\psi_t(W))_{W\in\mathcal{W}'}$ such that conditions 1-5 of lemma 29 are fulfilled and $(\psi_t(W))_{W\in\mathcal{W}'_H}$ is a solution of $\sigma_{\mathbb{W}}(\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e}))$. Conditions 1-5 can be expressed by the following constraints. For every $W \in \mathcal{W}'_H$:

$$\mathsf{C}_I(W) := \mathbb{H} \quad \text{if } W \in \mathcal{W}' - \hat{\mathcal{W}}; \;\; \mathsf{C}_I(W) := I(\mathbb{H}) \quad \text{if } W \in \hat{\mathcal{W}}_0, \tag{202}$$

$$\mathsf{C}_I(W_k) := \{h \in I(\mathbb{H}) \mid a_k h b_k = h^{-1}\}, \;\; \mathsf{C}_I(\bar{W}_k) := \{h \in I(\mathbb{H}) \mid a_k^{-1} h b_k^{-1} = h^{-1}\}, \quad \text{if } W_k \in \hat{\mathcal{W}}_0, \tag{203}$$

$$\mathsf{C}_\gamma(W) := \{h \in \mathbb{H} \mid \gamma(W) \subseteq \gamma(h)\}; \;\; \mathsf{C}_\mu(W) := \{h \in \mathbb{H} \mid \mu_1(W) = \mu_{\mathcal{A}}(h)\}, \tag{204}$$

$$\mathsf{C}_\delta(W) := \{h \in \mathbb{H} \mid \forall t \in \gamma(W), \delta(t, W) \subseteq \delta(t, h)\}, \tag{205}$$

For every $W \in \mathcal{W}' - \mathcal{W}'_H$:

$$\mathsf{C}_I(W) := \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid W \in \mathrm{dom}(\mathbb{I}') \Rightarrow s \in \mathrm{dom}(\mathbb{I}_t)\}, \tag{206}$$

$$\mathsf{C}_\gamma(W) := \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \gamma(W) \subseteq \gamma(s)\}; \;\; \mathsf{C}_\mu(W) := \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \mu_1(W) = \mu_{\mathcal{A}}(s)\} \tag{207}$$

$$\mathsf{C}_\delta(W) := \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \forall t \in \gamma(W), \delta(t, W) \subseteq \delta(t, s)\}. \tag{208}$$

Finally, for every $W \in \mathcal{W}'$:

$$\mathsf{C}(W) := \mathsf{C}_I(W) \cap \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W). \tag{209}$$

The values of the $\mathsf{C}_*(W)$ have been modified in such a way that, now, every subset $\mathsf{C}(W)$ with $W \in \mathcal{W}'_H$ belongs to EQR($\mathbb{H}$) while every subset $\mathsf{C}(W)$ with $W \in \mathcal{W}' - \mathcal{W}'_H$ is recognized by some t-automaton with labels in EQR($\mathbb{H}$). Line 6 amounts thus to:

- find a solution in $\mathbb{H}$ to the system $\sigma_{\mathbb{W}}(\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e}))$ with the additional constraints $\mathsf{C}(W)$ for $W \in \mathcal{W}'_H$- this is an instance of Q_2
- find an element in the set $\mathsf{C}(W)$ for $W \in \mathcal{W}' - \mathcal{W}'_H$- this reduces to finitely many instances of Q_2.

We have thus proved proposition 28. By induction over the size of the associated subgroups, we obtain

Theorem 31. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$*

9. Equations with positive rational constraints over $\mathbb{G}$

9.1. *Positive rational constraints*

No negation in the rat constraints. We adapt the above reductions by the following replacements:
- weak-homorphism ψ_t
- but $\mathbb{H}_t$ is replaced by $\mathbb{H}_{t+}$ i.e. we do not require the image of the generators to be reduced; since $\mathcal{A}$ is supposed $\approx$-compatible, such a ψ_t leads to a solution of the equations (this is obvious) and also satisfies the rational constraints. We thus obtain a proof of the

Theorem 32. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with positive rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with positive rational constraints in $\mathbb{H}$.*

9.2. *Basic constraints*

The set of constraints here is $\mathcal{C} =: \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{H}, \mathbb{G}\}$. This set turns out to be useful for the decidability of the positive first-order theory of $\mathbb{G}$. Here also we use:
- weak-homorphism ψ_t
- the AB-structure $\mathbb{H}_t$ is replaced by $\mathbb{H}_{t+}$.

Theorem 33. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with basic constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with constants in $\mathbb{H}$.*

9.3. *Constants*

The set of constraints is now $\mathcal{C} := \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{G}\}$. The following theorem is an immediate corollary of theorem 33.

Theorem 34. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with constants in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with constants in $\mathbb{H}$.*

10. Equations and disequations with rational constraints over $\mathbb{G}$

We recall that the notion of systems of equations and disequations with rational constraints over a monoid has been defined in §2.4.2.

10.1. *Rational constraints*

Let us show how to reduce a system of equations and disequations (with rational constraints), over $\mathbb{G}$ to a systems of equations (with rational constraints) over $\mathbb{H}_t$ together with a system of equations/disequations (with rational constraints) over $\mathbb{H}$.

Let us start with a system of equations with rational constraints, over $\mathbb{G}$, which is in normal form (see proposition 19):

$$((E_i)_{1\leq i\leq n}, (\bar{E}_i)_{n+1\leq i\leq 2n}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}}) \tag{210}$$

The equations E_i have the form

$$E_i : \quad (U_{i,1}, U_{i,2}U_{i,3}) \text{ for all } 1 \leq i \leq n \tag{211}$$

while the disequations $\bar{E}_i$ have the form

$$\bar{E}_i : \quad (U_{i,1}, U_{i,2}) \text{ for all } n+1 \leq i \leq 2n \tag{212}$$

where, for every $i \in [1, n]$, the symbols $U_{i,1}, U_{i,2}, U_{i,3}, U_{n+i,1}, U_{n+i,2}$ belong to the alphabet of unknowns $\mathcal{U}$. Let us consider the alphabet $\mathcal{V}_0 := [1, 2n] \times [1, 3] \times [1, 9]$ and the alphabet $\mathcal{W}$ constructed from this $\mathcal{V}_0$ in §3.5. We consider all the vectors $(W_{i,j,k})$ where $1 \leq i \leq 2n, 1 \leq j \leq 3, 1 \leq k \leq 9$ of elements of $\mathcal{W} \cup \{1\}$ and all triple $(e_{i,1,2}, e_{i,2,3}, e_{i,3,1}) \in (A \cup B)^3$ such that: the vectors

$$(W_{i,j,k})_{1\leq i\leq n, 1\leq j\leq 3, 1\leq k\leq 9}, \quad (e_{i,1,2}, e_{i,2,3}, e_{i,3,1})_{1\leq i\leq n}$$

fulfill conditions (94-104) and their counterpart for disequations

$$(W_{i,j,k})_{n+1\leq i\leq 2n, 1\leq j\leq 2, 1\leq k\leq 9}, \quad (e_{i,1,2})_{n+1\leq i\leq 2n}$$

fulfill the analogous conditions:

$$p_1(W_{i,j,k}) = \quad (i,j,k) \in \mathcal{V}_0 \quad \text{for } W_{i,j,k} \neq 1 \tag{213}$$

$$\gamma(\prod_{k=1}^{9} W_{i,j,k}) = \quad (1, H, b, 1, 1) \quad \text{for some } b \in \{0,1\} \tag{214}$$

$$\mu(\prod_{k=1}^{9} W_{i,j,k}) = \quad \mu_{\mathcal{U}}(U_{i,j}) \tag{215}$$

$$\gamma(\prod_{k=1}^{4} W_{i,1,k}) = \quad \gamma(\textstyle\prod_{k=1}^{4} W_{i,2,k}) \tag{216}$$

$$W_{i,j,5} \in \mathcal{W} \wedge \gamma(W_{i,j,5}) \text{ is a } \quad \text{H-type} \tag{217}$$

$$e_{i,1,2} \in \mathrm{Gi}(W_{i,1,5}) = \mathrm{Gi}(W_{i,2,5}) \tag{218}$$

A vector $(\vec{W}, \vec{e})$ fulfilling (94-104) for the indices $i \in [1, n]$ and (213-218) for the indices $i \in [n+1, 2n]$, is called an *admissible* vector. For every admissible vector $(\vec{W}, \vec{e})$ we define the following equations and disequations:

$$\prod_{k=1}^{9} W_{i,j,k} = \prod_{k=1}^{9} W_{i',j',k} \qquad \text{if } U_{i,j} = U_{i',j'} \tag{219}$$

$$W_{i,1,1}W_{i,1,2}W_{i,1,3}W_{i,1,4}e_{i,1,2} = W_{i,2,1}W_{i,2,2}W_{i,2,3}W_{i,2,4} \tag{220}$$

for all $1 \leq i \leq 2n$

$$W_{i,2,6}W_{i,2,7}W_{i,2,8}W_{i,2,9} = e_{i,2,3}\overline{W}_{i,3,4}\overline{W}_{i,3,3}\overline{W}_{i,3,2}\overline{W}_{i,3,1} \tag{221}$$

$$W_{i,1,5}W_{i,1,6}W_{i,1,7}W_{i,1,8} = e_{i,1,3}W_{i,3,6}W_{i,3,7}W_{i,3,8}W_{i,3,9} \tag{222}$$

$$W_{i,1,5} = e_{i,1,2}W_{i,2,5}e_{i,2,3}W_{i,3,5}e_{i,3,1} \tag{223}$$

for all $1 \leq i \leq n$

$$\bigwedge_{d \in \mathrm{Ge}(W_{i,1,5})} W_{i,1,5} \cdot d \neq e_{i,1,2} \cdot W_{i,2,5} \tag{224}$$

for all $n+1 \leq i \leq 2n$ such that $\tau\mathrm{e}(W_{i,1,5}) = \tau\mathrm{e}(W_{i,2,5})$.

We denote by $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ the sequence of equations (219-222), by $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ the sequence of equations and disequations (223-224). For every $(i,j) \in [1,n] \times [1,3] \cup [n+1] \times [1,2]$ we denote by $\overline{\mathrm{i},\mathrm{j}}$ the smallest pair such that $U_{i,j} = U_{\overline{\mathrm{i},\mathrm{j}}}$. By $\sigma_{\vec{W},\vec{e}} : \mathcal{U}^* \to \mathbb{W}$ we denote the unique monoid-homomorphism such that,

$$\sigma_{\vec{W},\vec{e}}(U_{i,j}) = \prod_{k=1}^{9} W_{\overline{\mathrm{i},\mathrm{j}},k}.$$

Lemma 35. *Let $\mathcal{S} = ((E_i)_{1 \leq i \leq n}, (\bar{E}_i)_{n+1 \leq i \leq 2n}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}})$ be a system of equations and disequations over $\mathbb{G}$, with rational constraint. Let us suppose that $\mathcal{S}$ is in normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{S}$ if and only if, there exists an admissible choice $(\vec{W}, \vec{e})$ of variables of $\mathcal{W}_t$ and elements of $A \cup B$ and an AB-homomorphism

$$\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$$

solving simultaneously the system $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ of equations over $\mathbb{H}_t$ and the system $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ of equations and disequations over $\mathbb{H}$, and such that

$$\sigma = \sigma_{\vec{W},\vec{e}} \circ \sigma_t \circ \bar{\pi}_{\mathbb{G}}.$$

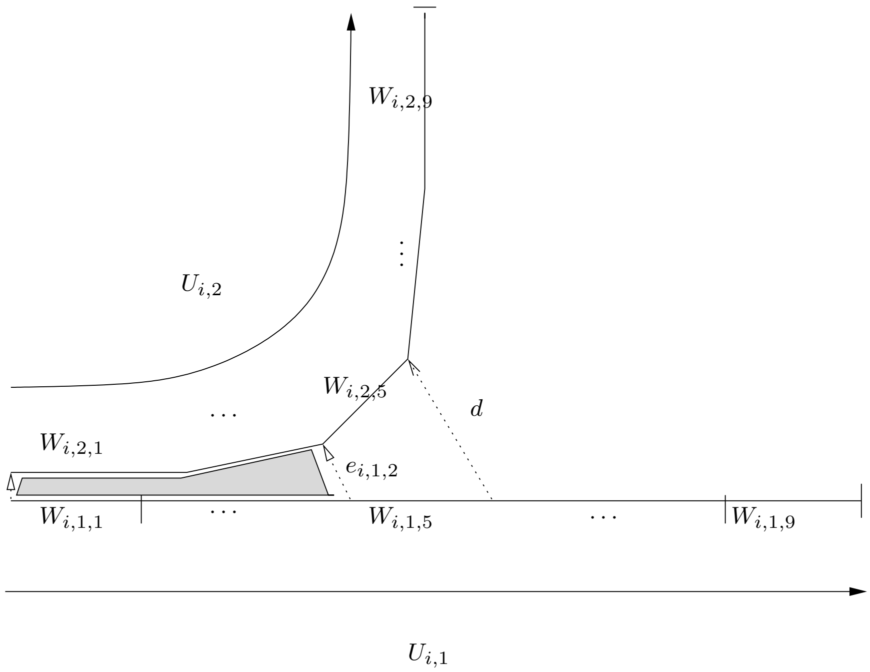

Fig. 15. Disequation cut into 3 parts

10.1.1. *From $\mathbb{G}$-solutions to t-solutions*

Let $\sigma : \mathcal{U}^* \to \mathbb{G}$ be a monoid homomorphism solving the system $\mathcal{S}$. For every $1 \leq i \leq n$ we construct the vector $(W_{i,*,*}, e_{i,*,*})$ as in §5.2.1. Let us fix now some disequation from $\mathcal{S}$, i.e. some integer $n+1 \leq i \leq 2n$. Let us choose, for every $j \in [1,2]$, some $s_{i,j} \in \mathrm{Red}(\mathbb{H}, t)$ such that:

$$\sigma(U_{i,j}) = \pi_G(s_{i,j}).$$

Let us consider some decomposition of the form (5) for $s_{i,2}, s_{i,1}$:

$$s_{i,2} = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell, \tag{225}$$

$$s_{i,1} = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_\lambda} h'_\lambda \cdots t^{\alpha'_{\ell'}} h'_{\ell'}, \tag{226}$$

We know that $s_{i,1} \not\approx s_{i,2}$. Let us distinguish the possible forms for $s_{i,1}$, as represented on figures 16-18.

Case 1: there exists some integer $\lambda \in [2, \ell]$, some $e_{i,1,2} \in B(\alpha_{\lambda-1})$ such that

$$h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1} = h'_0 \cdots t^{\alpha_{\lambda-1}} h'_{\lambda-1} e_{i,1,2}, \ \ \alpha_\lambda = \alpha'_\lambda$$

$$e_{i,1,2} h_{\lambda-1} \neq h'_{\lambda-1} d \quad \text{for all } d \in A(\alpha_\lambda).$$

We consider the following factors of $s_{i,2}, s_{i,1}$:

$$P_{i,2} = h_0 \cdots t^{\alpha_{\lambda-1}}, \ \ M_{i,2} = h_{\lambda-1}, \ \ S_{i,2} = t^{\alpha_\lambda} \cdots t^{\alpha_\ell} h_\ell.$$

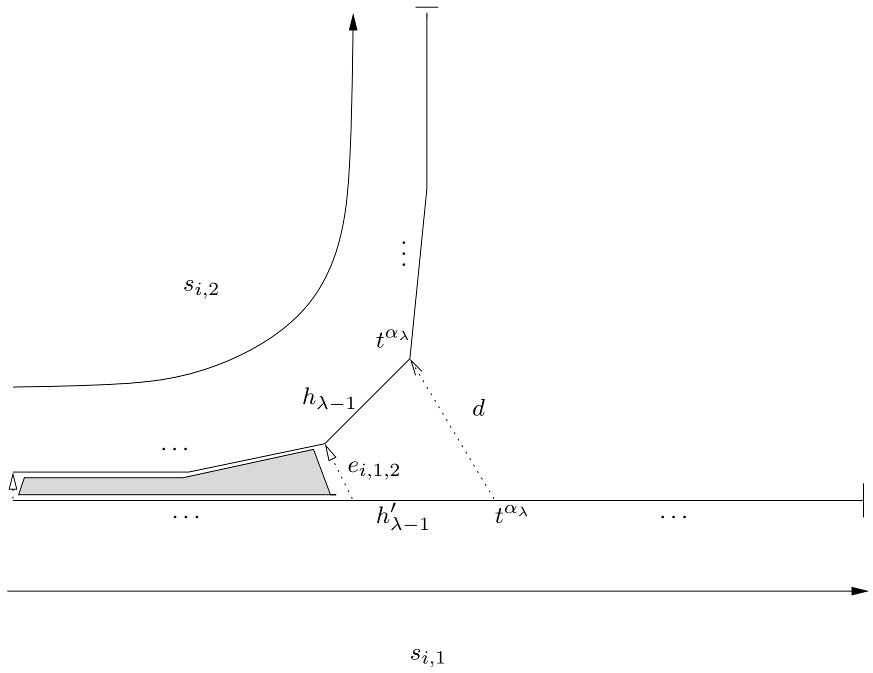

Fig. 16. Disequations, case 1

$$P_{i,1} = h'_0 \cdots t^{\alpha_{\lambda-1}}, \;\; M_{i,1} = h'_{\lambda-1}, \;\; S_{i,1} = t^{\alpha_\lambda} \cdots t^{\alpha'_{\ell'}} h'_{\ell'}.$$

Following the lines of §5.2.1, these reduced sequences can be cut into nine factors $(P_{i,j,k})$, $1 \leq j \leq 2, 1 \leq k \leq 9$, and subsequently lifted to nine letters $(W_{i,j,k})$, $1 \leq j \leq 2, 1 \leq k \leq 9$, such that the vector $(W_{i,*,*}, e_{i,1,2})$ fulfills conditions (213-218) and the classes $([P_{i,*,*}]_\sim)$ fulfill equations and disequations (220),(224). We can define $\sigma_t(W_{i,j,k}) = [P_{i,j,k}]_\sim$; σ_t can be extended into an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$, $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$ and such that

$$\sigma = \sigma_{\vec{W},\vec{e}} \circ \sigma_t \circ \bar{\pi}_{\mathbb{G}}.$$

Case 2: there exists $\lambda \in [2, \ell]$, some $e_{i,1,2} \in B(\alpha_{\lambda-1})$ such that

$$h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1} = h'_0 \cdots t^{\alpha_{\lambda-1}} h'_{\lambda-1} e_{i,1,2}, \;\; \alpha_\lambda = -\alpha'_\lambda$$

We consider the factors $P_{i,2}, M_{i,2}, S_{i,2}, P_{i,1}, M_{i,1}$ defined by the same formulas as in case 1, and define

$$S_{i,1} = t^{-\alpha_\lambda} \cdots t^{\alpha'_{\ell'}} h'_{\ell'}.$$

This time we obtain a vector $(W_{i,*,*})$ such that $\tau\mathrm{e}(W_{i,1,5}) \neq \tau\mathrm{e}(W_{i,2,5})$. It follows there is no disequation (224) associated to this index i. The vector $(W_{i,*,*}, e_{i,1,2})$

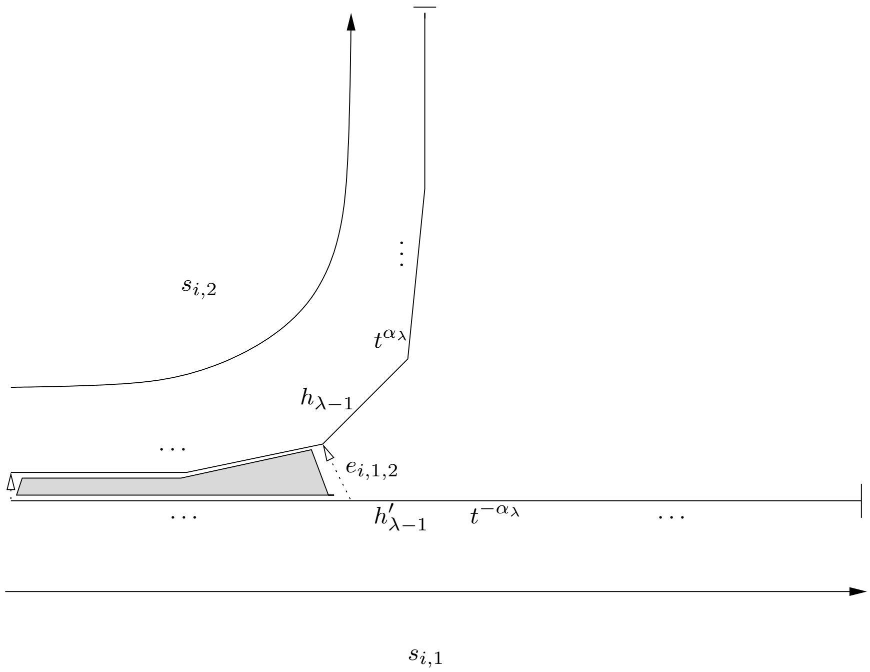

Fig. 17. Disequations, case 2

fulfills conditions (213-218) and the classes $[P_{i,*,*}]_\sim$) fulfill equation (220).
Case 3: there exists $\lambda \in [2, \ell]$, such that

$$s_{i,1} = h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1}.$$

This case can be treated similarly as case 2. We just define $S_{i,1} = 1$ and, correspondingly $W_{i,1,k} = 1$, for $6 \leq k \leq 9$.
Case 4: there exists some integer $\lambda \in [2, \ell']$, such that

$$s_{i,2} = h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1}.$$

This case is obtained from Case 3 by exchanging $s_{i,1}$ with $s_{i,2}$.
It remains to treat some degenerated cases.
Case 5: $\lambda = 1 \leq \ell$, fulfills one of the conditions defining cases 1-3 (except beeing smaller than 2) or $\lambda = 1 \leq \ell'$ fulfills the conditions defining case 4 (except beeing smaller than 2).
We take: $P_{i,1} = P_{i,2} = 1$, $e_{i,1,2} = 1$, $S_{i,1} = 1$ (under the condition of case 3), $S_{i,2} = 1$ (under the condition of case 4).
The construction is ended as in the degenerated cases of §5.2.1.
Case 6: $\ell = \ell' = 0$.
We take: $P_{i,1} = P_{i,2} = 1$, $e_{i,1,2} = 1$, $S_{i,1} = S_{i,2} = 1$.
The construction is ended as in the degenerated cases of §5.2.1.

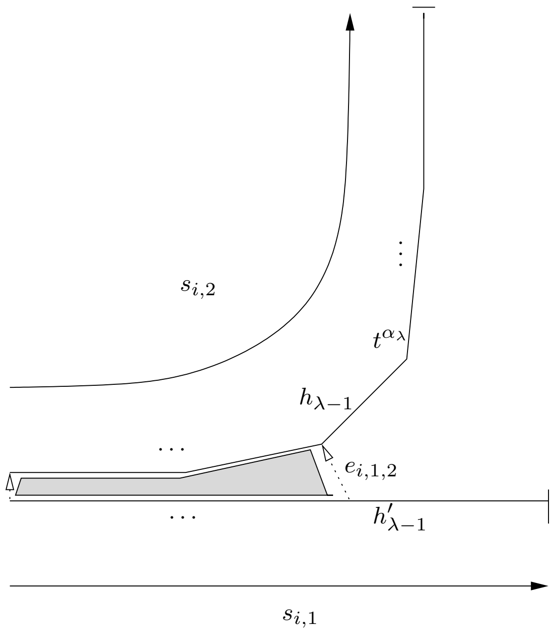

Fig. 18. Disequations, case 3

10.1.2. *From t-solutions to $\mathbb{G}$-solutions*

Let $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ be an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$.
Owing to the proofs of §5.2.2, we just have to prove that for every $i \in [n+1, 2n]$,

$$\sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,1})) \not\approx \sigma_t(\sigma_{\vec{W},\vec{e}}(U_{i,2})).$$

Using equation (219) and the definition of $\sigma_{\vec{W},\vec{e}}$, the above unequalityis equivalent with

$$\sigma_t(\prod_{k=1}^{9} W_{i,1,k}) \not\approx \sigma_t(\prod_{k=1}^{9} W_{i,2,k}). \tag{227}$$

Equation (220) states that

$$\sigma_t(\prod_{k=1}^{4} W_{i,1,k})e_{i,1,2} \approx \sigma_t(\prod_{k=1}^{4} W_{i,2,k}). \tag{228}$$

Let us distinguish several cases according to the values of $\tau\mathrm{e}(W_{i,j,5})$. Since, by (217), $\gamma(W_{i,j,5})$ are H-types, one of the following cases must occur.
Case 1: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (1,1)$
By (214), $\gamma(\prod_{k=6}^{9} W_{i,j,k}) = (1,1,0,1,1)$ for $j \in [1,2]$, which implies that, for every

$j \in [1,2]$:

$$\sigma_t(\prod_{k=1}^{9} W_{i,j,k}) = \sigma_t(\prod_{k=1}^{5} W_{i,j,k}), \tag{229}$$

But disequation (224) reads:

$$\sigma_t(W_{i,1,5}) \neq_{\mathbb{H}_t} e_{i,1,2} \cdot \sigma_t(W_{i,2,5}). \tag{230}$$

The conjunction of (228)(229)(230) proves inequality (227).
Case 2: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (A, T)$
Since σ_t is a solution of equation (220) and disequation (224), two representatives of $\sigma_t(\prod_{k=1}^{5} W_{i,2,k})$, (resp. $\sigma_t(\prod_{k=1}^{5} W_{i,1,k})$) cannot be prefixes of two reduced sequences which are equivalent modulo $\sim$. Thus (227) is established.
Case 3: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (B, T)$
Same argument as for case 2.
Case 4: $\tau\mathrm{e}(W_{i,2,5}) \neq \tau\mathrm{e}(W_{i,1,5})$
This shows that the projections of $\sigma_t(\prod_{k=1}^{9} W_{i,2,k})$, $\sigma_t(\prod_{k=1}^{9} W_{i,1,k})$ on $\{t, \bar{t}\}^*$ are non-equal, so that, a fortiori, inequality (227) holds.

10.2. *Positive rational constraints*

Here we choose $\mathcal{C} = RAT(\mathbb{G})$. We adapt the above reductions by the following replacements:
- weak-homomorphism ψ_t
- $\mathbb{H}_t$ is still used; for every $W \in \mathcal{W}'$, $\mathsf{C}(W)$ is recognized by a finite t-automaton with some labels in $DEQR_+(\mathbb{H})$, where $DEQR_+(\mathbb{H})$ is the set of subsets of $\mathbb{H}$ which are definable by systems of equations and disequations with positive rational constraints.

10.3. *Basic constraints*

The set of constraints here is $\mathcal{C} =: \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{H}, \mathbb{G}\}$. Here we use:
- weak-homomorphism ψ_t
- the AB-structure $\mathbb{H}_t$ is still used. for every $W \in \mathcal{W}'$, $\mathsf{C}(W)$ is recognized by a finite t-automaton with some labels in $DEQ(\mathbb{H})$, where $DEQ(\mathbb{H})$ is the set of subsets of $\mathbb{H}$ which are definable by systems of equations and disequations.

10.4. *Constants*

The set of constraints here is $\mathcal{C} =: \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{G}\}$. This case follows directly from the above case.

11. Equations over an amalgamated product

Embedding into an HNN-extension.
With rational constraints: corollary of theorem **??** and the transfer theorem for free-products ([Diekert-Lohrey]? `mrkus checks this`).

Theorem 36. *Let $\mathbb{H}_1, \mathbb{H}_2$ be monoids,two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, an isomorphism $\varphi : A_1 \to A_2$. The satisfiability problem for systems of equations with rational constraints in the amalgamted product $\langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1)\rangle$ is Turing-reducible to the pair of problems (S_1, S_2) where*
1- S_1 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}_1$
2- S_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}_2$

With constants: reduces to rational constraints over $\mathbb{G}$ where the labels of the t-automaton belong to $\mathcal{C} = \{\{m\} \mid m \in \mathbb{H}_1 \cup \mathbb{H}_2\} \cup \{\mathbb{H}_1, \mathbb{H}_2\}$. Thus finally reduces to systems of equations on $\mathbb{H}_1 * \mathbb{H}_2$ with the same set of constraints. These systems reduce to systems of equations with constants in $\mathbb{H}_1$ and to systems of equations with constants in $\mathbb{H}_2$, by [Diekert-Lohrey, FSTTCS 2003,`markus checks`].

Theorem 37. *Let $\mathbb{H}_1, \mathbb{H}_2$ be monoids, two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, an isomorphism $\varphi : A_1 \to A_2$. The satisfiability problem for systems of equations with constants in the amalgamated product $\langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1)\rangle$ is Turing-reducible to the pair of problems (S_1, S_2) where*
1- S_1 is the SAT-problem for systems of equations with constants in $\mathbb{H}_1$
2- S_2 is the SAT-problem for systems of equations with constants in $\mathbb{H}_2$

The two following variants of theorem 36 also follow from previous results of [Diekert-Lohrey 2003] combined with the results about HNN-extensions of this paper, via the embedding (11).

Theorem 38. *Let $\mathbb{H}_1, \mathbb{H}_2$ be monoids,two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, an isomorphism $\varphi : A_1 \to A_2$. The satisfiability problem for systems of equations and disequations with rational constraints in the amalgamated product $\langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1)\rangle$ is Turing-reducible to the pair of problems (S_1, S_2) where*
1- S_1 is the SAT-problem for systems of equations and disequations with rational constraints in $\mathbb{H}_1$
2- S_2 is the SAT-problem for systems of equations and disequations with rational constraints in $\mathbb{H}_2$

Theorem 39. *Let $\mathbb{H}_1, \mathbb{H}_2$ be monoids,two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, an isomorphism $\varphi : A_1 \to A_2$. The satisfiability problem for systems of equations and disequations with constants in the amalgamted product $\langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1)\rangle$*

is Turing-reducible to the pair of problems (S_1, S_2) *where*
1- S_1 *is the SAT-problem for systems of equations and disequations with constants in* $\mathbb{H}_1$
2- S_2 *is the SAT-problem for systems of equations and disequations with constants in* $\mathbb{H}_2$

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