On the Confluence of Trace Rewriting Systems

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February 26, 1999

Abstract

Trace rewriting systems, i.e., rewriting systems over trace monoids, generalize both semi–Thue systems and vector replacement systems. In [NO88], a particular trace monoid M is constructed such that confluence is undecidable for the class of length–reducing trace rewriting systems over M. In this paper, we show that this result holds for every trace monoid, which is neither free nor free commutative. Confluence for length–reducing semi–Thue systems is shown to be P–complete. Furthermore we introduce a restricted notion of confluence, called (α, β)–confluence, where $\alpha, \beta \geq 1$. We prove that (α, β)–confluence is decidable for trace rewriting sytems and use this result in order to obtain new classes of trace rewriting systems with a decidable confluence problem.

1 Introduction

The theory of *free partially commutative monoids* generalizes both, the theory of free monoids and the theory of free commutative monoids. In computer science, free partially commutative monoids are commonly called *trace monoids* and their elements are called *traces*. Both notions are due to Mazurkiewicz [Maz77], who recognized trace monoids as a model of concurrent processes. [DR95] gives an extensive overview about current research trends in trace theory.

The relevance of trace theory for computer science can be explained as follows. Assume a finite alphabet Σ. An element of the free monoid over Σ, i.e., a finite word over Σ, may be viewed as the sequence of actions of a sequential process. In addition to a finite alphabet Σ, the specification of a trace monoid (over Σ) requires a binary and symmetric independence relation on Σ. If two symbols a and b are independent then they are allowed to commute. Thus, the two words $sabt$ and $sbat$, where s and t are arbitrary words, denote the same trace. This trace may be viewed as the sequence of actions of a concurrent process where the two independent actions a and b may occur concurrently and thus may be observed either in the order ab or in the order ba.

This point of view makes it interesting to consider *trace rewriting systems*, see [Die90]. A trace rewriting system is a finite set of rules, where the left–hand side and right–hand side of each rule are traces. Trace rewriting systems generalize both, semi–Thue systems (see [BO93] for a detailed study) and vector replacement systems which are equivalent to Petri nets. Considered in the above framework of concurrent processes, a trace rewriting system may be viewed as a set of transformations that translate sequences of actions of one process into sequences of actions of another process. Thus, trace rewriting systems may for instance serve as a formal model of abstraction.

For all kinds of rewriting systems, the notion of a terminating system and the notion of a confluent system are of central interest. Together, these two properties guarantee the existence of unique normalforms. Unfortunately, in general both properties are undecidable for trace rewriting systems. More precisely, termination and confluence are only decidable for trace rewriting systems over free commutative monoids, i.e., vector replacement sytems, see [HL78], [BO84], [VRL98] [1]. For the confluence problem this situation changes if only terminating systems are considered. It is a classical result that for the class of terminating semi–Thue systems, confluence is decidable, see [BO81]. Moreover if only length–reducing semi–Thue systems are considered, then confluence can be even decided in polynomial time, see [BO81], [KKMN85]. Unfortunately even for length–reducing trace rewriting systems, confluence is undecidable in general. More precisely, in [NO88] a concrete trace monoid M is presented such that confluence is undecidable for the class of length–reducing trace rewriting systems over M. Therefore it remains the problem to determine those trace monoids for which confluence is decidable for the class of terminating trace rewriting systems. This question will be solved in Section 3, where we prove that confluence of terminating systems is decidable only for free or free commutative monoids. This result will be obtained by a reduction to the case of trace monoids with three generators, see Section 3.1. Thus, we have obtained a sharp borderline between decidability and undecidability in terms of the underlying trace monoid.

Another possible parameter for variation is the length of the involved derivations. More precisely, we say that a trace rewriting system is (α, β)–confluent if the following holds for all traces u, v, and w: If u and w can be reached from v by at most α many rewrite steps then there exists a trace v' that can be reached from u and w by at most β many rewrite steps. In Section 4 we will prove that (α, β)–confluence is decidable for trace rewriting systems. Finally in Section 5 we will present some conditions for terminating trace rewriting systems that imply the equivalence of confluence and $(1, \alpha)$–confluence for a fixed value α that can be determined effectively. In this way we obtain new classes of trace rewriting systems with an decidable confluence problem. Other decidability criteria for the confluence problem are presented in [Die90]. In Section 5 we briefly compare these criteria with our criteria.

2 Preliminaries

In this section we introduce some notions concerning trace monoids and trace rewriting systems. For a more detailed study, see [Die90] and [DR95].

2.1 Basic notions

For a set A, the powerset of A is denoted by 2^A. The identity relation $\{(a, a) \mid a \in A\}$ on A is denoted by Id_A. Given a function $f : A \longrightarrow B$ and a subset $C \subseteq A$, we denote by $f|_C : C \longrightarrow B$ the restriction of f to C. Given binary relations $E, F \subseteq A \times A$, the relational composition of E and F is denoted by EF. The *transitive closure* of E is denoted by E^+ and the *reflexive and transitive closure* of E is denoted by E^*. Given an alphabet Σ, Σ^* denotes the set of all finite words of elements of Σ as well as the *free monoid* over Σ. The empty word is denoted by 1. As usual $\Sigma^+ = \Sigma^* \backslash \{1\}$. For $\Gamma \subseteq \Sigma$ we define a projection morphism $\pi_\Gamma : \Sigma^* \longrightarrow \Gamma^*$ by $\pi_\Gamma(a) = a$ if $a \in \Gamma$ and $\pi_\Gamma(a) = 1$ otherwise. The length of the word s is denoted by $|s|$. The number of different occurrences of a letter $a \in \Sigma$ in s is denoted by $|s|_a$. We define $\Sigma^n = \{s \in \Sigma^* \mid |s| = n\}$. The set of all letters that occur in the word s is denoted by $alph(s) = \{a \in \Sigma \mid |s|_a > 0\}$.

An *independence alphabet* is an undirected graph (Σ, I), where Σ is a finite alphabet and $I \subseteq \Sigma \times \Sigma$ is an irreflexive and symmetric relation, called an *independence relation*. The complement $I^c = (\Sigma \times \Sigma) \backslash I$ is called a *dependence relation*. It is reflexive and symmetric. The pair (Σ, I^c) is called a *dependence alphabet*. Given an independence alphabet (Σ, I) we define the *trace monoid* $\mathbb{M}(\Sigma, I)$ as the quotient monoid $\Sigma^*/\!\equiv_I$, where $\equiv_I$ denotes the least equivalence relation that contains all pairs of the form $(sabt, sbat)$ for $(a, b) \in I$ and $s, t \in \Sigma^*$, which is a congruence on Σ^*. An element of $\mathbb{M}(\Sigma, I)$, i.e., an equivalence class of words, is called a *trace*. The trace that contains the word s is denoted by $[s]_I$ or briefly $[s]$. The neutral element of $\mathbb{M}(\Sigma, I)$ is the empty trace $[1]_I$ which will be also denoted by 1. Concatenation of traces is defined by $[s]_I[t]_I = [st]_I$. *Monoid morphisms* are defined in the usual way. Since for all words $s, t \in \Sigma^*$, $s \equiv_I t$ implies $|s| = |t|$,

[1]The decidability of termination for vector replacement systems seems to be folklore. It can be proved easily by using Dicksons Lemma.

$|s|_a = |t|_a$, and $\mathit{alph}(s) = \mathit{alph}(t)$, we can define $|[s]_I| = |s|$, $|[s]_I|_a = |s|_a$, and $\mathit{alph}([s]_I) = \mathit{alph}(s)$. The independence relation I can be lifted to $\mathbb{M}(\Sigma, I)$ by $u\ I\ v$ if $\mathit{alph}(u) \times \mathit{alph}(v) \subseteq I$. For a trace $u \in \mathbb{M}(\Sigma, I)$, we denote by $\mathit{min}(u) = \{a \in \Sigma \mid \exists s \in \Sigma^* : u = [as]_I\}$ the set of all minimal letters of u. The set $\mathit{max}(u)$ of all maximal letters of u is defined analogously. For the rest of this section let (Σ, I) be an arbitrary independence alphabet and let $M = \mathbb{M}(\Sigma, I)$.

Note that if $I = (\Sigma \times \Sigma)\backslash Id_\Sigma$ then M is isomorphic to the *free commutative monoid* $\mathbb{N}^{|\Sigma|}$ over $|\Sigma|$ many generators. In this case we will denote traces in M also by $|\Sigma|$–dimensional vectors over $\mathbb{N}$. On the other hand if $I = \emptyset$ or equivalently the dependence alphabet $(\Sigma, I^c) = (\Sigma, \Sigma \times \Sigma)$ is the complete graph on Σ then M is isomorphic to the free monoid Σ^* over Σ. More generally, if (Σ, I^c) is the disjoint union of complete graphs (Σ_i, D_i) for $i \in \{1, \ldots, n\}$ then M is isomorphic to $\prod_{i=1}^n \Sigma_i^*$. In this case we denote traces in M also by tuples of words. If $\Gamma \subseteq \Sigma$ then $(\Gamma, I \cap \Gamma \times \Gamma)$ is called an *induced subgraph* of (Σ, I). In this case the trace monoid $N = \mathbb{M}(\Gamma, I \cap \Gamma \times \Gamma)$ is a submonoid of M and we may view $\pi_\Gamma : \Sigma^* \longrightarrow \Gamma^*$ also as a trace morphism $\pi_\Gamma : M \longrightarrow N$. A *clique covering* of the dependence alphabet (Σ, I^c) is sequence $(\Sigma_1, \ldots, \Sigma_n)$ such that $\Sigma = \bigcup_{i=1}^n \Sigma_i$ and $I^c = \bigcup_{i=1}^n \Sigma_i \times \Sigma_i$. We may always assume that $\Sigma_i \subseteq \Sigma_j$ if and only if $i = j$ for all $i, j \in \{1, \ldots, n\}$. Given a clique covering $\Pi = (\Sigma_1, \ldots, \Sigma_n)$ of (Σ, I^c), we will use the abbreviation $\pi_i = \pi_{\Sigma_i}$. Furthermore $\pi_\Pi : M \longrightarrow \prod_{i=1}^n \Sigma_i^*$ denotes the trace morphism that is defined by $\pi(u) = (\pi_1(u), \ldots, \pi_n(u))$ for every $u \in M$. A fundamental fact about π_Π is expressed in the following lemma following lemma [CP85].

Lemma 2.1. Let $\Pi = (\Sigma_1, \ldots, \Sigma_n)$ be a clique covering of the dependence alphabet (Σ, I^c). Then $\pi_\Pi : M \longrightarrow \prod_{i=1}^n \Sigma_i^*$ is injective.

In particular M is isomorphic to its image under π_Π. Elements of this image, i.e., tuples $(s_1, \ldots, s_n)$ such that there exists a trace $u \in M$ with $\pi_i(u) = s_i$ for all $i \in \{1, \ldots, n\}$ are also called *reconstructible tuples*, see [CM85]. Note that if $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ are reconstructible then also $(s_1t_1, \ldots, s_nt_n)$ is reconstructible. A necessary condition for reconstructibility of $(s_1, \ldots, s_n)$ is that $\pi_j(s_i) = \pi_i(s_j)$ for all $i, j \in \{1, \ldots, n\}$. But this condition is not sufficient for reconstructibility, see Example 2.3.

Lemma 2.2. Let $u \in M$ and let $\Pi = (\Sigma_1, \ldots, \Sigma_n)$ be a clique covering of the dependence alphabet (Σ, I^c). For every $i \in \{1, \ldots, n\}$ let $\pi_i(u) = s_it_i$ be a factorization of $\pi_i(u)$. Then $\pi_i(s_j) = \pi_j(s_i)$ for all $i, j \in \{1, \ldots, n\}$ if and only if $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ are both reconstructible.

Proof. The if–direction is clear. We prove the only if–direction by an induction on $|u|$. The case $u = 1$ is clear. Thus assume that $u = av$ for some $a \in \Sigma$. Thus

$$\pi_i(u) = \pi_i(a)\pi_i(v) = \begin{cases} a\pi_i(v) & \text{if } a \in \Sigma_i \\ \pi_i(v) & \text{if } a \notin \Sigma_i, \end{cases}$$

which implies $\pi_i(v) = s_it_i$ if $a \notin \Sigma_i$ and $a\pi_i(v) = s_it_i$ if $a \in \Sigma_i$. W.l.o.g. assume that for some $m \in \{0, \ldots, n\}$ it holds $\{i \mid a \in \Sigma_i\} = \{1, \ldots, m\}$. We separate two cases.
Case 1: There exists an $i \in \{1, \ldots, m\}$ such that $s_i = 1$. Assume that for some $j \in \{1, \ldots, m\}$ it holds $s_j \neq 1$. Since $a\pi_j(v) = s_jt_j$ it follows $s_j = as$ for some $s \in \Sigma_j^*$. Thus $1 = \pi_j(s_i) = \pi_i(s_j) = \pi_i(as) = a\pi_i(s)$ which gives a contradiction. We conclude that for all $i \in \{1, \ldots, m\}$ it holds $s_i = 1$ and hence $t_i = a\pi_i(v)$. Since

$$\pi_i(v) = \begin{cases} s_i\pi_i(v) & \text{if } 1 \leq i \leq m \\ s_it_i & \text{if } m+1 \leq i \leq n \end{cases}$$

the induction hypothesis implies that $(s_1, \ldots, s_n)$ and $(\pi_1(v), \ldots, \pi_m(v), t_{m+1}, \ldots, t_n)$ are reconstructible. In particular $\pi_\Pi(w) = (\pi_1(v), \ldots, \pi_m(v), t_{m+1}, \ldots, t_n)$ for some $w \in M$. Thus $\pi_\Pi(aw) = (a\pi_1(v), \ldots, a\pi_m(v), t_{m+1}, \ldots, t_n) = (t_1, \ldots, t_n)$, i.e., $(t_1, \ldots, t_n)$ is also reconstructible.
Case 2: For all $i \in \{1, \ldots, m\}$ it holds $s_i \neq 1$. Thus $s_i = as'_i$ and $\pi_i(v) = s'_it_i$ for $i \in \{1, \ldots, m\}$. It follows

$$\pi_i(v) = \begin{cases} s'_it_i & \text{for } 1 \leq i \leq m \\ s_it_i & \text{for } m+1 \leq i \leq n. \end{cases}$$

Since furthermore for all $i, j \in \{1, \dots, m\}$ and all $k \in \{m+1, \dots, n\}$ it holds $\pi_i(s'_j) = \pi_j(s'_i)$ and $\pi_k(s'_i) = \pi_k(as'_i) = \pi_k(s_i) = \pi_i(s_k)$, the induction hypothesis implies that $(s'_1, \dots, s'_m, s_{m+1}, \dots, s_n)$ and $(t_1, \dots, t_n)$ are reconstructible. In particular $\pi_\Pi(w) = (s'_1, \dots, s'_m, s_{m+1}, \dots, s_n)$ for some $w \in M$. Thus $\pi_\Pi(aw) = (as'_1, \dots, as'_m, s_{m+1}, \dots, s_n) = (s_1, \dots, s_n)$. □

In particular, it is not the case that for all factorizations $\pi_i(u) = s_i t_i$ there exists a factorization $u = vw$ with $\pi_i(v) = s_i$ and $\pi_i(w) = t_i$ for all $i \in \{1, \dots, n\}$.

Example 2.3. Let $\Sigma = \{a, b, c, d, e\}$ and $I = \{(a,d), (d,a), (b,c), (c,b), (a,e), (e,a), (b,e), (e,b)\}$. In $\mathbb{M}(\Sigma, I)$ we have for instance, $u = [adbcedcb]_I = [dacebdbc]_I$, $min(u) = \{a, d\}$ and $max(u) = \{b, c\}$. A clique covering for the dependence alphabet (Σ, I^c) is for instance

$$\Pi = (\{a,b\}, \{b,d\}, \{c,d,e\}, \{a,c\}).$$

Then $\pi_\Pi(u) = (abb, dbdb, dcedc, acc)$ which is therefore a reconstructible tuple. The factorization $\pi_\Pi(u) = (ab, db, d, a)(b, db, cedc, cc)$ satisfies the conditions of Lemma 2.2. And indeed it holds $u = [adb]_I[cedcb]_I$, $\pi_\Pi([adb]_I) = (ab, db, d, a)$, and $\pi_\Pi([cedcb]_I) = (b, db, cedc, cc)$.

On the other hand, the tuple $(s_1, s_2, s_3, s_4) = (ab, bd, dc, ca)$ is not reconstructible. But note that it holds $\pi_i(s_j) = \pi_j(s_i)$ for all $i, j \in \{1, 2, 3, 4\}$.

Another fundamental fact about traces is known as Levi's lemma for traces [CP85].

Lemma 2.4. Let $u_1, u_2, v_1, v_2 \in M$ and let $u_1 u_2 = v_1 v_2$. Then there exist four traces $w_{1,1}$, $w_{1,2}$, $w_{2,1}$ and $w_{2,2}$ such that

$$u_1 = w_{1,1}w_{1,2}, \quad u_2 = w_{2,1}w_{2,2}, \quad v_1 = w_{1,1}w_{2,1}, \quad v_2 = w_{1,2}w_{2,2}, \quad w_{1,2}\, I\, w_{2,1}.$$

The next lemma follows by induction form Levis's lemma for traces.

Lemma 2.5. Let $u_1, \dots, u_m, v_1, \dots, v_n \in M$. Then it holds $u_1 u_2 \dots u_m = v_1 v_2 \dots v_n$ if and only if there exist $w_{i,j} \in M$ for all $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ such that

(1) $u_i = w_{i,1} w_{i,2} \dots w_{i,n}$ for every $1 \le i \le m$,

(2) $v_j = w_{1,j} w_{2,j} \dots w_{m,j}$ for every $1 \le j \le n$, and

(3) $w_{i,j}\, I\, w_{k,l}$ if $1 \le i < k \le m$ and $1 \le l < j \le n$.

This situation in the lemma can be visualized by a diagram of the following kind (where $m = n = 5$), where the i–th column corresponds to u_i, the j–th row corresponds to v_j, and the square where the i–th column and the j–th row intersect corresponds to $w_{i,j}$.

v_5	$w_{1,5}$	$w_{2,5}$	$w_{3,5}$	$w_{4,5}$	$w_{5,5}$
v_4	$w_{1,4}$	$w_{2,4}$	$w_{3,4}$	$w_{4,4}$	$w_{5,4}$
v_3	$w_{1,3}$	$w_{2,3}$	$w_{3,3}$	$w_{4,3}$	$w_{5,3}$
v_2	$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$w_{4,2}$	$w_{5,2}$
v_1	$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$w_{4,1}$	$w_{5,1}$
	u_1	u_2	u_3	u_4	u_5

Proof. We use induction over $m + n$. The case $m = 1$ or $n = 1$ is trivial. Thus let $m > 1$ and $n > 1$. Lemma 2.4 applied to the identity $(u_1 \cdots u_{m-1})u_m = (v_1 \cdots v_{n-1})v_n$ gives four traces x, u, v and $w_{m,n}$ such that

$$u_1 u_2 \cdots u_{m-1} = xv, \quad v_1 v_2 \cdots v_{n-1} = xu, \quad u_m = u w_{m,n}, \quad v_n = v w_{m,n}, \quad u\, I\, v.$$

Next we apply the induction hypothesis to the identity $u_1 u_2 \cdots u_{m-1} = xv$. We obtain traces $y_1, y_2, \dots, y_{m-1}$ and $w_{1,n}, w_{2,n}, \dots w_{m-1,n}$ such that

$$x = y_1 y_2 \cdots y_{m-1}, \quad v = w_{1,n} w_{2,n} \cdots w_{m-1,n}, \quad u_i = y_i w_{i,n}\ (1 \le i \le m-1), \quad y_k\, I\, w_{i,n} \text{ if } i < k.$$

Similarly, by the induction hypothesis applied to the identity $v_1 v_2 \cdots v_{n-1} = xu$ there exist traces $z_1, z_2, \ldots, z_{n-1}$ and $w_{m,1}, w_{m,2}, \ldots w_{m,n-1}$ such that

$$x = z_1 z_2 \cdots z_{m-1}, \quad u = w_{m,1} w_{m,2} \cdots w_{m,n-1}, \quad v_j = z_j w_{m,j} \ (1 \le j \le n-1) \quad z_i \, I \, w_{m,j} \text{ if } i > j.$$

Thus $y_1 y_2 \cdots y_{m-1} = x = z_1 z_2 \cdots z_{m-1}$. The induction hypothesis applied to this identity gives traces $w_{i,j}$ $(1 \le i \le m-1,\ 1 \le j \le n-1)$ such that

- $y_i = w_{i,1} w_{i,2} \ldots w_{i,n-1}$ for every $1 \le i \le m-1$,
- $z_j = w_{1,j} w_{2,j} \ldots w_{m-1,j}$ for every $1 \le j \le n-1$, and
- $w_{i,j} \, I \, w_{k,l}$ if $1 \le i < k \le m-1$ and $1 \le l < j \le n-1$.

Altogether we now obtain the following.

- $u_i = y_i w_{i,n} = w_{i,1} w_{i,2} \ldots w_{i,n-1} w_{i,n}$ for every $1 \le i \le m-1$.
- $u_m = u w_{m,n} = w_{m,1} w_{m,2} \cdots w_{m,n-1} w_{m,n}$
- $v_j = z_j w_{m,j} = w_{1,j} w_{2,j} \ldots w_{m-1,j} w_{m,j}$ for every $1 \le j \le n-1$.
- $v_n = v w_{m,n} = w_{1,n} w_{2,n} \cdots w_{m-1,n}, w_{m,n}$

Finally we have to verify that $w_{i,j} \, I \, w_{k,l}$ if $1 \le i < k \le m$ and $1 \le l < j \le n$. For the case $k < m$ and $j < n$ this was already stated.

- $1 \le i < k \le m-1$ and $1 \le l < n$: Since $y_k \, I \, w_{i,n}$ and $w_{k,l}$ is a factor of y_k it holds $w_{i,n} \, I \, w_{k,l}$.
- $1 \le i < m$ and $1 \le l < n$: Then $w_{i,n}$ is a factor of v and $w_{m,l}$ is a factor of u. Since $u \, I \, v$ we have $w_{i,n} \, I \, w_{m,l}$.
- $1 \le i < m$ and $1 \le l < j < n$: It holds $z_j \, I \, w_{m,l}$. Since $w_{i,j}$ is a factor of z_j it holds $w_{i,j} \, I \, w_{m,l}$.

Now we have covered all possibilities. $\square$

2.2 Recognizable trace languages

For an introduction into the field of recognizable trace languages, we refer the reader for instance to chapter 6 of [DR95]. In this section let M be an arbitrary monoid and let (Σ, I) be an arbitrary independence alphabet. An M–*automaton* is a triple $\mathcal{A} = (Q, h, F)$ where Q is a finite monoid, $h : M \longrightarrow Q$ is a monoid morphism, and $F \subseteq Q$. The M–automaton $\mathcal{A}$ *recognizes* the set $L(\mathcal{A}) = h^{-1}(F)$. A subset $L \subseteq M$ is called *recognizable* if there exists an M–automaton $\mathcal{A}$ with $L = L(\mathcal{A})$. The set of all recognizable subsets of M is denoted by $REG(M)$. Sets in $REG(\mathbb{M}(\Sigma, I))$ are called *recognizable trace languages*. The sets in $REG(\Sigma^*)$ are exactly the regular languages over Σ. If $\mathcal{A} = (Q, h, F)$ is an $\mathbb{M}(\Sigma, I)$–automaton then h is uniquely determined by the values $h(a)$ for $a \in \Sigma$. Thus $\mathcal{A}$ is a finite object. The following facts are well–known.

Fact 2.6. The following closure properties hold:

(1) $REG(M)$ is closed under Boolean operations.

(2) $REG(\mathbb{M}(\Sigma, I))$ is effectively closed under Boolean operations as well as concatenation, i.e., given $\mathbb{M}(\Sigma, I)$–automata $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{M}(\Sigma, I)$-automata that recognize the sets $\mathbb{M}(\Sigma, I) \backslash L(\mathcal{A})$, $L(\mathcal{A}) \cap L(\mathcal{B})$, $L(\mathcal{A}) \cup L(\mathcal{B})$, and $L(\mathcal{A})L(\mathcal{B})$, respectively, can be effectively computed from $\mathcal{A}$ and $\mathcal{B}$.

(3) If $L \subseteq \mathbb{M}(\Sigma, I)$ is finite then $L \in REG(\mathbb{M}(\Sigma, I))$.

(4) If $\Gamma \subseteq \Sigma$ then $\{u \in \mathbb{M}(\Sigma, I) \mid \mathit{alph}(u) \subseteq \Gamma\} \in \mathit{REG}(\mathbb{M}(\Sigma, I))$ and an $\mathbb{M}(\Sigma, I)$-automaton for this set can be constructed effectively.

Fact 2.7. The following problem is decidable.
INPUT: An $\mathbb{M}(\Sigma, I)$–automaton (Q, h, F).
QUESTION: $L(\mathcal{A}) = \emptyset$?

Lemma 2.8. ([Dub86]) Let $(\Sigma_1, \ldots, \Sigma_n)$ be a clique covering of the dependence alphabet (Σ, I^c) and let $L_i \in \mathit{REG}(\Sigma_i^*)$ for every $i \in \{1, \ldots, n\}$. Then $L = \{u \in \mathbb{M}(\Sigma, I) \mid \bigwedge_{i=1}^n \pi_i(u) \in L_i\} \in \mathit{REG}(\mathbb{M}(\Sigma, I))$.

Proof. For every $i \in \{1, \ldots, n\}$ let $\mathcal{A}_i = (Q_i, h_i, F_i)$ be a Σ_i^*–automaton with $L_i = L(\mathcal{A}_i)$. Thus $L_i = h_i^{-1}(F_i)$. Let $Q = \prod_{i=1}^n Q_i$ and let $F = \prod_{i=1}^n F_i$. Define the monoid morphism $h : \mathbb{M}(\Sigma, I) \longrightarrow Q$ by $h(u) = (h_1(\pi_1(u)), \ldots, h_n(\pi_n(u)))$ for every $u \in \mathbb{M}(\Sigma, I)$. We claim that the automaton (Q, h, F) recognizes L. We have $u \in h^{-1}(F)$ if and only if $(h_1(\pi_1(u)), \ldots, h_n(\pi_n(u))) \in \prod_{i=1}^n F_i$ if and only if $\pi_i(u) \in h_i^{-1}(F_i) = L_i$ for all $i \in \{1, \ldots, n\}$ if and only if $u \in L$. $\square$

2.3 Trace Rewriting Systems

We start this section with some definitions concerning abstract reduction systems. In the following let $\to$ be an arbitrary binary relation (for which we use infix notation) on an arbitrary set A. The inverse relation $\to^{-1}$ of $\to$ is also denoted by $\leftarrow$. For $n \geq 0$, the relation $\to^n$ is inductively defined by (i) $\to^0 = \mathit{Id}_A$ and (ii) $\to^{n+1} = \to^n\to$. Furthermore we define $\to^{\leq n} = \bigcup_{i=0}^n \to^i$ We call $\to$ *terminating* if there does not exist an infinite chain $a_1 \to a_2 \to a_3 \to \cdots$ in A. We call the pair $(a, b) \in A \times A$ *confluent* (with respect to $\to$) if $a \to^* c \leftarrow^* b$ for some $c \in A$. Let $\alpha \geq 1$. We call the pair (a, b) *α–confluent* (with respect to $\to$) if $a \to^\alpha c \leftarrow^\alpha b$ for some $c \in A$. The relation $\to$ is called *confluent* if $a \leftarrow^* c \to^* b$ implies $a \to^* d \leftarrow^* b$ for some $d \in A$. The relation $\to$ is called *locally confluent* if $a \leftarrow c \to b$ implies $a \to^* d \leftarrow^* b$ for some $d \in A$. Finally, for $\alpha, \beta \geq 1$, $\to$ is called *(α, β)–confluent* if $a \leftarrow^{\leq \alpha} c \to^{\leq \alpha} b$ implies $a \to^{\leq \beta} d \leftarrow^{\leq \beta} b$ for some $d \in A$. Obviously, if $\to$ is confluent then $\to$ is locally confluent and if $\to$ is (α, β)–confluent for some $\alpha, \beta \geq 1$ then $\to$ is also locally confluent. A $(1, \alpha)$–confluent relation will be briefly called α–confluent. A 1–confluent relation is also called *strongly confluent* in the literature. It is easy to see that if $\to$ is strongly confluent then $\to$ is also confluent ([New43]). Newman's lemma ([New43]) states that if $\to$ is terminating then $\to$ is confluent if and only if $\to$ is locally confluent.

Lemma 2.9. If there exists an $\alpha \geq 1$ such that $\to$ is (α, α)–confluent then $\to$ is confluent.

Proof. If $\to$ is (α, α)–confluent, then the relation $\to^{\leq \alpha}$ is strongly confluent and thus confluent. Now assume that $a \leftarrow^* c \to^* b$. Divide these derivations into subderivations of at most α many rewrite steps, i.e., $a \leftarrow^{\leq \alpha} \ldots \leftarrow^{\leq \alpha} c \to^{\leq \alpha} \ldots \to^{\leq \alpha} b$. Since $\to^{\leq \alpha}$ is confluent it follows $a \to^{\leq \alpha} \ldots \to^{\leq \alpha} d \leftarrow^{\leq \alpha} \ldots \leftarrow^{\leq \alpha} b$, i.e., $a \to^* d \leftarrow^* b$ for some $d \in A$. $\square$

A *trace rewriting system*, briefly TRS, over the trace monoid $M = \mathbb{M}(\Sigma, I)$ is a non–empty finite subset of $M \times M$. In the rest of this section let $\mathcal{R}$ be an arbitrary TRS over an arbitrary trace monoid $M = \mathbb{M}(\Sigma, I)$. An element $(l, r) \in \mathcal{R}$ is also denoted by $l \to r$. The set $\{l \mid \exists r \in M : (l, r) \in \mathcal{R}\}$ of all left–hand sides of $\mathcal{R}$ is denoted by $\mathit{dom}(\mathcal{R})$. The set $\{r \mid \exists l \in M : (l, r) \in \mathcal{R}\}$ of all right–hand sides of $\mathcal{R}$ is denoted by $\mathit{ran}(\mathcal{R})$. Given $c = (l, r) \in \mathcal{R}$ and $s, t \in M$, we write $s \to_c t$ if $s = ulv$ and $t = urv$ for some $u, v \in M$. We write $s \to_{\mathcal{R}} t$ if there exists a $c \in \mathcal{R}$ with $s \to_c t$. If $I = \emptyset$, i.e., $M \simeq \Sigma^*$ then $\mathcal{R}$ is also called a *semi–Thue system* over Σ. On the other hand, if $I = (\Sigma \times \Sigma) \backslash \mathit{Id}_\Sigma$, i.e., $M \simeq \mathbb{N}^{|\Sigma|}$ then $\mathcal{R}$ is also called a *vector replacement system*. We say that the TRS $\mathcal{R}$ is terminating (confluent, locally confluent, (α, β)–confluent) if $\to_{\mathcal{R}}$ is terminating (confluent, locally confluent, (α, β)–confluent). We say that $\mathcal{R}$ is *length–reducing* if $|l| > |r|$ for every $(l, r) \in \mathcal{R}$. Obviously, if $\mathcal{R}$ is length reducing, then $\mathcal{R}$ is also terminating. We say that $\mathcal{R}$ is *special* if $\mathit{ran}(\mathcal{R}) = \{1\}$ and $1 \notin \mathit{dom}(\mathcal{R})$. Let $\mathit{CONFL}(M)$ denote the set of all length–reducing and confluent TRSs over M and for $\alpha, \beta > 0$ let $\mathit{CONFL}(\alpha, \beta, M)$ denote the set of all (α, β)–confluent TRSs over M. The topic of this paper is the question whether these sets are decidable. Trivially, if $CONFL(M)$ is undecidable then confluence is also undecidable for the class of all terminating TRSs over M. The well–known critical pair lemma for semi–Thue systems ([NB72]) states that a semi–Thue system is locally confluent if and only if a finite set of so called *critical pairs* are confluent. These critical pairs result from overlapping left–hand sides and can be calculated effectively, see the end of Section 2.3.1. Therefore it is decidable whether a terminating semi–Thue system is locally confluent and thus confluent, see [BO81]. In particular, for every finite alphabet Γ, $CONFL(\Gamma^*)$ is decidable. More precisely, $CONFL(\Gamma^*)$ can be decided in time $O(n^3)$ where the length of the input $\mathcal{R}$ (a length–reducing semi–Thue system) is $\sum\{|l| + |r| \mid (l, r) \in \mathcal{R}\}$, see [KKMN85]. In contrast to this result, Narendran and Otto [NO88] have presented a trace monoid $\mathbb{M}(\Gamma, J)$ for which $CONFL(\mathbb{M}(\Gamma, J))$ is undecidable. This result will be generalized in Section 3 where we prove that $CONFL(M)$ is decidable if and only if M is a free monoid or a free commutative monoid. On the other hand, in Section 4 we will prove that $CONFL(\alpha, \beta, M)$ is always decidable. Note that a locally confluent semi–Thue system $\mathcal{R}$ is always α–confluent for some $\alpha \geq 1$, which can be computed effectively. For α we can choose any $\beta \geq 1$ such that for all finitely many critical pairs (s_1, s_2) it holds $s_1 \to_{\mathcal{R}}^{\leq\beta} s \;{}_{\mathcal{R}}\!\!\leftarrow^{\leq\beta} s_2$ for some s. For TRSs this does not hold anymore, as the following example shows.

Example 2.10. Let M be the trace monoid $M = \mathbb{M}(\{a, b, c\}, \{(a, c), (c, a)\})$ and let $\mathcal{R} = \{[ba] \to 1, [ab] \to 1, [c] \to 1\}$, which is a special TRS. Then for every $n \geq 0$:

$$[c^n b] \;{}_{\mathcal{R}}\!\!\leftarrow\; [bac^n b] = [bc^n ab] \to_{\mathcal{R}} [bc^n]$$

Of course $[c^n b] \to_{\mathcal{R}}^n [b] \;{}_{\mathcal{R}}\!\!\leftarrow^n [bc^n]$ but there does not exist a $k < n$ and a trace $u \in M$ such that $[c^n b] \to_{\mathcal{R}}^k u \;{}_{\mathcal{R}}\!\!\leftarrow^k [bc^n]$. Thus, for every $\alpha \geq 1$, $\mathcal{R}$ is not α–confluent. On the other it can be shown that $\mathcal{R}$ is confluent, see Example 2.15.

Lemma 2.9 leads to the question whether there are TRSs that are $(\alpha + 1, \alpha + 1)$–confluent but not (α, α)–confluent. At least for $\alpha = 1$ this is true as the following example shows. We leave the general question as an open problem.

Example 2.11. Consider the non–terminating TRS $\mathcal{R} = \{aa \to 1, aa \to a, 1 \to aa, 1 \to 1\}$ over the free (and free commutative) monoid $\{a\}^* \simeq \mathbb{N}$. In the following we denote the word a^n simply by n. Thus $\mathcal{R} = \{2 \to 0, 2 \to 1, 0 \to 2, 0 \to 0\}$. We denote the four rules of $\mathcal{R}$ by -2, -1, $+2$, and 0 respectively. For instance $n \to_{-1} m$ if $m = n - 1$ and $n \geq 2$. The TRS $\mathcal{R}$ is not $(1, 1)$–confluent because $2 \to_{-2} 0$ and $2 \to_{-1} 1$ but we have only $0 \to_{\mathcal{R}} 2$ and $1 \to_{\mathcal{R}} 3$.

On the other hand we claim that $\mathcal{R}$ is $(2, 2)$–confluent and thus confluent. Let us abbreviate $\to_{\mathcal{R}}$ by $\to$. In order to prove that $\mathcal{R}$ is $(2, 2)$–confluent, it suffices to show that for each situation $k \leftarrow^2 m \to^2 n$ it holds $k \to^2 m' \leftarrow^2 n$ for some $m' \in \mathbb{N}$ (since $0 \in \mathcal{R}$ it suffices to consider the relation $\to^2$ instead of $\to^{\leq 2}$. This is the only purpose of the rule 0. In the following we will always omit the trivial steps $n \to_0 n$.). Since $k \to^2 m' \leftarrow^2 n$ implies $k + d \to^2 m' + d \leftarrow^2 n + d$ for every $d \in \mathbb{N}$ it suffices to consider finitely many situations $k \leftarrow^2 m \to^2 n$, namely one for each choice of $c_1, d_1, c_2, d_2 \in \{-2, -1, +3, 0\}$ in $k \;{}_{c_1}\!\!\leftarrow\; {}_{d_1}\!\!\leftarrow\; m \to_{d_2} \to_{c_2} n$ Fortunately we can reduce the number of situations that have to be checked considerably.

First w.l.o.g. $c_1, d_1, c_2, d_2 \neq +2$, because otherwise we can assume w.l.o.g that $d_1 = +2$. Since $k = c_1 + d_1 + m = c_1 + 2 + m \geq m$ we have $k \to_{d_2} \to_{c_2} k + d_2 + c_2 = m + 2 + c_1 + d_2 + c_2$. Similarly $n \to_{+2} n + 2 \to_{c_1} n + 2 + c_1 = m + d_2 + c_2 + 2 + c_1$. Thus, for the further consideration we can assume that $c_1, d_1, c_2, d_2 \neq +2$. Furthermore we can assume $d_1 \neq d_2$. Finally note that $m \to_{-2} m - 2 \to_{-1} m - 3$ implies $m \geq 4$, and thus $m \to_{-1} m - 1 \to_{-2} m - 3$. Thus we can exclude the case $-1 \in \{c_1, d_1\} \cap \{c_2, d_2\}$. The following situations remain:

- $0 \;{}_{-2}\!\!\leftarrow\; {}_{-2}\!\!\leftarrow\; 4 \to_{-1} \to_{-2} 1$: It holds $0 \to_{+2} 2 \to_{-1} 1$.
- $0 \;{}_{-2}\!\!\leftarrow\; {}_{-2}\!\!\leftarrow\; 4 \to_{-1} \to_{-1} 2$: It holds $2 \to_{-2} 0$.
- $1 \;{}_{-2}\!\!\leftarrow\; 3 \to_{-1} \to_{-2} 0$: It holds $0 \to_{+2} 2 \to_{-1} 1$.
- $1 \;{}_{-2}\!\!\leftarrow\; 3 \to_{-1} \to_{-1} 1$: trivial

- $3 \;{}_{-1}\!\!\leftarrow 4 \rightarrow_{-2}\rightarrow_{-2} 0$: It holds $3 \rightarrow_{-1} 2 \;{}_{+2}\!\!\leftarrow 0$.
- $1 \;{}_{-1}\!\!\leftarrow 2 \rightarrow_{-2} 0$: It holds $0 \rightarrow_{+2} 2 \rightarrow_{-1} 1$.

2.3.1 A critical pair lemma

For trace rewriting systems strange phenomena may arise, which cannot occur for ordinary semi–Thue systems. For instance if there exist two disjoint occurrences of left–hand sides l_1 and l_2 in a word t then we can first replace the occurrence of l_1 by the corresponding right–hand side. In the resulting word, the occurrence of l_2 still exists and thus may be replaced by the corresponding right–hand side. If the two rules are applied in the reverse order, the result will be the same. This simple fact does not hold for TRSs in general, as the following example shows.

Example 2.12. Let M be the trace monoid from Example 2.10 Consider the TRS $\mathcal{R}_1 = \{[c] \rightarrow [b], [aa] \rightarrow 1\}$ over M. In the trace $[caa] = [aca]$ there exist unique disjoint occurrences of $[c]$ and $[aa]$. But $[aca] \rightarrow_{\mathcal{R}_1} [aba]$ by an application of the rule $[c] \rightarrow [b]$. In the resulting trace $[aba]$ there does not exist an occurrence of $[aa]$.

Another example is the one–rule TRS $\mathcal{R}_2 = \{[ac] \rightarrow [b]\}$ over the same trace monoid. In $u = [aacc]$ there exist four occurrences of the left–hand side $[ac]$. But via the step $[aacc] \rightarrow_{\mathcal{R}_2} [abc]$, the unique occurrence of $[ac]$ that was disjoint of the replaced occurrence is destroyed.

In this section we present a class of TRSs such that the above phenomenon cannot occur. Let us introduce the following technical property.

> The TRS $\mathcal{R}$ satisfies condition (A) if
>
> (A1) for all $(l \rightarrow r) \in \mathcal{R}$ and all $a \in \Sigma$, if $a\,I\,l$ then $a\,I\,r$ and
>
> (A2) for all $(l_0 \rightarrow r_0), (l_1 \rightarrow r_1) \in \mathcal{R}$ and all factorizations $l_0 = p_0q_0$, $l_1 = p_1q_1$ with $p_i \neq 1 \neq q_i$ for $i \in \{0,1\}$, $p_0\,I\,p_1$, and $q_0\,I\,q_1$ it holds: There exist factorizations $r_0 = s_0t_0$, $r_1 = s_1t_1$ such that $a\,I\,p_i$ implies $a\,I\,s_i$ and $a\,I\,q_i$ implies $a\,I\,t_i$ for all $a \in \Sigma$, $i \in \{0,1\}$.

The TRS $\mathcal{R}_0$ from Example 2.12 does not satisfy property (A1) for the rule $[c] \rightarrow [b]$: We have $a\,I\,c$ but not $a\,I\,b$. Note that $\mathcal{R}_0$ satisfies property (A2). The TRS $\mathcal{R}_1$ from the same example satisfies (A1) but (A2) is not satisfied. Choose $p_0 = [a]$, $q_0 = [c]$, $p_1 = [c]$, and $q_1 = [a]$. Thus $p_0\,I\,p_1$ and $q_0\,I\,q_1$. But either $s_0 = [b]$ or $t_0 = [b]$. In the first case we have $c\,I\,p_0$ but $c\,I^c\,s_0$, in the second case we have $a\,I\,q_0$ but $a\,I^c\,t_0$. On the other hand, the TRS from Example 2.10 satisfies condition (A). In fact, every special TRS satisfies condition (A). The condition (A) will become important in Lemma 2.14.

As already mentioned earlier, for a semi–Thue system $\mathcal{R}$ it is possible to construct a finite set of critical pairs, which result from overlapping left–hand sides of $\mathcal{R}$, such that $\mathcal{R}$ is locally confluent if and only if all critical pairs are confluent. In [Die90], the definition of critical pairs for semi–Thue system was extended to general TRSs such that again a TRS $\mathcal{R}$ is locally confluent if and only if all its critical pairs are confluent. But unfortunately the set of critical pairs of a TRS is in general infinite. However, since confluence is already undecidable for the class of all terminating TRSs ([NO88]), this is not a specific feature of the definition proposed in [Die90]) but an unavoidable restriction. In the next definition we introduce the notions of a *critical situation* and the notion of a *critical pair* for a TRS that satisfies condition (A). This definition slightly differs from the definition of critical pairs in [Die90]. In particular it is restricted to TRSs that satisfy condition (A). This restriction is motivated by our intended applications in Section 3 and 5. But again the set of critical situations of a TRS is in general infinite.

Definition 2.13. The set $Crit(\mathcal{R})$ of all *critical situations* of $\mathcal{R}$ is the set of all triples (t_0, t, t_1) such that there exist rules $(l_0, r_0), (l_1, r_1) \in \mathcal{R}$ and seven traces p_i, q_i, w_i $(i \in \{0,1\})$ and $s \neq 1$ such that

(1) $l_0 = p_0 s q_0$, $l_1 = p_1 s q_1$

(2) $p_0 \, I \, p_1$, $q_0 \, I \, q_1$, $w_0 \, I \, w_1$, $s \, I \, w_0 w_1$, $w_0 \, I \, q_0 p_1$, $w_1 \, I \, p_0 q_1$

(3) $a \, I^c \, p_i$ and $b \, I^c \, q_{1-i}$ for all $a \in \mathit{min}(w_i)$, $b \in \mathit{max}(w_i)$ and all $i \in \{0,1\}$.

(4) $t = p_0 w_0 p_1 s q_1 w_1 q_0 = p_1 w_1 p_0 s q_0 w_0 q_1$, $t_0 = p_1 w_1 r_0 w_0 q_1$, $t_1 = p_0 w_0 r_1 w_1 q_0$

We say that this critical situation *results from* the rules $l_0 \to r_0$ and $l_1 \to r_1$. We define the set $\mathit{CP}(\mathcal{R})$ of *critical pairs* of $\mathcal{R}$ as $\mathit{CP}(\mathcal{R}) = \{(t_0, t_1) \mid \exists t : (t_0, t, t_1) \in \mathit{Crit}(\mathcal{R})\}$ and we define the set $\mathit{CT}(\mathcal{R})$ of *critical traces* of $\mathcal{R}$ as $\mathit{CT}(\mathcal{R}) = \{t \mid \exists t_0, t_1 : (t_0, t, t_1) \in \mathit{Crit}(\mathcal{R})\}$.

Lemma 2.14. If $\mathcal{R}$ satisfies condition (A) then $\mathcal{R}$ is locally confluent (α–confluent, respectively) if and only if all pairs in $\mathit{CP}(\mathcal{R})$ are confluent (α–confluent, respectively).

Proof. Let $\mathcal{R}$ satisfies condition (A). Note that $(t_0, t, t_1) \in \mathit{Crit}(\mathcal{R})$ implies $t_0 \ {}_{\mathcal{R}}\!\leftarrow \ t \to_{\mathcal{R}} t_1$. Thus one direction of the lemma is obvious. Now assume that all pairs in $\mathit{CP}(\mathcal{R})$ are confluent (α–confluent, respectively). Consider an arbitrary situation $t_0 \ {}_{\mathcal{R}}\!\leftarrow \ t \to_{\mathcal{R}} t_1$. We have to show that the pair (t_0, t_1) is confluent (α–confluent, respectively). There exist rules $(l_0 \to r_0), (l_1 \to r_1) \in \mathcal{R}$ and traces $u_0, u_1, v_0, v_1 \in M$ with $t_0 = u_0 r_0 v_0 \ {}_{\mathcal{R}}\!\leftarrow \ u_0 l_0 v_0 = t = u_1 l_1 v_1 \to_{\mathcal{R}} u_1 r_1 v_1$. Lemma 2.5 applied to the identity $u_0 l_0 v_0 = u_1 l_1 v_1$ gives traces p_i, q_i, w_i, y_i ($i \in \{0,1\}$), and s such that $l_i = p_i s q_i$, $p_0 \, I \, p_1$, $q_0 \, I \, q_1$, $w_0 \, I \, w_1$, $s \, I \, w_0 w_1$, $w_0 \, I \, q_0 p_1$, and $w_1 I p_0 q_1$, see the following diagram.

v_1	w_1	q_0	y_1
l_1	p_1	s	q_1
u_1	y_0	p_0	w_0
	u_0	l_0	v_0

It holds $u_0 r_0 v_0 = (y_0 p_1 w_1) r_0 (w_0 q_1 y_1)$ and $u_1 r_1 v_1 = (y_0 p_0 w_0) r_1 (w_1 q_0 y_1)$. It suffices to prove that the pair $(p_1 w_1 r_0 w_0 q_1, p_0 w_0 r_1 w_1 q_0)$ is confluent (α–confluent) since this implies that also the pair $(y_0 p_1 w_1) r_0 (w_0 q_1 y_1), (y_0 p_0 w_0) r_1 (w_1 q_0 y_1))$ is confluent (α–confluent).

Let us first consider the case $s = 1$. Only for this case we will need condition (A). Thus $l_0 = p_0 q_0$ and $l_1 = p_1 q_1$. We claim that the pair $(p_1 w_1 r_0 w_0 q_1, p_0 w_0 r_1 w_1 q_0)$ is 1–confluent. First assume that $p_0 = 1$. We claim that $p_1 w_1 r_0 w_0 q_1 \to_{\mathcal{R}} w_0 r_1 w_1 r_0 \ {}_{\mathcal{R}}\!\leftarrow \ w_0 r_1 w_1 q_0$. Since $l_0 = p_0 q_0 = q_0$ this can be deduced as follows:

$$
\begin{aligned}
p_1 w_1 r_0 w_0 q_1 = \; & p_1 w_1 w_0 q_1 r_0 && (\text{since } w_0 q_1 \, I \, q_0 = l_0 \text{ which implies } w_0 q_1 \, I \, r_0 \text{ by (A1)}) \\
= \; & w_0 p_1 q_1 w_1 r_0 && (\text{since } w_0 \, I \, w_1,\ w_0 \, I \, p_1, \text{ and } w_1 \, I \, q_1) \\
\to_{\mathcal{R}} \; & w_0 r_1 w_1 r_0 \ {}_{\mathcal{R}}\!\leftarrow w_0 r_1 w_1 q_0
\end{aligned}
$$

Thus, we may assume that $p_0 \neq 1$. Similarly we may assume that also $q_0 \neq 1$, $p_1 \neq 1$, and $q_1 \neq 1$. But then condition (A2) implies that there exist factorizations $r_0 = s_0 t_0$, $r_1 = s_1 t_1$ such that $a \, I \, p_i$ implies $a \, I \, s_i$ and $a \, I \, q_i$ implies $a \, I \, t_i$ for all $a \in \Sigma$. In particular it holds

$$p_1 \, I \, s_0,\ w_1 \, I \, s_0,\ p_0 \, I \, s_1,\ w_0 \, I \, s_1,\ q_1 \, I \, t_0,\ w_0 \, I \, t_0,\ q_0 \, I \, t_1,\ w_1 \, I \, t_1.$$

Furthermore $p_1 \, I \, s_0$ implies $s_1 \, I \, s_0$ and $q_1 \, I \, t_0$ implies $t_1 \, I \, t_0$. Thus, we obtain

$$
\begin{aligned}
& p_1 w_1 r_0 w_0 q_1 = p_1 w_1 s_0 t_0 w_0 q_1 = s_0 w_0 p_1 q_1 w_1 t_0 \to_{\mathcal{R}} s_0 w_0 s_1 t_1 w_1 t_0 = \\
& \qquad s_1 w_1 s_0 t_0 w_0 t_1 \ {}_{\mathcal{R}}\!\leftarrow s_1 w_1 p_0 q_0 w_0 t_1 = p_0 w_0 s_1 t_1 w_1 q_0 = p_0 w_0 r_1 w_1 q_0.
\end{aligned}
$$

In the following we assume that $s \neq 1$. We prove confluence (α–confluence) of the pair $(y_0 p_1 w_1 r_0 w_0 q_1 y_1, y_0 p_0 w_0 r_1 w_1 q_0 y_1)$ by an induction on $|w_0 w_1|$. If $a \, I^c \, p_i$ for all $a \in \mathit{min}(w_i)$ and $a \, I^c \, q_{1-i}$ for all $a \in \mathit{max}(w_i)$ ($i \in \{0,1\}$) then $(p_1 w_1 r_0 w_0 q_1, p_1 w_1 l_0 w_0 q_1, p_0 w_0 r_1 w_1 q_0) \in \mathit{Crit}(\mathcal{R})$ and the pair $(p_1 w_1 r_0 w_0 q_1, p_0 w_0 r_1 w_1 q_0)$ is confluent (α–confluent, respectively). Note that this case also includes the induction hypothesis $w_0 w_1 = 1$. Thus assume that for instance $w_0 = a w_0'$ (the other cases can be dealt analogously) and $a \, I \, p_0$. Since $w_0 \, I \, s q_0$ it follows $a \, I \, l_0$. Since $\mathcal{R}$

satisfies condition (A1) it follows $a\,I\,r_0$. Thus $p_1w_1r_0aw_0'q_1 = ap_1w_1r_0w_0'q_1$ and $p_0aw_0'r_1w_1q_0 = ap_0w_0'r_1w_1q_0$. By the induction hypothesis the pair $(p_1w_1r_0w_0'q_1, p_0w_0'r_1w_1q_0)$ is confluent (α–confluent, respectively). Thus the same holds for the pair $(ap_1w_1r_0w_0'q_1, ap_0w_0'r_1w_1q_0)$. $\square$

Example 2.15. Let us use Lemma 2.14 in order to prove that TRS $\mathcal{R} = \{[ba] \to 1, [ab] \to 1, [c] \to 1\}$ from Example 2.10, where only a and c are independent, is confluent. Since $\mathcal{R}$ satisfies condition (A), Lemma 2.14 can be applied. Let $l_0 = p_0sq_0$ and $l_1 = p_1sq_1$ be left–hand sides of $\mathcal{R}$, where $s \neq 1$, $p_0\,I\,p_1$, $q_0\,I\,q_1$. If we exclude the trivial case $l_0 = s = l_1$ then only the following two cases may occur.
case 1: $l_0 = [ab]$, $l_1 = [ba]$, $s = [b]$, $p_0 = [a] = q_1$, and $p_1 = 1 = q_0$:
We have to consider all pairs $(p_1w_1w_0q_1, p_0w_0w_1q_0) = (w_1w_0[a], [a]w_0w_1)$, where w_0 and w_1 satisfy the conditions (2) and (3) from Definition 2.13. In particular, $s\,I\,w_0w_1$. Since $s = [b]$ this implies $w_0 = 1 = w_1$. Thus, we have to consider the pair $([a], [a])$ which is trivially confluent.
case 2: $l_0 = [ba]$, $l_1 = [ab]$, $s = [a]$, $p_0 = [b] = q_1$, and $p_1 = 1 = q_0$:
We have to consider all pairs $(p_1w_1w_0q_1, p_0w_0w_1q_0) = (w_1w_0[b], [b]w_0w_1)$, where w_0 and w_1 satisfy the conditions (2) and (3) from Definition 2.13. In particular $s\,I\,w_0w_1$. Since $s = [a]$ it follows $w_0 = [c^m]$ and $w_1 = [c^n]$ for some $m, n \geq 0$. But we cannot have $m \geq 1$ and $n \geq 1$ since this would contradict $w_0\,I\,w_1$. From $w_1\,I\,p_0q_1$ and $p_0 = [b] = q_1$ it follows $w_1 = 1$. Thus, for every $m \geq 0$ we have to consider the pair $([c^mb], [bc^m])$, which is obviously confluent due to the rule $[c] \to 1$.

2.3.2 Codings between trace rewriting systems

In order to prove prove Theorem 3.12 from Section 3.2 it will be useful to consider mappings σ between TRSs over a trace monoid M' and TRSs over a second trace monoid M such that $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent. The following lemma gives a condition for σ that ensures this property. Let $\sigma : M' \to M$ be a morphism between trace monoids and let $\mathcal{R}$ be a TRS over M'. Define the TRS $\sigma(\mathcal{R})$ over M by $\sigma(\mathcal{R}) = \{\sigma(l) \to \sigma(r) \mid (l \to r) \in \mathcal{R}\}$.

Lemma 2.16. Let $\sigma : M' \to M$ be a morphism between trace monoids and let $\mathcal{R}$ be a TRS over M'. Assume that the following properties hold.

- σ is injective.
- $\sigma(\mathcal{R})$ satisfies condition (A) and is terminating.
- If $l \in \mathit{dom}(\mathcal{R})$ and $\sigma(s) = u_1\sigma(l)u_2$ then $u_1 = \sigma(u_1')$ and $u_2 = \sigma(u_2')$ for some $u_1', u_2' \in M'$.
- If $u \in \mathit{CT}(\sigma(\mathcal{R}))$ then $u = \sigma(u')$ for some $u' \in M'$.

Then $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent.

Proof. We first prove the following statement.

$$\text{If } \sigma(u') \to_{\sigma(\mathcal{R})} v \text{ then there exists a } v' \in M' \text{ such that } v = \sigma(v') \text{ and } u' \to_{\mathcal{R}} v'. \tag{1}$$

Let $\sigma(u') = u_1\sigma(l)u_2$ and $v = u_1\sigma(r)u_2$, where $(l, r) \in \mathcal{R}$. The third assumption from the lemma implies that there exist $u_1', u_2' \in M'$ such that $u_1 = \sigma(u_1')$ and $u_2 = \sigma(u_2')$. Thus $\sigma(u') = \sigma(u_1'lu_2')$, which implies $u' = u_1'lu_2'$ since σ is injective. Therefore $u' \to_{\mathcal{R}} u_1'ru_2'$ and $\sigma(u_1'ru_2') = u_1\sigma(r)u_2 = v$.

Now assume that $\sigma(\mathcal{R})$ is confluent and let $u' \to_{\mathcal{R}}^* u_1'$ and $u' \to_{\mathcal{R}}^* u_2'$. Since σ is a monoid morphism it follows $\sigma(u') \to_{\sigma(\mathcal{R})}^* \sigma(u_1')$ and $\sigma(u') \to_{\sigma(\mathcal{R})}^* \sigma(u_2')$. Since $\sigma(\mathcal{R})$ is confluent there exists an $v \in M$ such that $\sigma(u_1') \to_{\sigma(\mathcal{R})}^* v$ and $\sigma(u_2') \to_{\sigma(\mathcal{R})}^* v$. An inductive extension of (1) yield $v_1', v_2' \in M'$ such that $v = \sigma(v_1')$, $u_1' \to_{\mathcal{R}}^* v_1'$ and $v = \sigma(v_2')$, $u_2' \to_{\mathcal{R}}^* v_2'$. Since σ is injective, it follows $v_1' = v_2'$ Thus, $\mathcal{R}$ is confluent.

Finally, assume that $\mathcal{R}$ is confluent. Since $\sigma(\mathcal{R})$ is terminating it suffices to prove that $\sigma(\mathcal{R})$ is locally confluent. Let $(u_1, u, u_2) \in \mathit{Crit}(\sigma(\mathcal{R}))$ be arbitrary. Since $\sigma(\mathcal{R})$ satisfies condition (A) it suffices by Lemma 2.14 to show that the pair (u_1, u_2) is confluent with respect to $\sigma(\mathcal{R})$. By the fourth assumption from the lemma, $u = \sigma(u')$ for some $u' \in M'$. Thus $\sigma(u') \to_{\sigma(\mathcal{R})} u_1$ and $\sigma(u') \to_{\sigma(\mathcal{R})} u_2$. By (1) there exist $u_1', u_2' \in M'$ such that $u_1 = \sigma(u_1')$, $u_2 = \sigma(u_2')$ and $u' \to_{\mathcal{R}} u_1'$, $u' \to_{\mathcal{R}} u_2'$. Confluence of $\mathcal{R}$ implies that there exists a $v' \in M'$ such that $u_1' \to_{\mathcal{R}}^* v'$ and $u_2' \to_{\mathcal{R}}^* v'$, which implies $u_1 = \sigma(u_1') \to_{\sigma(\mathcal{R})}^* \sigma(v')$ and $u_2 = \sigma(u_2') \to_{\sigma(\mathcal{R})}^* \sigma(v')$. Thus, $\sigma(\mathcal{R})$ is confluent. $\square$

For a finite alphabet Σ and two words $s_1, s_2 \in \Sigma^*$ we say that a word $t \in \Sigma^*$ is an *overlapping* of the words s_1 and s_2 either (i) $t = s_i = us_{1-i}v$ for some $u, v \in \Sigma^*$ and $i \in \{0,1\}$ or $t = s_iu = vs_{1-i}$ and $|s_i| > |v|$ for some $i \in \{0,1\}$, $u, v \in \Sigma_i^*$, see the following diagrams:

<table>
<tr><td colspan="3">s_i</td></tr>
<tr><td>u</td><td>s_{1-i}</td><td>v</td></tr>
</table>

<table>
<tr><td>v</td><td colspan="2">s_{1-i}</td></tr>
<tr><td colspan="2">s_i</td><td>u</td></tr>
</table>

Usually in formal language theory a large finite alphabet $\{b_1, \ldots, b_n\}$ is encoded into the two–element alphabet $\{b_1, b_2\}$ via the morphism ϕ defined by $b_i \mapsto b_1 b_2^i$ for $i \in \{1, \ldots, n\}$. This is an injective morphism from $\{b_1, \ldots, b_n\}^*$ to $\{b_1, b_2\}^*$ Four our purpose this coding is not suitable since the third condition of the previous lemma is not always satisfied. For instance if $j > i$ then $\phi(b_j) = \phi(b_i)b_2^{j-i}$ but of course b_2^{j-i} is not in the range of ϕ. What is needed is a coding which does not generate new overlappings between left–hand sides. In the next lemma, such a coding for free monoids is presented. In order to use it later also for non–free monoids, we include some additional symbols $a_1, \ldots, a_m$ which are mapped to itself.

Lemma 2.17. Let $m \geq 0, n \geq 2$ and let $\Sigma = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ and $\Gamma = \{a_1, \ldots, a_m, b_1, b_2\}$. Define the morphism $\phi : \Sigma^* \to \Gamma^*$ by

$$\phi(a_i) = a_i \text{ for } i \in \{1, \ldots, m\} \text{ and } \phi(b_i) = b_1 b_2 b_1^{i+1} b_2^{n-i+2} \text{ for } i \in \{1, \ldots, n\}.$$

Then it holds

- if $\phi(s) = s_1\phi(l)s_2$ and $l \neq 1$ then $s_1 = \phi(s_1')$ and $s_2 = \phi(s_2')$ for some $s_1', s_2' \in \Sigma^*$ and
- if $\phi(l_1) = s_1 s$ and $\phi(l_2) = ss_2$ then $s_1 s s_2 = \phi(s')$ for some $s' \in \Sigma^*$

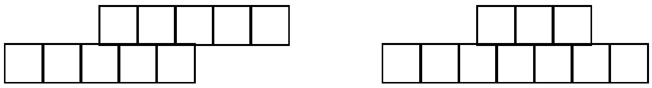
Proof. Assume that $\phi(s) = s_1\phi(l)s_2$ and $l \neq 1$. First we prove that $s_1 = \phi(s_1')$ for some $s_1' \in \Sigma^*$. Choose the factorization $s_1 = \phi(u)t$ with u maximal (which exists since $s_1 = \phi(1)s_1$). Thus $\phi(s) = \phi(u)t\phi(l)s_2$ which implies $t\phi(l)s_2 = \phi(v)$ for some $v \in \Sigma^+$. We claim that $t = 1$ which implies $s_1 = \phi(u)$. Assume that $t \neq 1$. If $v = a_i \cdots$ for some $i \in \{1, \ldots, m\}$ then $t = a_i \cdots$ which can be excluded due to the maximality of u. Thus $v = b_i \cdots$ for some $i \in \{1, \ldots n\}$. Therefore $t\phi(l)s_2 = b_1 b_2 b_1^{i+1} b_2^{n-i+2} \cdots$ and t must be a proper prefix of $b_1 b_2 b_1^{i+1} b_2^{n-i+2}$. The cases $t = b_1$ and $t = b_1 b_2 b_1^{i+1} b_2^j$ for $0 \leq j < n-i+2$ can be excluded since this would imply that $\phi(l) = b_2 \cdots$ (note that $l \neq 1$ and thus also $\phi(l) \neq 1$) which is not possible. The case $t = b_1 b_2 b_1^j$ for $0 \leq j \leq i-1$ can be excluded since otherwise $\phi(l) = b_1 b_1 \cdots$. Finally the remaining case $t = b_1 b_2 b_1^i$ is impossible since otherwise $\phi(l) = b_1 b_2 b_2 \ldots$.

Now we prove that $s_2 = \phi(s_2')$ for some $s_2' \in \Sigma^*$. Choose the factorization $s_2 = t\phi(u)$ with u maximal. Thus $\phi(s) = s_1\phi(l)t\phi(u)$ which implies $s_1\phi(l)t = \phi(v)$ for some $v \in \Sigma^+$. We claim that $t = 1$ which implies $s_2 = \phi(u)$. Assume that $t \neq 1$. If $v = \cdots a_i$ for some $i \in \{1, \ldots, m\}$ then $t = \cdots a_i$ which contradicts the maximality of u. Thus $v = \ldots b_i$ for some $i \in \{1, \ldots n\}$. Therefore $s_1\phi(l)t = \cdots b_1 b_2 b_1^{i+1} b_2^{n-i+2}$ and t must be a proper suffix of $b_1 b_2 b_1^{i+1} b_2^{n-i+2}$. The cases $t = b_2 b_1^{i+1} b_2^{i+1}$ and $t = b_1^j b_2^{i+1}$ for $0 \leq j \leq i$ can be excluded since this would imply that $\phi(l) = \cdots b_1$ which is not possible. If $t = b_1^{i+1} b_2^{i+1}$ then $\phi(l) = \cdots b_1 b_2$ which is also not possible. Finally we can exclude the cases $t = b_2^j$ for $1 \leq j \leq n-i+1$, since otherwise $\phi(l) = \ldots b_1 b_2 b_1^{i+1} b_2^k$ where $i + 1 + k < n + 3$.

Now we prove the second claim of the lemma. Assume that $s_1 s = \phi(l_1)$ and $ss_2 = \phi(l_2)$. The case $s = 1$ is clear, thus assume that $s \neq 1$. If we can show that $s = \phi(s')$ for some $s' \in \Sigma^*$, i.e., $s_1\phi(s') = \phi(l_1)$ and $\phi(s')s_2 = \phi(l_2)$, then the first statement of the lemma and $s \neq 1$ imply $s_1 = \phi(s_1')$ and $s_2 = \phi(s_2')$ for some $s_1', s_2' \in \Sigma^*$ and thus $t = \phi(s_1'\phi(s')\phi(s_2') = \phi(s_1' s' s_2')$. Now $s = \phi(s')$ for some $s' \in \Sigma^*$ can be proven as follows. Choose the factorization $s = \phi(u)v$ with

u maximal. Thus $ss_2 = \phi(u)vs_2 = \phi(l_2)$ and $vs_2 = \phi(w)$ for some $w \in \Sigma^*$. We claim that $v = 1$. Assume that $v \neq 1$ and thus also $w \neq 1$. If $w = a_i \cdots$ for some $i \in \{1, \ldots, m\}$ then $v = a_i \cdots$ which contradicts the maximality of u. Thus $w = b_i \ldots$ for some $i \in \{1, \ldots, n\}$ and $vs_2 = b_1 b_2 b_1^{i+1} b_2^{n-i+2} \ldots$. Hence v must be a proper prefix of $b_1 b_2 b_1^{i+1} b_2^{n-i+2}$. But this cannot be the case since v is a suffix of $\phi(l_1) = s_1 s = s_1 \phi(u) v$. $\square$

The lemma says that overlappings of two words of the following kind (where each square represents a word $\phi(x)$ with $x \in \Sigma$ are not possible.



Thus each overlapping between two words $\phi(s_1)$ and $\phi(s_2)$, where $s_1 \neq 1 \neq 2$ results from an overlapping of s_1 and s_2. Note that if $m = 0$ in the previous lemma and $\mathcal{R}$ is a length–reducing semi–Thue system then also $\phi(\mathcal{R})$ is length–reducing, because ϕ satisfies $|\phi(b_i)| = n + 5$ for all $i \in \{1, \ldots, n\}$.

3 Undecidability of confluence for length–reducing TRS

In this section we will prove that $\mathit{CONFL}(M)$ is decidable if and only if M is free or free commutative, which generalizes a result of [NO88]. For this it will be useful to study also the set $\mathit{CONFL}_{\neq 1}(M)$ that consist of all $\mathcal{R} \in \mathit{CONFL}(M)$ such that $1 \notin \mathit{ran}(\mathcal{R})$. Trivially, if $\mathit{CONFL}_{\neq 1}(M)$ is undecidable then $\mathit{CONFL}(M)$ is also undecidable. Our proof will proceed in two steps. In a first step (Section 3.1) we will prove the undecidability $\mathit{CONFL}_{\neq 1}(\mathbb{M}(\Sigma, I))$ for the two smallest independence alphabets (Γ, J) that do not result in a free or free commutative monoid. In a second step (Section 3.2) we show that $\mathit{CONFL}_{\neq 1}(\mathbb{M}(\Sigma, I))$ is undecidable if $\mathit{CONFL}_{\neq 1}(\mathbb{M}(\Gamma, J))$ is undecidable for some induced subgraph (Γ, J) of (Σ, I).

3.1 Independence alphabets with three vertices

If (Σ, I) is an independence alphabet with $|\Sigma| = 2$ then $M = \mathbb{M}(\Sigma, I)$ is either isomorphic to the free monoid $\{a, b\}^*$ or isomorphic to the free commutative monoid $\mathbb{N}^2 = \{a\}^* \times \{b\}^*$. In both cases $\mathit{CONFL}(M)$ is decidable. If $|\Sigma| = 3$ then there exist up to isomorphism two cases, where M is neither free nor free commutative. The first case arises from the independence alphabet

and will be considered in Section 3.1.1. The corresponding trace monoid M is isomorphic to $\{a, b\}^* \times \{c\}^*$. The second case arises from the independence alphabet

and will be considered in Section 3.1.2. In the corresponding trace monoid M, exactly two letters are allowed to commute. In both cases we will prove that $\mathit{CONFL}_{\neq 1}(M)$ (and thus $\mathit{CONFL}(M)$) is undecidable.

3.1.1 The case $a - c - b$

In this section we prove that $\mathit{CONFL}(\{c\}^* \times \{a, b\}^*)$ is undecidable. In a first step we prove that $\mathit{CONFL}(\{c\}^* \times \Gamma^*)$ is undecidable for some alphabet Γ that contains more then two letters. The result for $\{c\}^* \times \{a, b\}^*$ follows by a simple coding. A trace $t \in \Sigma_1^* \times \Sigma_2^*$ will be also denoted by $(t^{(1)}, t^{(2)})$, where $t^{(i)} \in \Sigma_i^*$. The proof of the following lemma is very easy and is left to the reader.

Lemma 3.1. A TRS $\mathcal{R}$ over $\Sigma_1^* \times \Sigma_2^*$ satisfies condition (A) if for every $(l, r) \in \mathcal{R}$ and every $i \in \{1, 2\}$, if $l^{(i)} = 1$ then $r^{(i)} = 1$.

Lemma 3.2. Let $\mathcal{R}$ be a TRS over the direct product $\Sigma_1^* \times \Sigma_2^*$ of free monoids. Then for every $(t_0, t, t_1) \in \mathit{Crit}(\mathcal{R})$ there exist rules $(l_0, r_0), (l_1, r_1) \in \mathcal{R}$ such that for both $i = 1$ and $i = 2$ one of the two following three cases holds and furthermore for either $i = 1$ or $i = 2$ the second or third case holds:

(1) $l_0^{(i)} \neq 1 \neq l_1^{(i)}$ and $t^{(i)} = l_j^{(i)} s l_{1-j}^{(i)}$, $t_j^{(i)} = r_j^{(i)} s l_{1-j}^{(i)}$, $t_{1-j}^{(i)} = l_j^{(i)} s r_{1-j}^{(i)}$ for some $j \in \{0,1\}$, $w \in \Sigma_i^*$

(2) $t^{(i)} = l_j^{(i)} = u l_{1-j}^{(i)} v$, $t_j^{(i)} = r_j^{(i)}$, $t_{1-j}^{(i)} = u r_{1-j}^{(i)} v$ for some $u, v \in \Sigma_i^*$ and $j \in \{0,1\}$

(3) $t^{(i)} = l_j^{(i)} u = v l_{1-j}^{(i)}$, $t_j^{(i)} = r_j^{(i)} u$, $t_{1-j}^{(i)} = v r_{1-j}^{(i)}$, and $|l_j^{(i)}| > |v|$ for some $j \in \{0,1\}$, $u, v \in \Sigma_i^*$

Note that in the second and third case $t^{(i)}$ is an overlapping of $l_1^{(i)}$ and $l_2^{(i)}$. In the first case we say that $t^{(i)}$ is disjointly generated by $l_1^{(i)}$ and $l_2^{(i)}$, see the following picture, where we omitted the superscript (i):

l_i	s	l_{1-i}

Proof. Let $(t_0, t, t_1) \in \mathit{Crit}(\mathcal{R})$. By Definition 2.13 there exist rules $(l_0, r_0), (l_1, r_1) \in \mathcal{R}$ and pairs $p_i, q_i, w_i, s \in \Sigma_1^* \times \Sigma_2^*$ with $t = p_1 w_1 l_0 w_0 q_1 = p_0 w_0 l_1 w_1 q_0$, $t_j = p_{1-j} w_{1-j} r_j w_j q_{1-j}$ $(j \in \{0,1\})$ and

- $l_j = p_j s q_j$ for $j \in \{0,1\}$, $s \neq (1,1)$,
- $p_0 \, I \, p_1$, $q_0 \, I \, q_1$, $w_0 \, I \, w_1$, $s \, I \, w_0 w_1$, $w_0 \, I \, q_0 p_1$, $w_1 \, I \, p_0 q_1$
- $p_j \, I \, w_j$ or $q_{1-j} \, I \, w_j$ implies $w_j = 1$ for $j \in \{0,1\}$.

We can separate the following cases.
case 1: $s^{(1)} \neq 1 \neq s^{(2)}$ then $s \, I \, w_0 w_1$ implies $w_0 = w_1 = (1,1)$, i.e. $t = p_1 l_0 q_1 = p_0 l_1 q_0$, $t_j = p_{1-j} r_j q_{1-j}$ for $j \in \{0,1\}$. Let $i \in \{1,2\}$. Then $t^{(i)} = p_1^{(i)} l_0^{(i)} q_1^{(i)} = p_0^{(i)} l_1^{(i)} q_0^{(i)}$. Furthermore $p_0 \, I \, p_1$ implies $p_j^{(i)} = 1$ for some $j \in \{0,1\}$. Similarly $q_j^{(i)} = 1$ for some $j \in \{0,1\}$. In each case, $t^{(i)}$ is an overlapping of $l_0^{(i)}$ and $l_1^{(i)}$.
case 2: $s^{(2)} = 1$ and $s^{(1)} \neq 1$ (the case $s^{(1)} = 1$, $s^{(2)} \neq 1$ is symmetrically). Since $s \, I \, w_0 w_1$ it follows $w_0^{(1)} = 1$ and $w_1^{(1)} = 1$. As in case 1, it follows that $t^{(1)}$ is an overlapping of $l_0^{(1)}$ and $l_1^{(1)}$. Furthermore, since $w_0^{(2)} \neq 1 \neq w_1^{(2)}$ contradicts $w_0 \, I \, w_1$ we may assume w.l.o.g. that $w_1 = (1,1)$ (the case $w_0 = (1,1)$ is symmetric). If also $w_0 = (1,1)$ then also $t^{(2)}$ is an overlapping of $l_0^{(2)}$ and $l_1^{(2)}$ (see the first case). Hence, assume that $w_0^{(2)} \neq 1$. Thus neither $p_0 \, I \, w_0$ nor $q_1 \, I \, w_0$ (because for instance $p_0 \, I \, w_0$ would imply $w_0 = 1$). It follows $p_0^{(2)} \neq 1 \neq q_1^{(2)}$. Hence $p_1 \, I \, p_2$ and $q_1 \, I \, q_2$ imply $p_1^{(2)} = 1 = q_0^{(2)}$. Now we have

$$t^{(2)} = p_0^{(2)} w_0^{(2)} p_1^{(2)} s^{(2)} q_1^{(2)} w_1^{(2)} q_0^{(2)} = p_0^{(2)} w_0^{(2)} q_1^{(2)} = l_0^{(2)} w_0^{(2)} l_1^{(2)}$$

with $l_0^{(2)} = p_0^{(2)} \neq 1 \neq q_1^{(2)} = l_1^{(2)}$. Thus also $t^{(2)}$ is disjointly generated by $l_0^{(2)}$ and $l_1^{(2)}$. $\square$

In the following let $\mathcal{M} = (Q, \Sigma, \Box, \delta, q_0, q_f)$ be a universal deterministic one–tape Turing machine, where Q is the finite set of states, Σ is the tape alphabet with $Q \cap \Sigma = \emptyset$, $\Box \in \Sigma$ is the blank symbol, $\delta : Q \backslash \{q_f\} \times \Sigma \to Q \times \Sigma \times \{L, R\}$ is the total transition function, q_0 is the initial state and $q_f \neq q_0$ is the final state. Since $\mathcal{M}$ is universal, the problem whether $\mathcal{M}$ halts on a given input $w \in (\Sigma \backslash \{\Box\})^+$ is undecidable. Note that $\mathcal{M}$ terminates if and only if it reaches the final state q_f. Let $\Gamma = Q \cup \Sigma \cup \{0, \triangleright, \triangleleft, A, B, \$\}$, where $0, \triangleright, \triangleleft, A, B, \$ \notin Q \cup \Sigma$. Given a word $w \in (\Sigma \backslash \{\Box\})^+$, we define a TRS $\mathcal{R}_w$ over $\{C\}^* \times \Gamma^*$ by the rules of Figure 1. Note that $\mathcal{R}_w$ is length–reducing and that all right–hand sides are non empty. Since we excluded the case $w = 1$, we do not have to consider the pair $(1, \triangleright q \triangleleft \$\$)$ in the last group of rules. The corresponding rule would have the form $(1, \triangleright q \triangleleft \$\$) \to (1, \triangleright q' \Box a' \triangleleft)$ if $\delta(q, \Box) = (q', a', L)$ which is not length–reducing. Let $\mathcal{R}_0$ be the system that consists of the rules (1a) to (1c) and let $\mathcal{R}_{\mathcal{M}}$ be the system that consists of the rules (4a), (4b) and (5a) to (5e).

Rules for the absorbing symbol 0:

(1a) $(1, x0) \to (1, 0)$ for $x \in \Gamma$
(1b) $(1, 0x) \to (1, 0)$ for $x \in \Gamma$
(1c) $(C, 0) \to (1, 0)$ for $x \in \Gamma$

Rules for deleting non well–formed words:

(2a) $(1, \triangleleft y) \to (1, 0)$ for $y \in \Gamma \backslash \{\$\}$
(2b) $(1, \triangleleft \$ y) \to (1, 0)$ for $y \in \Gamma \backslash \{\$\}$
(2c) $(1, x \triangleright) \to (1, 0)$ for $x \in \Gamma$

Main rules: Let $n = |w| + 2$

(3a) $(C, \triangleright A^n) \to (1, \triangleright q_0 w \triangleleft)$
(3b) $(C, B) \to (1, 0)$

Rules for shifting \$-symbols to the left:

(4a) $(1, a\beta\$\$) \to (1, a\$\beta)$ for $a \in \Sigma$, $\beta \in \Sigma \cup \{\triangleleft\}$
(4b) $(1, a\$\beta\$\$) \to (1, a\$\$\beta)$ for $a \in \Sigma$, $\beta \in \Sigma \cup \{\triangleleft\}$

Rules for simulating $\mathcal{M}$: Let $a, a', b \in \Sigma$, and $q \in Q \backslash \{q_f\}$, $p \in Q$.

(5a) $(1, q \triangleleft \$\$) \to (1, a'p\triangleleft)$ if $\delta(q, \Box) = (p, a', R)$
(5b) $(1, bq \triangleleft \$\$) \to (1, pba'\triangleleft)$ if $\delta(q, \Box) = (p, a', L)$
(5c) $(1, qa\$\$) \to (1, a'p)$ if $\delta(q, a) = (p, a', R)$
(5d) $(1, bqa\$\$) \to (1, pba')$ if $\delta(q, a) = (p, a', L)$
(5e) $(1, \triangleright qa\$\$) \to (1, \triangleright p\Box a')$ if $\delta(q, a) = (p, a', L)$

Figure 1: The TRS $\mathcal{R}_w$

Lemma 3.3. $\mathcal{R}_w$ is confluent if and only if $\mathcal{M}$ does not terminate on input w.

Proof. The following proof is a variant of the one presented in [NO88]. First assume that $\mathcal{M}$ terminates on input w. Then there exists an $m \geq 1$ such that

$$(1, \triangleright q_0 w \triangleleft \$^m B) \to^+_{\mathcal{R}_\mathcal{M}} (1, \triangleright u q_f a_1 \$^2 a_2 \$^2 \dots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B) = (1, t),$$

where $u \in \Sigma^*, l \geq 0, a_1, \dots, a_l \in \Sigma$ and $k \geq 2$. Since $\mathcal{M}$ cannot move from the final state q_f, the pair $(1, t)$ is irreducible. Thus we obtain

$$(1, t) \ {}_{\mathcal{R}_w}\!\leftarrow^+ (1, \triangleright q_0 w \triangleleft \$^m B) \ {}_{(3a)}\!\leftarrow (C, \triangleright A^n \$^m B) \to_{(3b)} (1, \triangleright A^n \$^m 0) \to^*_{(1a)} (1, 0).$$

Since the left–most pair and right–most pair are both irreducible, it follows that $\mathcal{R}_w$ is not confluent.

Now assume that $\mathcal{M}$ does not terminate on input w. By Lemma 3.1, $\mathcal{R}_w$ satisfies condition (A). Thus, we can apply Lemma 2.14 in order to prove that $\mathcal{R}_w$ is confluent. We have to consider all $(t_1, t, t_2) \in \mathit{Crit}(\mathcal{R}_w)$. By Lemma 3.2 the following cases may occur.

(1) Note that all critical situations of the form $(vC, r) \ {}_c\!\leftarrow (CvC, l) \to_c (Cv, r)$, where $c = ((C, l) \to (1, r)) \in \mathcal{R}_w$ and $v \in \{C\}^*$ are trivially confluent since $vC = Cv$.

(2) Note that if $(s, s_1 0 s_2) \to_{\mathcal{R}_w} (s', t')$ then also $t' = s_1' 0 s_2'$ for some $s_1', s_2' \in \Gamma^*$. Since every pair of the form $(s, s_1 0 s_2)$ can be reduced to $(1, 0)$, this shows that all critical situations that result from one of the rules (1) to (1c) and any other rule are confluent.

(3) Also the situations $(1, 0) \ {}_{(2a)}\!\leftarrow (1, \triangleleft\triangleright) \to_{(2c)} (1, 0)$ and $(1, 0) \ {}_{(2b)}\!\leftarrow (1, \triangleleft\$\triangleright) \to_{(2c)} (1, \triangleleft 0)$ are trivially confluent.

(4) Let $c = ((l, yl') \to (r, r')) \in \mathcal{R}_w$ with $y \in \Gamma \backslash \{\$\}$. Then $(l, 0l') \ {}_{(2a)}\!\leftarrow (l, \triangleleft yl') \to_c (r, \triangleleft r')$ is a critical situation. Since r' must be of the from zr'' for some $z \in \Gamma \backslash \{\$\}$ and $r'' \in \Gamma^*$, we obtain $(r, \triangleleft zr'') \to_{(2a)} (r, 0r'') \to^*_{\mathcal{R}_0} (1, 0) \ {}_{\mathcal{R}_0}\!\leftarrow^* (l, 0l')$. Critical situations that arise from $(2b)$ and any other rule can be dealt similarly.

(5) Let $c = ((l, l'x) \to (r, r')) \in \mathcal{R}_w$, where $x \in \Gamma$. Then $(l, l'0) \ {}_{(2c)}\!\leftarrow (l, l'x\triangleright) \to_c (r, r'\triangleright)$ is a critical situation which can be dealt analogously to the previous case.

(6) $(C, 0A^n)\ {}_{(2c)}\!\!\leftarrow\ (C, x \triangleright A^n) \rightarrow_{(3a)} (1, x \triangleright q_0 w \triangleleft)$:

We obtain $(1, x \triangleright q_0 w \triangleleft) \rightarrow_{(2c)} (1, 0q_0 w \triangleleft) \rightarrow^*_{\mathcal{R}_0} (1, 0)\ {}_{\mathcal{R}_0}\!\!\leftarrow^*\ (C, 0A^n)$.

(7) $(1, 0qa\$\$)\ {}_{(2c)}\!\!\leftarrow\ (1, x \triangleright qa\$\$) \rightarrow_{(5e)} (1, x \triangleright q'\Box a')$:

We obtain $(1, x \triangleright q'\Box a') \rightarrow_{(2c)} (1, 0q'\Box a') \rightarrow^*_{\mathcal{R}_0} (1, 0)\ {}_{\mathcal{R}_0}\!\!\leftarrow^*\ (1, 0qa\$\$)$. Now we have considered all critical situations that can result from one of the rules (2a) to (2c) and any other rule.

(8) $(1, \triangleright q_0 w \triangleleft v \triangleright A^n)\ {}_{(3a)}\!\!\leftarrow\ (C, \triangleright A^n v \triangleright A^n) \rightarrow_{(3a)} (1, \triangleright A^n v \triangleright q_0 w \triangleleft)$ for an arbitrary $v \in \Gamma^*$:

Since in both pairs, the second component contains a factor of the form $x\triangleright$, we can apply rule (2c) to both pairs. After this, both pairs can be reduced to $(1, 0)$ with the rules (1a) to (1c).

(9) $(1, Bv \triangleright q_0 w \triangleleft)\ {}_{(3a)}\!\!\leftarrow\ (C, Bv \triangleright A^n) \rightarrow_{(3b)} (1, 0v \triangleright A^n)$ for an arbitrary $v \in \Gamma^*$:

In the left pair, a factor of the form $x\triangleright$ appears in the second component. Thus, we can apply rule (2c) and we can reduce both pairs to $(1, 0)$ with the rules (1a) to (1c).

(10) $(1, \triangleright q_0 w \triangleleft vB)\ {}_{(3a)}\!\!\leftarrow\ (C, \triangleright A^n vB) \rightarrow_{(3b)} (1, \triangleright A^n v0)$ for an arbitrary $v \in \Gamma^*$:

This is the main case and will be considered later.

(11) $(1, 0vB)\ {}_{(3b)}\!\!\leftarrow\ (C, BvB) \rightarrow_{(3b)} (1, Bv0)$ for an arbitrary $v \in \Gamma^*$: Both pairs can be reduced to $(1, 0)$.

These are all critical situations that can arise. Note that since $\mathcal{M}$ is deterministic, the rules (5a) to (5e) do not produce any critical situations. Thus, it remains to consider the critical situation

$$(1, \triangleright q_0 w \triangleleft vB)\ {}_{(3a)}\!\!\leftarrow\ (C, \triangleright A^n vB) \rightarrow_{(3b)} (1, \triangleright A^n v0)$$

for an arbitrary $v \in \Gamma^*$. Of course, the right pair can be reduced to $(1, 0)$. The truth of the following claim proves the truth of our first claim.

Claim: If $\mathcal{M}$ does not terminate on input w then $(1, \triangleright q_0 w \triangleleft vB) \rightarrow^*_{\mathcal{R}_w} (1, 0)$ for every $v \in \Gamma^*$.

First consider the case $v = \$^m$ for $m \geq 0$. We obtain

$$(1, \triangleright q_0 w \triangleleft \$^m B) \rightarrow^*_{\mathcal{R}_{\mathcal{M}}} (1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}}B),$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $\alpha_1, \ldots, \alpha_{l+1} \in \{0, 1\}$. Thus,

$$\begin{array}{l}(1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}}B) \rightarrow_{\{(2a),(2b)\}} \\ (1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l}0) \rightarrow^*_{(1a)} (1, 0).\end{array}$$

Now assume $v = \$^m yv'$, where $m \geq 0, y \in \Gamma \backslash \{\$\}$ and $v' \in \Gamma^*$. Similarly to the derivation above, we obtain

$$(1, \triangleright q_0 w \triangleleft \$^m xv'B) \rightarrow^*_{\mathcal{R}_{\mathcal{M}}} (1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}}yv'B),$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $\alpha_1, \ldots, \alpha_{l+1} \in \{0, 1\}$. Since $y \in \Gamma \backslash \{\$\}$ we obtain

$$\begin{array}{l}(1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}}yv'B) \rightarrow_{\{(2a),(2b)\}} \\ (1, \triangleright uqa_1\$^{\alpha_1}a_2\$^{\alpha_2} \ldots a_{l-1}\$^{\alpha_{l-1}}a_l\$^{\alpha_l}0v'B) \rightarrow^*_{\{(1a),(1b)\}} (1, 0),\end{array}$$

which concludes the proof. □

From the previous lemma it follows that $\mathit{CONFL}_{\neq 1}(\{C\}^* \times \Gamma^*)$ is undecidable. The following lemma strengthens this result.

Lemma 3.4. $\mathit{CONFL}_{\neq 1}(\{c\}^* \times \{a, b\}^*)$ is undecidable.

Proof. Consider the TRS $\mathcal{R}_w$ from the previous proof. We will use the coding ϕ from Lemma 2.17, where $m = 1$, $a_1 = C$, and $\{b_1, \ldots, b_n\} = \Gamma$. The morphism ϕ induces a morphism $\sigma : \{C\}^* \times \Gamma^* \to \{C\}^* \times \{b_1, b_2\}^*$ in the obvious way. But note that the fourth condition of Lemma 2.16 is not satisfied for the TRS $\sigma(\mathcal{R}_w)$. For instance $(C, \phi(B)s\phi(B)) \in CT(\sigma(\mathcal{R}_w))$ but if s does not belong to the range of σ then the same holds for the pair $(C, \phi(B)s\phi(B))$. We solve this problem by adding some rules to $\sigma(\mathcal{R}_w)$. Let $\mathcal{P}_w$ be the TRS $\sigma(\mathcal{R}_w)$ plus the following rules (note that each word $\phi(x)$ for $x \in \Gamma$ has the length $n + 5$):

(6a) $(1, x\phi(0)) \to (1, \phi(0))$ for $x \in \{b_1, b_2\}$

(6b) $(1, \phi(0)x) \to (1, \phi(0))$ for $x \in \{b_1, b_2\}$

(6c) $(1, x\phi(\triangleright)) \to (1, \phi(0))$ for $x \in \{b_1, b_2\}$

(6d) $(1, \phi(\triangleleft)s) \to (1, \phi(0))$ for all $s \in \{b_1, b_2\}^{n+5} \backslash \{\phi(x) \mid x \in \Gamma\}$

(6e) $(1, \phi(\triangleleft\$)s) \to (1, \phi(0))$ for all $s \in \{b_1, b_2\}^{n+5} \backslash \{\phi(x) \mid x \in \Gamma\}$

Then $\mathcal{P}_w$ is a length–reducing TRS with $1 \notin \mathit{ran}(\mathcal{P}_w)$ that satisfies condition (A). The following claim proves the Lemma:

Claim: $\mathcal{P}_w$ is confluent if and only if $\mathcal{M}$ does not terminate on input w.

First assume that $\mathcal{P}_w$ is confluent. Assume that $\mathcal{M}$ terminates on input x. Then for some $m \geq 1$

$$(1, \triangleright u q_f a_1 \$^2 a_2 \$^2 \ldots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B) \ {}_{\mathcal{R}_w}\!\leftarrow^+ \ (C, \triangleright A^n \$^m B) \to^+_{\mathcal{R}_w} (1, 0).$$

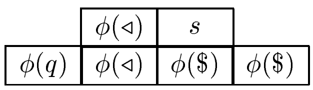
Application of the morphism σ gives

$$(1, \phi(\triangleright u q_f a_1 \$^2 a_2 \$^2 \ldots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B)) \ {}_{\mathcal{P}_w}\!\leftarrow^+ \ (C, \phi(\triangleright A^n \$^m B)) \to^+_{\mathcal{P}_w} (1, \phi(0)).$$

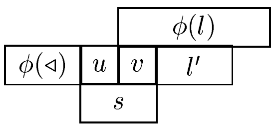
It is easy to see that the left and right pair in this derivation are irreducible with respect to $\mathcal{P}_w$. Thus $\mathcal{P}_w$ is not confluent which is a contradiction.

Now assume that $\mathcal{M}$ does not terminate on input w. By Lemma 3.3, $\mathcal{R}_w$ is confluent. It suffices to consider an arbitrary $(t_1, t, t_2) \in \mathit{Crit}(\mathcal{P}_w)$. The case that (t_1, t, t_2) results from two rules of $\mathcal{P}_w \backslash \sigma(\mathcal{R}_w)$ is clear, because in this case t_1 and t_2 both contain the factor $\phi(0)$. Thus t_1 and t_2 can both be reduced to $\phi(0)$ with rule $(6a)$, $(6b)$, and $\sigma(1c)$.

Next assume that (t_1, t, t_2) results from a rule $c \in \mathcal{P}_w \backslash \sigma(\mathcal{R}_w)$ and a rule from $\sigma(\mathcal{R}_w)$. Let us consider the case that c is the rule $(6d)$ (the other cases can be considered similarly). Let $(x, \phi(l)) \to (1, \phi(r))$ be a rule in $\sigma(\mathcal{R}_w)$, where $x \in \{C, 1\}$. Thus t must be of the form (x, t'), where t' is an overlapping of $\phi(l)$ and $\phi(\triangleleft)s$, where $|s| = n + 5$ and $s \notin \{\phi(x) \mid x \in \Gamma\}$. The case $t' = \phi(l)$ can be excluded since this would imply that the word s is of the form $\phi(x)$ for some $x \in \Gamma$, see the following diagram, where $l = q \triangleleft \$\$$ is the left–hand side of rule (5a):



Because of Lemma 2.17 the only possible overlapping of $\phi(\triangleleft)s$ and $\phi(l)$ is of the form $\phi(\triangleleft)u\phi(l)$ where $s = uv$, $\phi(l) = vl'$, and $u \neq 1 \neq v$, see the following diagram:



Thus $t = (x, \phi(\triangleleft)u\phi(l)) = (x, \phi(\triangleleft)sl')$, $t_1 = (1, \phi(\triangleleft)u\phi(r))$, and $t_2 = (1, \phi(0)l')$. Thus $t_2 \to^*_{(6b)} (1, \phi(0))$. We claim that the prefix of $u\phi(r)$ of length $n + 5$ (which exists) does not belong to $\{\phi(x) \mid x \in \Gamma\}$, which implies $t_1 = (1, \phi(\triangleleft)u\phi(r)) \to_{(6d)} \to^*_{(6b)} (1, \phi(0))$. But this is clear, because otherwise we would obtain an overlapping between $\phi(r)$ and some $\phi(x)$ for $x \in \Gamma$ of the following kind, where $r = yr'$, $y \in \Gamma$:

u	$\phi(y)$	$\phi(r')$
$\phi(x)$		

Finally assume that (t_1, t, t_2) results from two rules in $\sigma(\mathcal{R}_w)$. The critical situations of this type precisely correspond to the critical situations of $\mathcal{R}_w$ that we have considered in the cases (1) to (10) in the proof of Lemma 3.3. Apart from cases that correspond to the cases (7) to (10) it holds $t = \sigma(s)$, $t_1 = \sigma(s_1)$, $t_2 = \sigma(s_2)$ for some s, t_1 and t_2, which follows from Lemma 2.17. Since $\mathcal{R}_w$ is confluent it follows $s_i \to^*_{\mathcal{R}_w} u$ ($i \in \{1,2\}$) for some u and therefore $\sigma(s_i) \to^*_{\mathcal{P}_w} \sigma(u)$ Thus it remains to consider the four cases that correspond to cases (7) to (10) in the proof of Lemma 3.3. The cases (7) to (9) can be dealt analogously to the corresponding cases from the proof of Lemma 3.3 using the new rules (6a), (6b), and (6c). Thus it remains to consider the critical pair $(t_1, t_2) = ((1, \phi(\triangleright q_0 w \triangleleft) v \phi(B)), (1, \sigma(\triangleright A^n) v \phi(0)))$. where $v \in \{b_1, b_2\}^*$. The case $v = \phi(v')$ for some $v' \in \Gamma$ is clear since in this case t_1 and t_2 belong to the range of σ. Thus assume that $v = \phi(v')s$ for some $v' \in \Gamma^*$ and some $s \in \{b_1, b_2\}^+$, where s can be assumed to have no prefix of the form $\phi(x)$ for some $x \in \Gamma$. The pair $(1, \sigma(\triangleright A^n) v \phi(0))$ can be reduced to $(1, \phi(0))$ with the rule (6a). We have to show that also $(1, \phi(\triangleright q_0 w \triangleleft) v \phi(B)) \to_{\mathcal{P}_w} (1, \phi(0))$ holds. If $v' = \$^m y v''$ for some $m \geq 0$, $y \neq \$$ and $v'' \in \Gamma^*$ we can again use the arguments from case 2 from the end of the proof of Lemma 3.3. Thus assume that $v' = \$^m$ for some $m \geq 0$, i.e., $v = \phi(\$)^m s$. We obtain

$$(1, \phi(\triangleright q_0 w \triangleleft \$^m) s \phi(B)) \to^*_{\mathcal{P}_w} (1, \phi(\triangleright u q a_1 \$^{\alpha_1} a_2 \$^{\alpha_2} \ldots a_{l-1} \$^{\alpha_{l-1}} a_l \$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}}) s \phi(B)),$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $\alpha_1, \ldots, \alpha_{l+1} \in \{0, 1\}$. Since $s \neq 1$ is not of the form $\phi(x)s'$ for some $x \in \Gamma$ and some $s' \in \Gamma^*$, the prefix of $s\phi(B)$ of length $n+5$ does not belong to $\{\phi(x) \mid x \in \Gamma\}$. Hence we can apply either rule ($6d$) or rule ($6e$). The resulting pair can be reduced to $(1, \phi(0))$ with the rules ($1c$), ($6a$), and ($6b$). $\square$

Note that the system $\mathcal{P}_w$ is not not length–increasing in both components, i.e., for all $(l_1, l_2) \to (r_1, r_2) \in \mathcal{P}_w$, $|l_1| \geq |r_1|$ and $|l_2| \geq |r_2|$. Together with Corollary 5.1 of Section 4 this gives a very sharp borderline between decidability and undecidability for the case of a direct product of free monoids.

3.1.2 The case $a - c \quad b$

In this section we will consider trace monoids of the following form.

Definition 3.5. Define the independence alphabet (Σ_n, I_n) by $(\{a, c, b_1, \ldots, b_n\}, \{(a,c),(c,a)\})$, where $n \geq 1$. Let $M_n = \mathbb{M}(\Sigma_n, I_n)$.

The aim of this section is to show that $\mathit{CONFL}_{\neq 1}(M_1)$ is undecidable. This result will be proven in three steps. In a first step we will show that there exists an $n \geq 2$ such that (a stricter version of) $\mathit{CONFL}_{\neq 1}(M_n)$ is undecidable (Lemma 3.7). In a second step the generators $b_1, \ldots, b_n$ of M_n will be encoded into two generators b_1 and b_2 (Lemma 3.8), which proves the undecidability of $\mathit{CONFL}_{\neq 1}(M_2)$. In a last step we show how to encode the two generators b_1 and b_2 into a single generator b_1 with the help of the two commuting letters a and c (Lemma 3.9). This two–step encoding makes the proof easier to follow. In order to make these codings possible we have to start in Lemma 3.7 with a stricter version of $\mathit{CONFL}_{\neq 1}(M_n)$. In both coding–steps we use Lemma 2.16 together with the next lemma that applies to traces which satisfy the following condition (B).

> A TRS $\mathcal{R}$ over M_n ($n \geq 1$) satisfies condition (B) if for every $l \in \mathit{dom}(\mathcal{R})$ it holds $\{b_1, \ldots, b_n\} \cap \mathit{alph}(l) \neq \emptyset$.

Obviously, a TRS $\mathcal{R}$ that satisfies condition (B) also satisfies condition (A), since for $l_1, l_2 \in \mathit{dom}(\mathcal{R})$ their cannot exist factorizations $l_1 = p_1 q_1$, $l_2 = p_2 q_2$ with $p_i \neq 1 \neq q_i$ for $i \in \{1,2\}$ and $p_1 \, I \, p_2$, $q_1 \, I \, q_2$ (since either p_1 or q_1 contains a letter from $\{b_1, \ldots, b_n\}$).

Lemma 3.6. Let $\mathcal{R}$ be a TRS over M_n for some $n > 0$ that satisfies condition (B). If $(t_1, t, t_2) \in$ *Crit*$(\mathcal{R})$ then there exist rules $(l_1 \to r_1), (l_2 \to r_2) \in \mathcal{R}$ such that one of the following six cases holds, where in each case $s \neq 1$, $\alpha, \beta, \gamma, \delta > 0$, $\{x, \underline{x}\} = \{y, \underline{y}\} = \{a, c\}$, and $p_1, p_2, q_1, q_2 \in M$.

(1) $l_1 = x^\alpha s y^\gamma, l_2 = \underline{x}^\beta s \underline{y}^\delta$, and $t = \underline{x}^\beta x^\alpha s y^\gamma \underline{y}^\delta = x^\alpha \underline{x}^\beta s \underline{y}^\delta y^\gamma$, $t_1 = \underline{x}^\beta r_1 \underline{y}^\delta$, $t_2 = x^\alpha r_2 y^\gamma$

(2) $l_1 = x^\alpha s, l_2 = \underline{x}^\beta s q_2$, and $t = \underline{x}^\beta x^\alpha s q_2 = x^\alpha \underline{x}^\beta s q_2$, $t_1 = \underline{x}^\beta r_1 q_2$, $t_2 = x^\alpha r_2$

(3) $l_1 = s x^\alpha, l_2 = p_2 s \underline{x}^\beta$, and $t = p_2 s x^\alpha \underline{x}^\beta = p_2 s \underline{x}^\beta x^\alpha$, $t_1 = p_2 r_1 \underline{x}^\beta$, $t_2 = r_2 x^\alpha$

(4) $l_1 = s q_1, l_2 = p_2 s$, and $t = p_2 s q_1$, $t_1 = p_2 r_1$, $t_2 = r_2 q_1$

(5) $l_1 = s$, $l_2 = p_2 s q_2$, and $t = l_2 = p_2 l_1 q_2$, $t_1 = p_2 r_1 q_2$, $t_2 = r_2$

(6) $l_1 = p_1 x^\alpha, l_2 = x^\alpha q_2$, and $t = p_1 x^\alpha \underline{x}^\beta q_2 = p_1 \underline{x}^\beta x^\alpha q_2$, $t_1 = r_1 \underline{x}^\beta q_2$, $t_2 = p_1 \underline{x}^\beta r_2$

Note that the cases (4) and (6) are not exclusive.

Proof. Let $(t_1, t, t_2) \in$ *Crit*$(\mathcal{R})$. Thus, there exist rules $(l_1 \to r_1), (l_2 \to r_2) \in \mathcal{R}$ such that

$$t = p_2 w_2 l_1 w_1 q_2 = p_1 w_1 l_2 w_2 q_1,\ \ t_1 = p_2 w_2 r_1 w_1 q_2,\ \ t_2 = p_1 w_1 r_2 w_2 q_1,$$

and

$$l_1 = p_1 s q_1,\ \ l_2 = p_2 s q_2,\ \ s \neq 1,\ \ p_1\, I\, p_2,\ \ q_1\, I\, q_2,\ \ w_1\, I\, w_2,\ \ s\, I\, w_1 w_2,\ \ w_1\, I\, q_1 p_2,\ \ w_2\, I\, p_1 q_2$$

We separate the following cases.
case 1: $w_1 = 1 = w_2$: Thus $t = p_2 l_1 q_2 = p_1 l_2 q_1$, $t_1 = p_2 r_1 q_2$, $t_2 = p_1 r_2 q_1$.
case 1.1: $p_1 \neq 1 \neq p_2$: Since $p_1\ I_n\, p_2$, $p_1 = x^\alpha$ and $p_2 = \underline{x}^\beta$ for some $\alpha, \beta > 0$.
case 1.1.1: $q_1 \neq 1 \neq q_2$: Thus, $q_1 = y^\gamma$ and $q_2 = \underline{y}^\delta$ for some $\gamma, \delta > 0$ and we obtain type (1).
case 1.1.2: $q_1 = 1$: We obtain type (2). The case $q_2 = 1$ is symmetric.
case 1.2: $p_1 = 1$ (the case $p_2 = 1$ will be symmetric).
case 1.2.1: $q_1 \neq 1 \neq q_2$: We obtain type (3).
case 1.2.2: $q_2 = 1$: We obtain type (4).
case 1.2.3: $q_1 = 1$: We obtain type (5).
case 2: $w_1 \neq 1$: Since $s \neq 1$ and $s I w_1$ it follows $s = x^\alpha$, $w_1 = \underline{x}^\beta$ for some $\alpha, \beta > 0$. But $s w_1 I w_2$ then implies $w_2 = 1$. Finally, we claim that also $p_2 = q_1 = 1$. Assume $p_2 \neq 1$. Since $w_1 I p_2$ and $w_1 = \underline{x}^\beta$ it follows $p_2 = x^\gamma$ for some $\gamma > 0$. Thus, $l_2 = p_2 s q_2 = x^\gamma x^\alpha q_2$. Since l_2 must contain a letter from $\{b_1, \ldots, b_n\}$ it follows $q_2 \neq 1$. Similarly either p_1 or q_1 must contain a letter from $\{b_1, \ldots, b_n\}$. But the first alternative contradicts $p_1 I p_2 \neq 1$, whereas the second alternative contradicts $q_1 I q_2 \neq 1$. Analogously, we can show that $q_1 = 1$. We obtain type (6). The case $w_2 \neq 1$ is symmetric. □

Lemma 3.7. There exists an $n \geq 1$ such that the following problem is undecidable.
INPUT: A length–reducing TRS $\mathcal{R}$ over M_n that satisfies condition (B) such that for all $(l \to r) \in \mathcal{R}$ it holds $\mathit{max}(l) \subseteq \{a, c\}$ and $r \neq 1$.
QUESTION: Is $\mathcal{R}$ confluent ?

Proof. Let $\mathcal{M}$ be the same Turing machine that we used in the proof of Lemma 3.3. Let $(\Sigma_n, I_n) = (Q \cup \Sigma \cup \{0, \triangleright, \triangleleft, A, B, C, \$\}, \{(\$, B), (B, \$)\})$. Given a word $w \in (\Sigma \backslash \{\Box\})^+$, we define a TRS $\mathcal{R}_w$ over M_n by the rules of Figure 2, where the value $\omega \geq 2$ will be specified later. Note that $\mathcal{R}_w$ is length–reducing and that all right–hand sides are non empty. Since we excluded the case $w = 1$, we do not have to consider the trace $\triangleright q \triangleleft \$\$$ in the last group of rules. Note that $\mathcal{R}_w$ satisfies the additional conditions required in the lemma. Let $\mathcal{R}_{\mathcal{M}}$ be the system that consists of the rules (4) and (5a) to (5e).
Claim: $\mathcal{R}_w$ is confluent if and only if $\mathcal{M}$ does not terminate on input w.

Rules for the absorbing symbol 0:
(1) $x0\$ \to 0\$$ for $x \in \Sigma_n$

Rules for deleting non well–formed words:
(2) $\triangleleft \$^k C\$ \to 0\$$ for $k \in \{0, \ldots, \omega - 1\}$

Main rules: Let $n = |w| + 3$.
(3a) $A^n B \to \triangleright q_0 w \triangleleft$
(3b) $BC\$ \to 0\$$

Rules for shifting \$-symbols to the left:
(4) $a\$^k \beta \$^\omega \to a\$^{k+1}\beta$ for $k \in \{0, \ldots, \omega - 1\}$, $a \in \Sigma$, and $\beta \in \Sigma \cup \{\triangleleft\}$

Rules for simulating $\mathcal{M}$: Let $a' \in \Sigma \backslash \{\Box\}$, $a, b \in \Sigma$, and $q \in Q \backslash \{q_f\}$, $p \in Q$.
(5a) $q \triangleleft \$^\omega \to pq'\triangleleft$ if $\delta(q, \Box) = (p, a', R)$
(5b) $bq \triangleleft \$^\omega \to pba'\triangleleft$ if $\delta(q, \Box) = (p, a', L)$
(5c) $qa\$^\omega \to a'p$ if $\delta(q, a) = (p, a', R)$
(5d) $bqa\$^\omega \to pba'$ if $\delta(q, a) = (p, a', L)$
(5e) $\triangleright qa\$^\omega \to \triangleright p\Box a'$ if $\delta(q, a) = (p, a', L)$

Figure 2: The TRS $\mathcal{R}_w$

First assume that $\mathcal{M}$ terminates on input w. Then there exists an $m \geq 0$ such that

$$\triangleright q_0 w \triangleleft \$^m C\$ \to^+_{\mathcal{R}_\mathcal{M}} \triangleright u q_f a_1 \$^\omega a_2 \$^\omega \ldots a_{l-1}\$^\omega a_l \$^\omega \triangleleft \$^k C\$ = u',$$

where $u \in \Sigma^*, l \geq 0, a_1, \ldots, a_l \in \Sigma$ and $k \geq \omega$. Since $\mathcal{M}$ cannot move out of the final state q_f, the trace u' is irreducible. Thus, we obtain

$$u' \;{}_{\mathcal{R}_w}\!\!\leftarrow^+ \triangleright q_0 w \triangleleft \$^m C\$ \;{}_{(3a)}\!\!\leftarrow A^n B \$^m C\$ = A^n \$^m BC\$ \to_{(3b)} A^n \$^m 0\$ \to^+_{(1)} 0\$.$$

Since the left– and right–most trace are both irreducible it follows that $\mathcal{R}_w$ is not confluent.

Now assume that $\mathcal{M}$ does not terminate on input w. In order to prove that $\mathcal{R}_w$ is confluent, we can apply Lemma 2.14 since $\mathcal{R}_w$ satisfies condition (B) and hence also condition (A). Thus, we have to consider all critical situations $(t_1, t, t_2) \in \mathit{Crit}(\mathcal{R})$. By Lemma 3.6, t, t_1, and t_2 have one of the six forms, enumerated in Lemma 3.6. There are only the following three cases (note that since $\mathcal{M}$ is deterministic, the rules in group (5) do not produce any critical situations).

- $B0\$ \;{}_{(1)}\!\!\leftarrow B\$0\$ = \$B0\$ \to_{(1)} \$0\$$ is a critical situation of type (2) following the enumeration in Lemma 3.6. But the left and right trace can both be reduced to $0\$$.
- Let $\{x, y\} = \{B, \$\}$ and $c = (l'x \to r) \in \mathcal{R}$ (all rules of $\mathcal{R}_w$ are of this form). Then for every $m \geq 0$,

  $$l'y^m 0\$ \;{}_{(1)}\!\!\leftarrow l'y^m x0\$ = l'xy^m 0\$ \to_c ry^m 0\$$$

  is a critical situation, where $l'y^m x0$ is of type (6) following the enumeration in Lemma 3.6. But the left and right trace can both be reduced to $0\$$ with rule (1).
- The last type of critical situation arises from the two main rules (3a) and (3b), namely

  $$\triangleright q_0 w \triangleleft \$^m C\$ \;{}_{(3a)}\!\!\leftarrow A^n B\$^m C\$ = A^n \$^m BC\$ \to_{(3b)} A^n \$^m 0\$,$$

  where $m \geq 0$ is arbitrary. Again $A^n B\$^m C$ is of type (6). The right trace can be reduced to $0\$$. Furthermore since $\mathcal{M}$ does not halt on input w, for every $m \geq 0$ it holds

  $$\triangleright q_0 w \triangleleft \$^m C\$ \to^*_{\mathcal{R}_\mathcal{M}} \triangleright uqv_1 \$^{\alpha_1} v_2 \$^{\alpha_2} \ldots v_{l-1} \$^{\alpha_{l-1}} v_l \$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}} C\$,$$

  where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $v_1, \ldots, v_l \in \Sigma$, and $\alpha_1, \ldots, \alpha_{l+1} \in \{0, \ldots, \omega - 1\}$. Thus,

  $$\triangleright uqv_1 \$^{\alpha_1} v_2 \$^{\alpha_2} \ldots v_{l-1} \$^{\alpha_{l-1}} v_l \$^{\alpha_l} \triangleleft \$^{\alpha_{l+1}} C\$ \to_{(2)} \triangleright uqv_1 \$^{\alpha_1} v_2 \$^{\alpha_2} \ldots v_{l-1} \$^{\alpha_{l-1}} v_l \$^{\alpha_l} 0\$ \to^*_{(1)} 0\$.$$

□

The undecidability of $CONFL(M_2)$ is also stated in [Ber96]. The following lemma is a strengthening of this result.

Lemma 3.8. The following problem is undecidable.
INPUT: A length–reducing TRS $\mathcal{R}$ over M_2 that satisfies condition (B) such that for all $(l \to r) \in \mathcal{R}$ it holds $max(l) \subseteq \{a, c\}$ and $r \neq 1$.
QUESTION: Is $\mathcal{R}$ confluent ?

Proof. In this (and the next) proof it is important to distinguish traces from words. Thus, all traces are written in the form $[s]$ for a word s. Let n be the number whose existence is stated in Lemma 3.7. Let $\phi : \Sigma_n^* \to \Sigma_2^*$ be the injective morphism from Lemma 2.17 where $\{a_1, \ldots, a_m\} = \{a, c\}$, i.e.,

$$\phi(a) = a, \quad \phi(c) = c, \quad \phi(b_i) = b_1 b_2 b_1^{i+1} b_2^{n-i+2} \text{ for } i \in \{1, \ldots, n\}.$$

Obviously ϕ defines a trace morphism $\sigma : M_n \to M_2$ by $\sigma([s]) = [\phi(s)]$. In the following let $\mathcal{R} = \mathcal{R}_w$ be a TRS from the proof of Lemma 3.7. Then $\sigma(\mathcal{R})$ satisfies also condition (B) and furthermore for every $(l \to r) \in \mathcal{R}$ it holds $max(l) \subseteq \{a, c\}$ and $r \neq 1$. Furthermore if we choose the value $\omega \geq 2$ in the proof of Lemma 3.7 big enough, the system $\sigma(\mathcal{R})$ will be also length–reducing. The rules in group (1) to (4) will be length–reducing independently of the value of ω since for each of these rules $l \to r$ it holds $|\pi_\Gamma(l)| \geq |\pi_\Gamma(r)|$ where Γ is the alphabet $\{b_1, \ldots, b_n\} = Q \cup \Sigma \cup \{0, \triangleright, \triangleleft, A, C\}$. But this does not hold for the rules in group (5). By Lemma 3.7 the following claim proves the lemma.
Claim: $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent.

We can apply Lemma 2.16 in order to prove the claim. Thus, we have to show that the four conditions of Lemma 2.16 hold. Injectivity of σ is easy to prove. In order to prove the third conditions of Lemma 2.16, we will prove the following more general statement for all $s_1, s_2 \in \Sigma_2^*$ and $s, l \in \Sigma_n^+$.

$$\text{If } [\phi(s)] = [s_1][\phi(l)][s_2] = [s_1 \phi(l) s_2] \text{ then } s_1 = \phi(s_1'),\ s_2 = \phi(s_2') \text{ for some } s_1', s_2' \in \Sigma_n^*. \quad (2)$$

Note that (2) need not hold for $l = 1$. Furthermore note the following simple fact.

$$\text{If } [\phi(s)] = [t] \text{ then there exists a } s' \in \Sigma_n^* \text{ such that } [s'] = [s] \text{ and } t = \phi(s'). \quad (3)$$

Assume that $[\phi(s)] = [s_1 \phi(l) s_2]$ Because of (3), we may assume that $\phi(s) = s_1 \phi(l) s_2$. But then Lemma 2.17 (for $\{a_1, \ldots, a_m\} = \{a, c\}$) implies $s_1 = \phi(s_1')$ and $s_2 = \phi(s_2')$ for some $s_1', s_2' \in \Sigma_n^*$.

It remains to prove the fourth condition of Lemma 2.16. Thus, assume that $[t] \in CT(\sigma(\mathcal{R}))$. We have to show that there exists a $t' \in \Sigma_n^*$ with $[\phi(t')] = [t]$. Since $\sigma(\mathcal{R})$ satisfies condition (B), we can apply Lemma 3.6 and it suffices to consider the six types for $[t]$ that we have enumerated in Lemma 3.6. The first three types and type (5) are easy, because for these types we have $[t] = [x^\alpha \phi(l_k) y^\gamma] = [\phi(x^\alpha l_k y^\gamma)]$ for some $x, y \in \{a, c\}$, $k \in \{1, 2\}$, and $\alpha, \beta \geq 0$.

Assume that $[t]$ is of type (4), i.e., $[t] = [psq]$, where $[\phi(l_1)] = [ps]$ and $[\phi(l_2)] = [sq]$. By (3), we may assume that $ps = \phi(l_1)$ and $sq = \phi(l_2)$. But then Lemma 2.17 implies $psq = \phi(s')$ for some $s' \in \Sigma_n^*$.

Finally assume that $[t]$ is of type (6), i.e., $[t] = [ux^\alpha y^\beta v]$, where $\{x, y\} = \{a, c\}$, $[\phi(l_1)] = [ux^\alpha]$, and $[\phi(l_2)] = [x^\alpha v]$. By (3), we may assume that $\phi(l_1) = ux^\alpha$ and $\phi(l_2) = x^\alpha v$. This implies $u = \phi(u')$ and $v = \phi(v')$ for some $u', v' \in \Sigma_n^*$ and therefore $[t] = [\phi(u') x^\alpha y^\beta \phi(v')] = [\phi(u' x^\alpha y^\beta v')]$. □

Lemma 3.9. $CONFL_{\neq 1}(M_1)$ is undecidable.

Proof. The proof follows the line of the proof of Lemma 3.8. In the following we denote the single additional generator b_1 of M_1 by b. We define an injective morphism $\phi : \Sigma_2^* \to \Sigma_1^*$ by

$$\phi(a) = a, \quad \phi(c) = c, \quad \phi(b_1) = bbab, \quad \phi(b_2) = bbcb.$$

ϕ defines an injective trace morphism $\sigma : M_2 \to M_1$ by $\sigma([s]) = [\phi(s)]$. In the following let $\mathcal{R}$ be a TRS from the proof of Lemma 3.8. Thus $\sigma(\mathcal{R})$ satisfies also condition (B) and furthermore for every $(l \to r) \in \sigma(\mathcal{R})$ it holds $max(l) \subseteq \{a, c\}$ and $r \neq 1$. Furthermore if we start in the proof of Lemma 3.7 with a value $\omega \geq 2$ that is big enough, the system $\sigma(\mathcal{R})$ will be also length–reducing. By Lemma 3.8 the following claim proves the lemma.
Claim: $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent.

We can apply Lemma 2.16 in order to prove the claim. Thus, it remains to show the second and third conditions of Lemma 2.16. Again the following property holds.

$$\text{If } [\phi(s)] = [t] \text{ then there exists an } s' \in \Sigma_2^* \text{ such that } [s'] = [s] \text{ and } t = \phi(s') \tag{4}$$

The second condition of Lemma 2.16, namely

$$\text{If } [l] \in dom(\mathcal{R}) \text{ and } [\phi(s)] = [s_1\phi(l)s_2] \text{ then } s_1 = \phi(s_1'),\ s_2 = \phi(s_2') \text{ for some } s_1', s_2' \in \Sigma_2^*, \tag{5}$$

will be proven by the same strategy that we used in the proof of Lemma 2.17. Let $[l] \in dom(\mathcal{R})$ and $[\phi(s)] = [s_1][\phi(l)][s_2] = [s_1\phi(l)s_2]$. Note that $|\phi(l)| \geq 2$ since $l \notin \{a, c\}$. Because of (4), we can assume that $\phi(s) = s_1\phi(l)s_2$. Choose the factorization $s_1 = \phi(u)t$ with u maximal (which exists since $s_1 = \phi(1)s_1$). Since $\phi(l) = s_1\phi(l)s_2 = \phi(u)t\phi(l)s_2$ it follows $t\phi(l)s_2 = \phi(v)$ for some $v \in \Sigma_2^+$. We claim that $t = 1$ which implies $s_1 = \phi(u)$. Assume that $t \neq 1$. If $v = a\cdots$ or $v = c\cdots$ then also $t = a\cdots$ or $t = c\cdots$ which contradicts the maximality of u. Thus $v = b_1\ldots$ or $v = b_2\cdots$. Assume w.l.o.g that $v = b_1w$, i.e., $t\phi(l)s_2 = bbab\phi(w)$ Thus t is a proper prefix of the word $bbab$. The case $t = b$ can be excluded since otherwise $\phi(l) = ba\cdots$ which is not possible. If $t = bb$ then $\phi(l)s_2 = ab\phi(w)$. Since $\phi(l) = ab$ is not possible we must have $w \neq 1$. If $w = a\cdots$ or $w = c\cdots$ then $\phi(l) = aba\cdots$ or $\phi(l) = abc\cdots$ which is not possible. If $w = b_1\cdots$ or $w = b_2\cdots$ then $\phi(l) = abbb\cdots$ which is also impossible.

Now we prove that $s_2 = \phi(s_2')$ for some $s_2' \in \Sigma_2^*$. Choose the factorization $s_2 = t\phi(u)$ with u maximal. Since $\phi(l) = s_1\phi(l)s_2 = s_1\phi(l)t\phi(u)$ it follows $s_1\phi(l)t = \phi(v)$ for some $v \in \Sigma_2^+$. We claim that $t = 1$ which implies $s_2 = \phi(u)$. Assume that $t \neq 1$. If $v = \cdots a$ or $v = \cdots c$ then $t = \cdots a$ or $t = \cdots c$ which contradicts the maximality of u. Thus $v = \cdots b_1$ or $v = \cdots b_2$. Assume w.l.o.g that $v = wb_1$, i.e., $s_1\phi(l)t = \phi(w)bbab$. Thus t is a proper suffix of $bbab$. The cases $t = b$ and $t = ab$ can be excluded because otherwise bb or bba would be a suffix of $\phi(l)$ which is not possible (note that $|\phi(l) \geq 2$). Thus $t = bab$, i.e., $s_1\phi(l) = \phi(w)b$. Since $\phi(l) \neq b$ but bb cannot be a suffix of $\phi(l)$ we must have either $w = \cdots a$ or $w = \cdots c$. Assume w.l.o.g. that $w = w_1a$. Thus $s_1\phi(l) = \phi(w_1)ab$. Again, since $\phi(l) \neq ab$ but aab or cab cannot be a suffix of $\phi(l)$ we must have $w_1 = \cdots b_i$ for $i = 1$ or $i = 2$. If $w_1 = \cdots b_1$ we obtain $s_1\phi(l) = \cdots bbabab$, i.e., $abab$ is a suffix of $\phi(l)$ which is not possible. The case $w_1 = \cdots b_2$ can be excluded similarly.

It remains to prove the third conditions of Lemma 2.16. Let $[t] \in CT(\sigma(\mathcal{R}))$. We have to show that there exists a $t' \in \Sigma_2^*$ with $[\phi(t')] = [t]$. Since $\sigma(\mathcal{R})$ satisfies condition (B), we can apply Lemma 3.6 and it suffices to consider the six types for $[t]$ that we have enumerated in Lemma 3.6. The first three types and type (5) can be dealt by the same arguments that we have applied in the proof of Lemma 3.8. Assume that $[t]$ is of type (4), i.e., $[t] = [s_1ss_2]$, where $[\phi(l_1)] = [s_1s]$ and $[\phi(2)] = [ss_2]$. By (4), we can assume that $\phi(l_1) = s_1s$ and $\phi(l_2) = ss_2$. Again it suffices to show that $s = \phi(s')$ for some $s' \in \Sigma_2^*$.

Choose the factorization $s = \phi(u)v$ with u maximal. Thus $ss_2 = \phi(u)vs_2 = \phi(l_2)$ and $vs_2 = \phi(w)$ for some $w \in \Sigma_n^*$. We claim that $v = 1$. Assume that $v \neq 1$ and thus also $w \neq 1$. If $w = a\cdots$ or $w = c\cdots$ then $v = a\cdots$ or $v = c\cdots$ which contradicts the maximality of u. Thus $w = b_1\ldots$ or $w = b_2\ldots$. Assume w.l.o.g. that $w = b_1\cdots$. Thus $vs_2 = bbab\ldots$. Hence v must be a proper prefix of $bbab$. If $v = bba$ then $\phi(l_1) = s_1s = s_1\phi(u)v = \ldots bba$ which is not possible. But if $v = b$ or $v = bb$ then similarly $\phi(l_1) = \cdots b$. But this gives a contradiction since $max([l_1]) \subseteq \{a, c\}$ implies $\phi(l_1) = \ldots a$ or $\phi(l_2) = \ldots c$. It should be noted that this the only point where we need the fact that $max([l]) \subseteq \{a, c\}$ for all left–hand sides $[l]$. Finally, type (6) for $[t]$ can be dealt analogously to the corresponding part of the proof of Lemma 3.8. $\square$

3.2 The general case

A confluent semi–Thue system is also confluent if we add an additional letter (that does not appear in the rules) to the alphabet. This trivial fact becomes wrong for TRSs. The following example is taken from [Die90], pp. 125.

Example 3.10. Consider the trace monoid M that is generated by the independence relation

$$a - c - f - b - d$$

and let $N \subset M$ the trace monoid that is generated by the independence relation

$$a - c \qquad b - d$$

Let $\mathcal{R} = \{ab \to c,\ cd \to a\}$. If we consider $\mathcal{R}$ over the submonoid N of M then $\mathcal{R}$ is confluent (which follows from [Die90], note the $\mathcal{R}$ does not satisfy condition (A) and thus we cannot apply Lemma 2.14). But if we consider $\mathcal{R}$ as a TRS over M then $\mathcal{R}$ is no longer confluent. To see this, consider the trace $[cabfd] = [afcdb]$, which can be rewritten to $[ccfd] = [cfcd]$ and $[afab]$. The only trace that can be derived from $[cfcd]$ is $[cfa]$, whereas the only trace that can be derived from $[afab]$ is $[afc]$. But since a and f are dependent it holds $[cfa] \neq [afc]$.

Thus, the following lemma is not a triviality.

Lemma 3.11. Let (Γ, I) be an independence alphabet and let $\Sigma \subseteq \Gamma$. Let $M = \mathbb{M}(\Sigma, I \cap \Sigma \times \Sigma) \subseteq \mathbb{M}(\Gamma, I) = N$. If $\mathit{CONFL}_{\neq 1}(N)$ is decidable then $\mathit{CONFL}_{\neq 1}(M)$ is also decidable.

Proof. Given a length–reducing TRS $\mathcal{R}$ over M such that $1 \notin \mathit{ran}(\mathcal{R})$, we will construct a length–reducing TRS $\mathcal{P}$ over N such that $1 \notin \mathit{ran}(\mathcal{P})$ and $\mathcal{R}$ is confluent if and only if $\mathcal{P}$ is confluent. The case $\Sigma = \Gamma$ is trivial. Thus, let us assume that there exists a $0 \in \Gamma \backslash \Sigma$. Let $\mathcal{P} = \mathcal{R} \cup \{[ab] \to [0] \mid a \in \Gamma \backslash \Sigma \ \vee\ b \in \Gamma \backslash \Sigma\}$. Note that $\mathcal{P}$ is length–reducing and $1 \notin \mathit{ran}(\mathcal{P})$. Assume that $\mathcal{R}$ is confluent and consider the situation $u_1 \ {}_{\mathcal{P}}\!\!\leftarrow u \to_{\mathcal{P}} u_2$. If $u \in M$ then we must have $u_1 \ {}_{\mathcal{R}}\!\!\leftarrow u \to_{\mathcal{R}} u_2$ and $u_1, u_2 \in M$. Confluence of $\mathcal{R}$ implies that $u_1 \to_{\mathcal{R}}^* v \ {}_{\mathcal{R}}^{*}\!\!\leftarrow u_2$ for some $v \in M$ and thus $u_1 \to_{\mathcal{P}}^* v \ {}_{\mathcal{P}}^{*}\!\!\leftarrow u_2$. If $u \notin M$ then u must be of the form $[s'as'']$ for some $a \in \Gamma \backslash \Sigma$. This must also hold for u_1 and u_2. Furthermore, since $1 \notin \mathit{ran}(\mathcal{R})$, it holds $u_i = 0$ or $|u_i| > 1$ for $i \in \{1, 2\}$. Thus u_1 and u_2 can be both reduced to 0. Now assume that $\mathcal{P}$ is confluent and consider the situation $u_1 \ {}_{\mathcal{R}}\!\!\leftarrow u \to_{\mathcal{R}} u_2$. Thus, $u_1 \ {}_{\mathcal{P}}\!\!\leftarrow u \to_{\mathcal{P}} u_2$ and confluence of $\mathcal{P}$ implies $u_1 \to_{\mathcal{P}}^* v \ {}_{\mathcal{P}}^{*}\!\!\leftarrow u_2$ for some $v \in N$. Since symbols from $\Gamma \backslash \Sigma$ do not appear in u_1 or u_2 and cannot be produced by rules from $\mathcal{P}$ it follows $u_1 \to_{\mathcal{R}}^* v \ {}_{\mathcal{R}}^{*}\!\!\leftarrow u_2$. $\square$

Now we are able to prove our first main result.

Theorem 3.12. *$\mathit{CONFL}(M)$ is decidable if and only if M is a free monoid or a free commutative monoid.*

Proof. The decidability of $\mathit{CONFL}(M)$ in the case of a free monoid or free commutative monoid is a well–known result, see [BO81] and [BL81]. Thus, assume that $M = \mathbb{M}(\Sigma, I)$ is neither free nor free commutative, i.e., $I \neq \emptyset$ and $I \neq (\Sigma \times \Sigma) \backslash \mathit{Id}_\Sigma$. If (Σ, I) contains an induced subgraph of the form

$$\bullet - \bullet - \bullet$$

then Lemma 3.4 and Lemma 3.11 imply the undecidability of $\mathit{CONFL}_{\neq 1}(M)$ (and thus $\mathit{CONFL}(M)$). Thus assume that not contain such an induced subgraph. This means that I is a transitive relation on Σ, i.e., (Σ, I) is a disjoint union of n cliques $(\Sigma_i, (\Sigma_i \times \Sigma_i) \backslash \mathit{Id}_{\Sigma_i})$ $(1 \leq i \leq n)$. If $n = 1$ then $I =\neq (\Sigma \times \Sigma) \backslash \mathit{Id}_\Sigma$ which we have excluded. If $|\Sigma_i| = 1$ for all $i \in \{1, \ldots, n\}$ then $I = \emptyset$ which we have also excluded. Thus there are at least two cliques and one of these cliques contains at least two nodes. Thus

$$\bullet - \bullet \qquad \bullet$$

must be an induced subgraph of (Σ, I), which is (Σ_1, I_1) from Definition 3.5. Lemma 3.9 and Lemma 3.11 imply that $\mathit{CONFL}_{\neq 1}(M)$ (and thus $\mathit{CONFL}(M)$) is undecidable. $\square$

Since confluence is decidable for the class of terminating semi–Thue systems and vector replacement systems it follows that for every trace monoid M, confluence is decidable for the class of terminating TRSs over M if and only if confluence is decidable for the class of length–reducing TRSs over M. The negative result of Theorem 3.12 can even be strengthened by using the following obvious lemma. Let $\mathit{TRS}_{>}(M)$ denote the set of all length–reducing TRSs over the trace monoid M.

Lemma 3.13. For every trace monoid M, the set $\mathit{TRS}_{>}(M) \setminus \mathit{CONFL}(M)$ is recursively enumerable.

Proof. A semi–algorithm that tests whether a TRS $\mathcal{R}$ over M belongs to $\mathit{TRS}_{>}(M) \setminus \mathit{CONFL}(M)$ may operate as follows: Generate all situations $u \;{}_{\mathcal{R}}\!\leftarrow\; w \rightarrow_{\mathcal{R}} v$. Since $\mathcal{R}$ is terminating it can be decided whether $u \rightarrow_{\mathcal{R}} w' \;{}_{\mathcal{R}}\!\leftarrow\; v$ for some $w' \in M$. $\square$

The following theorem follows immediately from the previous lemma and Theorem 3.12.

Theorem 3.14. If M is neither a free monoid nor a free commutative monoid then the set $\mathit{CONFL}(M)$ is not recursively enumerable.

4 Deciding (α, β)–confluence

In this section, we will prove the following result.

Theorem 4.1. $\mathit{CONFL}(\alpha, \beta, M)$ is decidable for all $\alpha, \beta \geq 1$ and for every trace monoid M.

In Section 5 we will prove some important consequences of Theorem 4.1. But also for itself, Theorem 4.1 is interesting. In the last section we proved that confluence for length–reducing TRSs is undecidable in most cases. Given a length–reducing TRS $\mathcal{R}$ over some trace monoid and a situation $v \;{}_{\mathcal{R}}\!\leftarrow\; u \rightarrow_{\mathcal{R}} w$, the traces v and w have a common descendant if and only if a common descendant is reached after less then $|u|$ many rewrite steps. For a monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$ let $\mathit{CONFL}(f, M)$ be the set of all TRSs over M such that $v \;{}_{\mathcal{R}}\!\leftarrow\; u \rightarrow_{\mathcal{R}} w$ implies $v \rightarrow_{\mathcal{R}}^{\leq f(|u|)} u' \;{}_{\mathcal{R}}\!\leftarrow^{\leq f(|u|)} w$ for some $u' \in M$. The above consideration and Theorem 3.12 imply that $\mathit{CONFL}(n \mapsto n, M)$ is undecidable if M is neither a free nor a free commutative monoid. For a free or free commutative monoids M, $\mathit{CONFL}(n \mapsto n, M)$ is easily seen to be decidable, since only finitely many critical pairs have to be considered. On the other hand, Theorem 4.1 implies that $\mathit{CONFL}(n \mapsto \alpha, M)$ is decidable for every constant $\alpha \geq 1$ and every trace monoid M. This sharpens the borderline between decidability and undecidability in another way.

The idea for the proof of Theorem 4.1 is to construct for every TRS $\mathcal{R}$ over M and every $\alpha, \beta \geq 1$ a logical sentence ϕ whose atomic subformulas are of the form $x \in L$ for a variable x and a recognizable trace language $L \subseteq M$ such that $\mathcal{R}$ is (α, β)–confluent if and only if φ is true in the trace monoid M. Due to the nice properties of recognizable trace languages that we have stated in Fact 2.6 and Fact 2.7, it can be decided effectively whether ϕ is true in M.

We start the proof of Theorem 4.1 by considering some simple logical formulas in trace monoids. Let *Var* be a countably infinite set of *variables.* Variables will be denoted by x, y, z, possibly with subscripts. Fix an independence alphabet (Σ, I) with $\mathit{Var} \cap \Sigma = \emptyset$ and let $M = \mathbb{M}(\Sigma, I)$. The set of all *patterns* is $(\mathit{Var} \cup \Sigma)^*$. Patterns will be denoted by S, T, $U, \ldots$, possibly with subscripts. The set of all variables that appear in a pattern S is denoted by $\mathit{Var}(S)$. A pattern S is called *linear* if every variable appears at most once in S. In the following all patterns are assumed to be linear. A linear pattern S with $\mathit{Var}(S) = \{x_1, \ldots, x_n\}$ is also denoted by $S(x_1, \ldots, x_n)$ (this does not mean that the variables $x_1, \ldots, x_n$ appear in S in this order. Linear patterns in Var^+ are called *types.* Thus a type is just a repetition–free list of variables. Types will be denoted by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, possibly with subscripts. Given a pattern S, we denote by $\mathit{typ}(S) = \pi_{\mathit{Var}}(S)$ the projection of S to the set of all variables, which is a type.

We will consider first–order formulas that are built up from atomic formulas of the form

- $S = T$, where S and T are linear patterns,
- $x \in L$, where $L \in \mathit{REG}(M)$ and $x \in \mathit{Var}$, and
- $\pi_\Gamma(x) = y$ where $\Gamma \subseteq \Sigma$ and $x, y \in \mathit{Var}$.

For $\Gamma \subseteq \Sigma$, we write $\mathit{alph}(x) \subseteq \Gamma$ instead of $x \in \{u \in M \mid \mathit{alph}(u) \subseteq \Gamma\}$ which is recognizable by Fact 2.6(4). A formula φ with free variables $x_1, \ldots, x_n$ is also denoted by $\varphi(x_1, \ldots, x_n)$. Sometimes we write $\bigvee_\alpha \varphi_\alpha$ to denote a finite disjunction, where the concrete number of disjuncts is not important. We use $\exists x_i (1 \leq i \leq m)$ as an abbreviation for the quantifier prefix $\exists x_1 \ldots \exists x_m$. A *substitution* is a function $\vartheta : Var \longrightarrow (Var \cup \Sigma)^*$. The homomorphic extension of ϑ to $(Var \cup \Sigma)^*$ is also denoted by ϑ. If $S(x_1, \ldots, x_m)$ is a linear pattern such that $\vartheta(S)$ is also linear, then we also write $S(\vartheta(x_1)/x_1, \ldots, \vartheta(x_m)/x_m)$ or simply $S(\ldots, \vartheta(x_i)/x_i, \ldots)$ instead of $\vartheta(S)$. A substitution ϑ is a *ground substitution* if $\vartheta(x) \in \Sigma^*$ for every $x \in Var$.

The interpretation of such formulas in M is the obvious one. For a ground substitution ϑ we write $(M, \vartheta) \models \varphi$ if φ is true in M if each variable x is set to $\vartheta(x)$. More precisely

- $(M, \vartheta) \models (S = T)$ if $[\vartheta(S)]_I = [\vartheta(T)]_I$,
- $(M, \vartheta) \models (x \in L)$ if $[\vartheta(x)]_I \in L$, and
- $(M, \vartheta) \models (\pi_\Gamma(x) = y)$ if $[\pi_\Gamma(\vartheta(x))]_I = [\vartheta(y)]_I$.

If φ is a sentence, i.e., does not contains free variables we simply write $M \models \varphi$. Two formulas $\varphi(x_1, \ldots, x_m)$ and $\phi(x_1, \ldots, x_m)$ are equivalent in M if for all ground substitutions ϑ it holds $(M, \vartheta) \models \varphi$ if and only if $(M, \vartheta) \models \phi$.

For a finite set X of variables and a function $\sigma : X \to 2^\Sigma$ we define an independence alphabet $(X, I(\sigma))$ by $I(\sigma) = \{(x, y) \mid \sigma(x) \times \sigma(y) \subseteq I\} \backslash Id_X$. The congruence relation $\equiv_{I(\sigma)}$ on X^* will be also denoted by $\equiv_\sigma$.

In the next two lemmas, we assume that $I = \emptyset$. Thus we are working in the free monoid $M = \Sigma^*$. The next lemma states a well–known fact about conjugated words, see e.g. [Lot83].

Lemma 4.2. Let $s, t \in \Sigma^+$ be non–empty words and let $x \in Var$. The word equation $sx = xt$ is equivalent in Σ^* to a formula $x \in L$ where $L \in \mathit{REG}(\Sigma^*)$.

Proof. Let $s, t \in \Sigma^+$. Then for $u \in \Sigma^*$ it holds $su = ut$ if and only if there exist $s_1, s_2 \in \Sigma^*$ with $s_2 \neq 1$, $s = s_1 s_2$, $t = s_2 s_1$, and $u = s^n s_1$ for some $n \geq 0$, see e.g. [Lot83]. The statement of the lemma is an immediate corollary of this fact. $\square$

The next lemma generalizes the previous lemma to the case of arbitrary many variables.

Lemma 4.3. Let $s_1, \ldots, s_{m+1}, t_1, \ldots, t_{m+1} \in \Sigma^*$. The word equation

$$s_1 x_1 s_2 \cdots x_m s_{m+1} = t_1 x_1 t_2 \cdots x_m t_{m+1} \tag{6}$$

is equivalent in Σ^* to a formula $\varphi(x_1, \ldots, x_n)$ of the form $\bigvee_\alpha \bigwedge_{i=1}^{m} x_i \in L_{i,\alpha}$, where $L_{i,\alpha} \in \mathit{REG}(\Sigma^*)$ for every $i \in \{1, \ldots, m\}$ and all α.

Proof. The lemma can be proven by induction on m. The case $m = 0$ is trivial. Assume $m > 0$. If neither s_1 is a prefix of t_1 nor t_1 is a prefix of s_1 then we may choose for φ the empty disjunction, i.e., the truth value false. Thus, assume w.l.o.g. that $s_1 = t_1 s$ and cancel t_1. It remains the equation $s x_1 s_2 \cdots x_m s_{m+1} = x_1 t_2 \cdots x_m t_{m+1}$. If $s = 1$ then we may also cancel x_1. Then by the induction hypothesis $s_2 x_2 s_3 \cdots x_m s_{m+1} = t_2 x_2 t_3 \cdots x_m t_{m+1}$ is equivalent to a finite disjunction $\bigvee_\alpha \bigwedge_{i=2}^{m} x_i \in L_{i,\alpha}$ where $L_{i,\alpha} \in \mathit{REG}(\Sigma^*)$ for every $i \in \{1, \ldots, m\}$ and all α. Then the original equation (6) is equivalent to $\bigvee_{\alpha}(x_1 \in \Sigma^* \wedge \bigwedge_{i=2}^{m} x_i \in L_{i,\alpha})$. If $s \neq 1$ then we may guess the position in $t_2 \ldots x_m t_{m+1}$ where sx_1 ends. More precisely $sx_1 s_2 \cdots x_m s_{m+1} = x_1 t_2 \cdots x_m t_{m+1}$ is equivalent to the finite disjunction of all formulas of the form

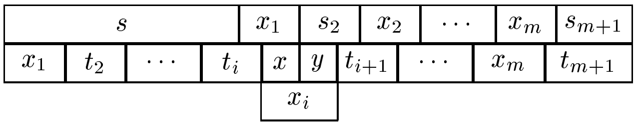
(1) $sx_1 = x_1 t_2 \cdots t_{i-1} x_{i-1} t' \wedge s_2 x_2 s_3 \cdots x_m s_{m+1} = t'' x_i t_{i+1} \cdots x_m t_{m+1}$ where $i \in \{2, \ldots, m+1\}$ and $t_i = t' t''$ and

(2) $\exists x, y : x_i = xy \wedge sx_1 = x_1 t_2 \cdots x_{i-1} t_i x \wedge s_2 x_2 s_3 \cdots x_m s_{m+1} = y t_{i+1} x_{i+1} t_{i+2} \cdots x_m t_{m+1}$ where $i \in \{2, \ldots, m\}$ and x, y are new variables.

Let us consider a formula of the second type (the first type can be dealt similarly). After substituting xy for x_i we obtain the formula

$$\exists x, y : x_i = xy \wedge sx_1 = x_1 t_2 \cdots x_{i-1} t_i x \wedge s_2 x_2 \cdots s_i x y s_{i+1} \cdots x_m s_{m+1} = y t_{i+1} x_{i+1} \cdots x_m t_{m+1},$$

see also the following figure.



It suffices to prove that a formula of this type is equivalent to a finite disjunction of the form required in the lemma. Note that for every solution of $sx_1 = x_1 t_2 \ldots x_{i-1} t_i x$, i.e., every tuple $(u_1, u_2, \ldots, u_{i-1}, u)$ with $su_1 = u_1 t_2 \ldots u_{i-1} t_i u$, it must hold $|s| = |t_2 u_2 \ldots u_{i-1} t_i u|$. But there exist only finitely many tuples $(u_2, \ldots, u_{i-1}, u)$ with the last property. Thus we can take the finite disjunction over all these tuples. Let $(u_2, \ldots, u_{i-1}, u)$ be one of these tuples and let $v = t_2 u_2 t_3 \ldots u_{i-1} t_i u$ and $w = s_2 u_2 s_3 \ldots u_{i-1} s_i u$. It suffices to prove that the formula

$$\exists y : x_i = uy \wedge sx_1 = x_1 v \wedge \bigwedge_{j=2}^{i-1} x_j \in \{u_j\} \wedge w y s_{i+1} x_{i+1} \ldots x_m s_{m+1} = y t_{i+1} x_{i+1} \ldots x_m t_{m+1}$$

is equivalent to a finite disjunction of the form required in the lemma. By Lemma 4.3 (note that $s \neq 1$ and thus also $v \neq 1$), the equation $sx_1 = x_1 v$ may be replaced by a formula $x_1 \in L_1$ where $L_1 \in REG(\Sigma^*)$. Furthermore by the induction hypothesis, the equation $w y s_{i+1} x_{i+1} \ldots s_m x_m s_{m+1} = y t_{i+1} x_{i+1} \ldots t_m x_m t_{m+1}$ (since $i \geq 2$ this equation contains $m - i + 1 < m$ many variables) is equivalent to a formula of the form $\bigvee_{\alpha}(y \in K_\alpha \wedge \bigwedge_{j=i+1}^{m} x_j \in L_{j,\alpha})$, where $L_{j,\alpha}, K_\alpha \in REG(\Sigma^*)$ for every $j \in \{i+1, \ldots m\}$ and all α. We obtain the formula

$$\exists y : x_i = uy \wedge x_1 \in L_1 \wedge \bigwedge_{j=2}^{i-1} x_j \in \{u_j\} \wedge \bigvee_{\alpha}(y \in K_\alpha \wedge \bigwedge_{j=i+1}^{m} x_j \in L_{j,\alpha})$$

which is equivalent to

$$\bigvee_{\alpha}(x_1 \in L_1 \wedge \bigwedge_{j=2}^{i-1} x_j \in \{u_j\} \wedge x_i \in uK_\alpha \wedge \bigwedge_{j=i+1}^{m} x_j \in L_{j,\alpha})$$

where $uK_\alpha = \{uu' \in \Sigma^* \mid u' \in K_\alpha\} \in REG(\Sigma^*)$. $\square$

In the following the independence alphabet (Σ, I) may be again arbitrary. The next lemma generalizes Lemma 4.3 to traces.

Lemma 4.4. Let $\sigma : \{x_1, \ldots, x_m\} \mapsto 2^\Sigma$, let $S(x_1, \ldots, x_m)$ and $T(x_1, \ldots, x_m)$ be linear patterns with $\mathit{typ}(S) \equiv_\sigma \mathit{typ}(T)$. Then the formula

$$S(x_1, \ldots, x_m) = T(x_1, \ldots, x_m) \;\wedge\; \bigwedge_{i=1}^{m} \mathit{alph}(x_i) \subseteq \sigma(x_i). \tag{7}$$

is equivalent in $\mathbb{M}(\Sigma, I)$ to a formula of the form $\bigvee_{\alpha} \bigwedge_{i=1}^{m} x_i \in L_{i,\alpha}$, where $L_{i,\alpha} \in \mathit{REG}(\mathbb{M}(\Sigma, I))$ for every $i \in \{1, \ldots, m\}$ and all α.

Proof. Let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet (Σ, I^c). W.l.o.g. assume that $S(x_1, \ldots, x_m)$ is of the form $S = s_1 x_1 s_2 x_2 \ldots s_m x_m s_{m+1}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ let $x_{i,j}$ be a new variable which will represent the projection of x_i to the clique Γ_j. Since the value of x_i is only allowed to contain symbols from $\sigma(x_i)$ we can replace $x_{i,j}$ by the empty word if $\sigma(x_i) \cap \Gamma_j = \emptyset$. Define the pattern $\chi_{i,j}$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ by (i) $\chi_{i,j} = 1$ if $\sigma(x_i) \cap \Gamma_j = \emptyset$ and (ii) $\chi_{i,j} = x_{i,j}$ otherwise. For $j \in \{1, \ldots, n\}$ let S_j be the pattern $\pi_{\Gamma_j}(s_1)\chi_{1,j}\pi_{\Gamma_j}(s_2)\chi_{2,j} \cdots \pi_{\Gamma_j}(s_m)\chi_{m,j}\pi_{\Gamma_j}(s_{m+1})$. Thus S_j represents the projection of S to the clique Γ_j. The pattern T_j is defined analogously. We claim that $\mathit{typ}(S_j) = \mathit{typ}(T_j)$ for every $j \in \{1, \ldots, n\}$:

> Since $\mathit{typ}(S) \equiv_\sigma \mathit{typ}(T)$ there exists an $\kappa \geq 0$ such that
>
> $$\mathit{typ}(S) = \mathcal{S}_0 \equiv_\sigma \mathcal{S}_1 \equiv_\sigma \cdots \equiv_\sigma \mathcal{S}_\kappa = \mathit{typ}(T)$$
>
> and for every $k \in \{0, \ldots \kappa - 1\}$ there exist types $\mathcal{T}_k$ and $\mathcal{U}_k$ and variables $y_k, z_k \in \{x_1, \ldots, x_m\}$ with $\mathcal{S}_k = \mathcal{T}_k y_k z_k \mathcal{U}_k$, $\mathcal{S}_{k+1} = \mathcal{T}_k z_k y_k \mathcal{U}_k$, and $y_k \, I_\sigma \, z_k$. For $k \in \{0, \ldots, \kappa\}$ and $j \in \{1, \ldots, n\}$ let $\mathcal{S}_{k,j}$ be the type that results from $\mathcal{S}_j$ by replacing the each variable x_i by $\chi_{i,j}$. Thus $\mathit{typ}(S_j) = \mathcal{S}_{0,j}$ and $\mathit{typ}(T_j) = \mathcal{S}_{\kappa,j}$. Therefore in order to prove $\mathit{typ}(S_j) = \mathit{typ}(T_j)$ it suffices to prove $\mathcal{S}_{k,j} = \mathcal{S}_{k+1,j}$ for every $k \in \{0, \ldots \kappa - 1\}$. For this it suffices to prove $\sigma(y_k) \cap \Gamma_j = \emptyset$ or $\sigma(z_k) \cap \Gamma_j = \emptyset$ for every $k \in \{0, \ldots \kappa - 1\}$. Assume that $\sigma(y_k) \cap \Gamma_j \neq \emptyset \neq \sigma(z_k) \cap \Gamma_j$. Thus there exist $a \in \Gamma_j \cap \sigma(y_k)$ and $b \in \Gamma_j \cap \sigma(z_k)$. Since $a, b \in \Gamma_j$ it follows $a \, I^c \, b$. But since $a \in \sigma(y_k)$ and $b \in \sigma(z_k)$ this contradicts $y_k \, I_\sigma \, z_k$.

Furthermore by Fact 2.1, (7) is equivalent to the formula

$$\exists x_{i,j} (1 \leq i \leq m, 1 \leq j \leq n) : \bigwedge_{j=1}^{n} (S_j = T_j \;\wedge\; \bigwedge_{i=1}^{m} \pi_{\Gamma_j}(x_i) = x_{i,j} \;\wedge\; \mathit{alph}(x_{i,j}) \in \sigma(x_i) \cap \Gamma_j). \tag{8}$$

Since $\mathit{alph}(x_{i,j}) \subseteq \Gamma_j$ and Γ_j is a clique for every $j \in \{1, \ldots, n\}$, the equation $S_j = T_j$ can be considered as a word equation in the free monoid Γ_j^*. Since $\mathit{typ}(S_j) = \mathit{typ}(T_j)$ for every $j \in \{1, \ldots, n\}$, by Lemma 4.3 the equation $S_j = T_j$ is equivalent in Γ_j^* to a formula $\bigvee_{\alpha} \bigwedge_{i=1}^{m} x_{i,j} \in L_{i,j,\alpha}$, where $L_{i,j,\alpha} \in \mathit{REG}(\Gamma_j^*)$ for all α. Thus, (8) is equivalent to

$$\exists x_{i,j} (1 \leq i \leq m, 1 \leq j \leq n) : \bigwedge_{j=1}^{n} \bigvee_{\alpha} \bigwedge_{i=1}^{m} (x_{i,j} \in L_{i,j,\alpha} \;\wedge\; \pi_{\Gamma_j}(x_i) = x_{i,j} \;\wedge\; \mathit{alph}(x_{i,j}) \in \sigma(x_i) \cap \Gamma_j),$$

which is equivalent to a finite disjunction of formulas of the form

$$\exists x_{i,j} (1 \leq i \leq m, 1 \leq j \leq n) : \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} x_{i,j} \in L_{i,j} \;\wedge\; \pi_{\Gamma_j}(x_i) = x_{i,j},$$

y_3	$z_{1,3}$	$s_{1,3}$	$z_{2,3}$	$s_{2,3}$	$z_{3,3}$
t_2	$t_{1,2}$		$t_{2,2}$		$t_{3,2}$
y_2	$z_{1,2}$	$s_{1,2}$	$z_{2,2}$	$s_{2,2}$	$z_{3,2}$
t_1	$t_{1,1}$		$t_{2,1}$		$t_{3,1}$
y_1	$z_{1,1}$	$s_{1,1}$	$z_{2,1}$	$s_{2,1}$	$z_{3,1}$
	x_1	s_1	x_2	s_2	x_3

Figure 3: The equation $x_1s_1x_2s_2x_3 = y_1t_1y_2t_2y_3$: The empty boxes represent traces that are not important.

i.e., $\bigwedge_{i=1}^{m} x_i \in \{u \in \mathbb{M}(\Sigma, I) \mid \bigwedge_{j=1}^{n} \pi_j(u) \in L_{i,j}\}$, where $L_{i,j} \subseteq \Gamma_j^*$ is a regular word language. By Fact 2.8 $\{u \in \mathbb{M}(\Sigma, I) \mid \bigwedge_{j=1}^{n} \pi_j(u) \in L_{i,j}\} \in REG(\mathbb{M}(\Sigma, I))$, which concludes the proof. $\square$

The following lemma follows directly from Lemma 2.5, see also Figure 3.

Lemma 4.5. Let $s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-1} \in \Sigma^*$, let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be pairwise different variables, and let $\sigma : \{x_1, \ldots, x_m, y_1, \ldots, y_n\} \to 2^\Sigma$. Then the formula

$$x_1s_1x_2s_2 \ldots s_{m-1}x_m = y_1t_1y_2t_2 \ldots t_{n-1}y_n \wedge \bigwedge_{i=1}^{m} alph(x_i) \subseteq \sigma(x_i) \wedge \bigwedge_{j=1}^{n} alph(y_j) \subseteq \sigma(y_j)$$

is equivalent in $\mathbb{M}(\Sigma, I)$ to a finite disjunction of formulas of the form

$$\begin{aligned}
&\exists z_{i,j}(1 \le i \le m, 1 \le j \le n) : \\
&\quad \bigwedge_{i=1}^{m} x_i = z_{i,1}t_{i,1}z_{i,2} \ldots t_{i,n-1}z_{i,n} \wedge \bigwedge_{j=1}^{n} y_j = z_{1,j}s_{1,j}z_{2,j} \ldots s_{m-1,j}z_{m,j} \wedge \\
&\quad \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} \mathrm{alph}(z_{i,j}) \subseteq \tau(z_{i,j})
\end{aligned}$$

where $\tau(z_{i,j}) \subseteq \Sigma$ for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $z_{i,j}\, I(\tau)\, z_{k,l}$ if $((i < k, l < j)$ or $x_i\, I(\sigma)\, x_k$ or $y_j\, I(\sigma)\, y_l)$ [2].

The next lemma is of central importance. Let $\underline{c} = (c_1, \ldots, c_\kappa)$ be a finite sequence of rules $c_i = (l_i \to r_i)$ with $l_i, r_i \in \mathbb{M}(\Sigma, I)$ for every $i \in \{1, \ldots, \kappa\}$. For $u, v \in \mathbb{M}(\Sigma, I)$, $u \to_{\underline{c}} v$ is defined inductively as follows. (i) If $\kappa = 0$, i.e., $\underline{c} = ()$ is the empty sequence then $u \to_{\underline{c}} v$ if $u = v$. (ii) If $\kappa > 0$ let $\underline{d} = (c_1, \ldots, c_{\kappa-1})$. Then $u \to_{\underline{c}} v$ if $u \to_{\underline{d}} w$ and $w \to_{c_\kappa} v$ for some $w \in \mathbb{M}(\Sigma, I)$. If $l_i, r_i \in \Sigma^*$ for every $i \in \{1, \ldots, \kappa\}$ then it is not difficult to construct finitely many equations $s_1x_1s_2 \ldots x_ms_{m+1} = t_1x_1t_2 \ldots x_mt_{m+1}$ with $s_i, t_i \in \Sigma^*$ for every $i \in \{1, \ldots, m+1\}$ such that for strings $s, t \in \Sigma^*$ it holds $s \to_{\underline{c}} t$ if and only if for some of the constructed equations $s_1x_1s_2 \ldots x_ms_{m+1} = t_1x_1t_2 \ldots x_mt_{m+1}$ and some $u_1, \ldots, u_m \in \Sigma^*$ it holds $s = s_1u_1s_2 \ldots u_ms_{m+1}$ and $t = t_1u_1t_2 \ldots u_mt_{m+1}$. The next lemma generalizes this fact to traces.

Lemma 4.6. Let $\kappa \ge 0$ be arbitrary and let $m = 2^\kappa$. There exist types $\mathcal{S}_\kappa$ and $\mathcal{T}_\kappa$ which contain both the same set of m variables such that the following holds: For each finite sequence $\underline{c} = (c_1, \ldots, c_\kappa)$, where $c_i \in \mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ for every $i \in \{1, \ldots, \kappa\}$, there exists a finite disjunction $\varphi(x, y)$ of the form

$$\bigvee_{\alpha} \exists x_1, \ldots x_m : x = S_\alpha(x_1, \ldots, x_m) \wedge y = T_\alpha(x_1, \ldots, x_m) \wedge \bigwedge_{i=1}^{m} \mathrm{alph}(x_i) \subseteq \sigma_\alpha(x_i), \qquad (9)$$

[2] Of course the words $s_{i,j}, t_{i,j}$, and the function τ may be different for each of these disjuncts.

where $\sigma_\alpha : \{x_1, \ldots, x_m\} \mapsto 2^\Sigma$ such that (i) $\mathit{typ}(S_\alpha) = \mathcal{S}_\kappa \equiv_{\sigma_\alpha} \mathcal{T}_\kappa = \mathit{typ}(T_\alpha)$ for all α and (ii) for every ground substitution ϑ it holds $(\mathbb{M}(\Sigma, I), \vartheta) \models \varphi(x, y)$ if and only if $[\vartheta(x)]_I \to_{\underline{c}} [\vartheta(y)]_I$.

Proof. The lemma can be proven by induction on $\kappa \geq 0$. If $\kappa = 0$ then we may choose $\mathcal{S}_0 = \mathcal{T}_0 = z$ and $\exists z : x = z \wedge y = z \wedge \mathit{alph}(z) \subseteq \Sigma$ for $\varphi(x, y)$.

If $\kappa > 0$ then by the induction hypothesis there exist types $\mathcal{S}_{\kappa-1}$ and $\mathcal{T}_{\kappa-1}$ which contain both the same set of $m = 2^{\kappa-1}$ many variables such that the conclusion of the lemma holds for $\kappa - 1$. Fix an arbitrary sequence $\underline{c} = (c_1, \ldots, c_\kappa)$, where $c_i \in \mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ for every $i \in \{1, \ldots, \kappa\}$. Let $\underline{d} = (c_1, \ldots, c_{\kappa-1})$. Then there exists a finite disjunction $\phi(x, y)$ of the form

$$\bigvee_\alpha \exists x_1, \ldots x_m : x = U_\alpha(x_1, \ldots, x_m) \wedge y = V_\alpha(x_1, \ldots, x_m) \wedge \bigwedge_{i=1}^m \mathrm{alph}(x_i) \subseteq \tau_\alpha(x_i) \tag{10}$$

such that (i) $\mathit{typ}(U_\alpha) = \mathcal{S}_{\kappa-1} \equiv_{\tau_\alpha} \mathcal{T}_{\kappa-1} = \mathit{typ}(V_\alpha)$ for all α and (ii) for every ground substitution ϑ it holds $(\mathbb{M}(\Sigma, I), \vartheta) \models \phi(x, y)$ if and only if $[\vartheta(x)]_I \to_{\underline{d}} [\vartheta(y)]_I$. W.l.o.g. assume that $\mathcal{T}_{\kappa-1} = x_1 x_2 \cdots x_m$. For every $i \in \{1, \ldots, m\}$ and $j \in \{1, 2\}$ let $z_{i,j}$ be a new variable. Note that these are $2m = 2^\kappa$ many new variables. We define $\mathcal{S}_\kappa = \mathcal{S}_{\kappa-1}(\ldots (z_{i,1} z_{i,2})/x_i \ldots)$ and $\mathcal{T}_\kappa = z_{1,1} z_{2,1} \cdots z_{m,1} z_{1,2} z_{2,2} \cdots z_{m,2}$. Let $c_\kappa = ([l]_I, [r]_I)$. Let $\varphi(x, y)$ be the finite disjunction

$$\bigvee_\alpha \exists x_1, \ldots, x_m, y_1, y_2 : x = U_\alpha \wedge V_\alpha = y_1 l y_2 \wedge y = y_1 r y_2 \wedge \bigwedge_{i=1}^m \mathrm{alph}(x_i) \subseteq \tau_\alpha(x_i), \tag{11}$$

where $y_1, y_2 \notin \{x_1, \ldots, x_m, x, y\}$. Then obviously for every ground substitution ϑ it holds $(\mathbb{M}(\Sigma, I), \vartheta) \models \varphi(x, y)$ if and only if $[\vartheta(x)]_I \to_{\underline{c}} [\vartheta(y)]_I$. We claim that $\varphi(x, y)$ is equivalent to a finite disjunction of the form (9). Fix an α and let $V = V_\alpha$, $U = U_\alpha$, and $\tau = \tau_\alpha$. It suffices to prove that the subformula

$$\exists x_1, \ldots, x_m, y_1, y_2 : x = U \wedge V = y_1 l y_2 \wedge y = y_1 r y_2 \wedge \bigwedge_{i=1}^m \mathrm{alph}(x_i) \subseteq \tau(x_i) \tag{12}$$

of $\varphi(x, y)$ is equivalent to a finite disjunction of the form (9). Note that $\mathit{typ}(V) = \mathcal{T}_{\kappa-1} = x_1 x_2 \ldots x_m$. By Lemma 4.5 we may replace the formula $V = y_1 l y_2 \wedge \bigwedge_{i=1}^m \mathit{alph}(x_i) \subseteq \tau(x_i)$ in (12) by a finite disjunction of the form

$$\begin{aligned}
&\bigvee_\beta \exists z_{i,j} (1 \leq i \leq m, 1 \leq j \leq 2) : \\
&\qquad y_1 = W_{1,\beta}(z_{1,1}, z_{2,1}, \ldots, z_{m,1}) \wedge y_2 = W_{2,\beta}(z_{1,2}, z_{2,2}, \ldots, z_{m,2}) \wedge \\
&\qquad \bigwedge_{i=1}^m x_i = z_{i,1} l_{i,\beta} z_{i,2} \wedge \mathit{alph}(z_{i,1}) \subseteq \sigma_\beta(z_{i,1}) \wedge \mathit{alph}(z_{i,2}) \subseteq \sigma_\beta(z_{i,2}),
\end{aligned}$$

where $\mathit{typ}(W_{i,\beta}) = z_{1,i} z_{2,i} \ldots z_{m,i}$ for all $i \in \{1, 2\}$ and all β and $z_{i,j}\, I(\sigma_\beta)\, z_{k,l}$ if $((i < k,\ j = 2,\ l = 1)$ or $x_i\, I(\tau)\, x_k)$ for all β (see also Figure 4, where $V = x_1 v_1 x_2 \ldots v_{m-1} x_m$ and $W_{i,\beta} = z_{1,i} v_{1,i} z_{2,i} \ldots v_{m-1,i} z_{m,i}$ for $i \in \{1, 2\}$). By doing some elementary logical transformations the formula that results from (12) by doing this replacement is equivalent to a finite disjunction of the form

$$\begin{aligned}
&\bigvee_\beta \exists z_{i,j} (1 \leq i \leq m, 1 \leq j \leq 2) \, \exists x_1, \ldots, x_m, y_1, y_2 : \\
&\qquad x = U \wedge y_1 = W_{1,\beta} \wedge y_2 = W_{2,\beta} \wedge y = y_1 r y_2 \wedge \\
&\qquad \bigwedge_{i=1}^m (x_i = z_{i,1} l_{i,\beta} z_{i,2} \wedge \mathit{alph}(z_{i,1}) \subseteq \sigma_\beta(z_{i,1}) \wedge \mathit{alph}(z_{i,2}) \subseteq \sigma_\beta(z_{i,2})).
\end{aligned} \tag{13}$$

y_2	$z_{1,2}$	$v_{1,2}$	$z_{2,2}$	$v_{2,2}$	$z_{3,2}$	$v_{3,2}$	$z_{4,2}$
l	$l_{1,\beta}$		$l_{2,\beta}$		$l_{3,\beta}$		$l_{4,\beta}$
y_1	$z_{1,1}$	$v_{1,1}$	$z_{2,1}$	$v_{2,1}$	$z_{3,1}$	$v_{3,1}$	$z_{4,1}$
	x_1	v_1	x_2	v_2	x_3	v_3	x_4

Figure 4: The equation $x_1v_1x_2v_2x_3v_3x_4 = y_1l_\alpha y_2$

If we define $S_\beta = U(\dots(z_{i,1}l_{i,\beta}z_{i,2})/x_i \dots)$ and $T_\beta = W_{1,\beta}\,rW_{2,\beta}$ we may eliminate the variables $x_1, \dots, x_m, y_1, y_2$ and (13) becomes equivalent to

$$\bigvee_\beta \exists z_{i,j}(1 \le i \le m, 1 \le j \le 2) : x = S_\beta \wedge y = T_\beta \wedge \bigwedge_{i=1}^{m} \mathit{alph}(z_{i,1}) \subseteq \sigma_\beta(z_{i,1}) \wedge \mathit{alph}(z_{i,2}) \subseteq \sigma_\beta(z_{i,2}).$$

Note that

$$\mathit{typ}(S_\beta) = \mathit{typ}(U)(\dots(z_{i,1}z_{i,2}/x_i \dots) = \mathcal{S}_{\kappa-1}(\dots z_{i,1}z_{i,2}/x_i \dots) = \mathcal{S}_\kappa$$

and

$$\mathit{typ}(T_\beta) = \mathit{typ}(W_{1,\beta}W_{2,\beta}) = z_{1,1}z_{2,1}\dots z_{m,1}z_{1,2}z_{2,2}\dots z_{m,2} = \mathcal{T}_\kappa.$$

Thus it remains to prove that

$$\mathcal{S}_\kappa = \mathcal{S}_{\kappa-1}(z_{i,1}z_{i,2}/x_i \dots) \equiv_{\sigma_\beta} z_{1,1}z_{2,1}\dots z_{m,1}z_{1,2}z_{2,2}\dots z_{m,2} = \mathcal{T}_\kappa \tag{14}$$

for all β. Since $\mathcal{S}_{\kappa-1} \equiv_\tau \mathcal{T}_{\kappa-1} = x_1 \dots x_m$ and $z_{i,j}\, I(\sigma_\beta)\, z_{k,l}$ if $x_i\, I(\tau)\, x_k$ it follows

$$\mathcal{S}_\kappa = \mathcal{S}_{\kappa-1}(\dots z_{i,1}z_{i,2}/x_i \dots) \equiv_{\sigma_\beta} \mathcal{T}_{\kappa-1}(\dots z_{i,1}z_{i,2}/x_i \dots) = z_{1,1}z_{1,2}z_{2,1}z_{2,2}\dots z_{m,1}z_{m,2}.$$

Since $z_{i,2}\, I(\sigma_\beta)\, z_{j,1}$ if $i < j$ it holds $z_{1,1}z_{1,2}z_{2,1}z_{2,2}\dots z_{m,1}z_{m,2} \equiv_{\sigma_\beta} z_{1,1}z_{2,1}\dots z_{m,1}z_{1,2}z_{2,2}\dots z_{m,2}$ which implies (14). $\square$

The previous lemma gives us the means to express the fact that a TRS is (α, β)–confluent as a logical formula of a particular type. In order to prove that is decidable whether a formula of that type is valid in $\mathbb{M}(\Sigma, I)$ we first prove the decidability of the validity of formulas of a simpler type. A positive Boolean formula is a Boolean formula that only contains the connectives $\wedge$ and $\vee$.

Lemma 4.7. Let φ be a first–order formula whose atomic subformulas are all of the form $x \in L$ for $L \in \mathit{REG}(\mathbb{M}(\Sigma, I))$. Then φ is equivalent to a positive Boolean combination ϕ of atomic formulas of the form $x \in L$, where $L \in \mathit{REG}(\mathbb{M}(\Sigma, I))$. Moreover ϕ can be calculated effectively from φ.

Proof. The lemma can be proven by induction on the structure of φ. The cases $\varphi \equiv (x \in L)$ is trivial. For $\varphi \equiv \neg\varphi'$ the induction hypothesis for φ' and Fact 2.6(2) have to be used. For $\varphi \equiv (\varphi_1 \wedge \varphi_2)$, the induction hypothesis suffices. It remains to consider the case $\varphi \equiv \exists x : \varphi'$. We may assume that $\varphi' \equiv \bigvee_{i=1}^m \varphi_i$ where φ_i is a finite conjunction of formulas of the form $x \in L$ with $L \in \mathit{REG}(\mathbb{M}(\Sigma, I))$. Since $\exists x : \bigvee_{i=1}^m \varphi_i$ is equivalent to $\bigvee_{i=1}^m \exists x : \varphi_i$ it suffices to prove that the conclusion of the lemma holds for a formula $\exists x : (x_1 \in L_1 \wedge \dots \wedge x_n \in L_n)$ where $L_i \in \mathit{REG}(\mathbb{M}(\Sigma, I))$ for every $i \in \{1, \dots, n\}$. The case $x \notin \{x_1, \dots, x_n\}$ is clear. Thus assume w.l.o.g. that $x = x_1$. By Fact 2.6(2) we may assume that $x \notin \{x_2, \dots, x_n\}$. Thus, $\exists x : (x \in L_1 \wedge x_2 \in L_2 \wedge \dots \wedge x_n \in L_n)$ is equivalent to $(\exists x : x \in L_1) \wedge x_2 \in L_2 \wedge \dots \wedge x_n \in L_n$ which is either equivalent to the truth value false (if $L_1 = \emptyset$) or equivalent to $x_2 \in L_2 \wedge \dots \wedge x_n \in L_n$. Moreover by Fact 2.7 it can be decided effectively which alternative holds. $\square$

For the proof of Theorem 4.1, the following lemma is not needed in its full generality. We formulate it in this general form since we intend to make further use of it in future works. In the following we also allow atomic formulas of the form $S \to_c T$, where S and T are patterns and $c = [l]_I \to [r]_I$ is a trace rewrite rule. The interpretation of such a formula is the obvious one.

Lemma 4.8. The following problem is decidable.
INPUT:

- An independence alphabet (Σ, I) and a finite set Δ which consists of finite sequences of trace rewrite rules over $\mathbb{M}(\Sigma, I)$.
- A function $\sigma : \{x_1, \dots, x_m\} \mapsto 2^\Sigma$ and sets $L_1, \dots, L_m \in \mathit{REG}(\mathbb{M}(\Sigma, I))$ such that for all $i \in \{1, \dots, m\}$ and all $u \in L_i$ it holds $\mathit{alph}(u) \subseteq \sigma(x_i)$.
- Linear patterns $S(x_1, \dots, x_m)$ and $T(x_1, \dots, x_m)$ with $\mathit{typ}(S) \equiv_\sigma \mathit{typ}(T)$.

QUESTION: Does $\mathbb{M}(\Sigma, I) \models \forall x_1 \in L_1, \dots, x_m \in L_m : \bigvee_{\underline{c} \in \Delta} S(x_1, \dots, x_m) \to_{\underline{c}} T(x_1, \dots, x_m)$ hold ?

Proof. Let (Σ, I), Δ, σ, $L_1, \dots, L_m$, $S(x_1, \dots, x_m)$, and $T(x_1, \dots, x_m)$ be of the form described in the lemma. By appending the trivial rule $1 \to 1$ to some sequences from Δ we may assume that all sequences in Δ have the same length κ. W.l.o.g. assume that $\mathit{typ}(S) = x_1 \cdots x_m$. By Lemma 4.6 the sentence

$$\forall x_1 \in L_1, \dots, x_m \in L_m : \bigvee_{\underline{c} \in \Delta} S(x_1, \dots, x_m) \to_{\underline{c}} T(x_1, \dots, x_m)$$

is equivalent to a sentence of the form

$$\begin{aligned} &\forall x_1 \in L_1, \dots, x_m \in L_m : \bigvee_{\alpha} \exists y_1, \dots, y_n : \\ &\qquad S = U_\alpha(y_1, \dots, y_n) \wedge T = V_\alpha(y_1, \dots, y_n) \wedge \bigwedge_{j=1}^{n} \mathit{alph}(y_j) \subseteq \sigma_\alpha(y_j), \end{aligned} \tag{15}$$

where $n = 2^\kappa$ and $y_1, \dots, y_n$ are new variables. Furthermore for all α it holds $\mathit{typ}(U_\alpha) = \mathcal{S}_\kappa \equiv_{\sigma_\alpha} \mathcal{T}_\kappa = \mathit{typ}(V_\alpha)$ where $\mathcal{S}_\kappa$ and $\mathcal{T}_\kappa$ are the types from Lemma 4.6. We may assume w.l.o.g. that $\mathit{typ}(U_\alpha) = y_1 \cdots y_n$ for all α. Since for all $i \in \{1, \dots, m\}$ and all $u \in L_i$ it holds $\mathit{alph}(u) \subseteq \sigma(x_i)$, (15) is equivalent to

$$\begin{aligned} &\forall x_1 \in L_1, \dots, x_m \in L_m : \bigvee_{\alpha} \exists y_1, \dots, y_n : \\ &\qquad S(x_1, \dots, x_m) = U_\alpha(y_1, \dots, y_n) \wedge T(x_1, \dots, x_m) = V_\alpha(y_1, \dots, y_n) \wedge \\ &\qquad \bigwedge_{j=1}^{n} \mathit{alph}(y_j) \subseteq \sigma_\alpha(y_j) \wedge \bigwedge_{i=1}^{m} \mathit{alph}(x_i) \subseteq \sigma(x_i). \end{aligned} \tag{16}$$

By Lemma 4.5 applied to $S(x_1, \dots, x_m) = U_\alpha(y_1, \dots, y_n) \wedge \bigwedge_{j=1}^{n} \mathit{alph}(y_j) \subseteq \sigma_\alpha(y_j) \wedge \bigwedge_{i=1}^{m} \mathit{alph}(x_i) \subseteq \sigma(x_i)$ the sentence (16) is equivalent to a sentence of the form

$$\begin{aligned} &\forall x_1 \in L_1, \dots, x_m \in L_m : \bigvee_{\alpha} \exists y_1, \dots, y_n : \bigvee_{\beta} \exists z_{i,j} (1 \le i \le m, 1 \le j \le n) : \\ &\qquad \bigwedge_{i=1}^{m} x_i = S_{i,\alpha,\beta}(z_{i,1}, \dots, z_{i,n}) \wedge \bigwedge_{j=1}^{n} y_j = T_{j,\alpha,\beta}(z_{1,j}, \dots, z_{m,j}) \wedge \\ &\qquad T(x_1, \dots, x_m) = V_\alpha(y_1, \dots, y_n) \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} \mathit{alph}(z_{i,j}) \subseteq \sigma_{\alpha,\beta}(z_{i,j}), \end{aligned}$$

where $typ(S_{i,\alpha,\beta}) = z_{i,1} \cdots z_{i,n}$, $typ(T_{j,\alpha,\beta}) = z_{1,j} \cdots z_{m,j}$ and $z_{i,j}\, I(\sigma_{\alpha,\beta})\, z_{k,l}$ if $((i < k,\ l < j)$ or $x_i\, I(\sigma)\, x_k$ or $y_j\, I(\sigma_\alpha)\, y_l)$ for all α, β. Let $T_{\alpha,\beta} = T(\ldots S_{i,\alpha,\beta}/x_i \ldots)$ and $V_{\alpha,\beta} = V_\alpha(\ldots T_{j,\alpha,\beta}/y_j \ldots)$ for all α, β. Then the above sentence is equivalent to

$$\begin{gathered}\forall x_1 \in L_1, \ldots, x_m \in L_m : \bigvee_{\alpha,\beta} \exists z_{i,j} (1 \le i \le m, 1 \le j \le n) : \\ \bigwedge_{i=1}^{m} x_i = S_{i,\alpha,\beta}(z_{i,1} \ldots z_{i,n}) \wedge T_{\alpha,\beta} = V_{\alpha,\beta} \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} alph(z_{i,j}) \subseteq \sigma_{\alpha,\beta}(z_{i,j}).\end{gathered} \tag{17}$$

We claim that $typ(T_{\alpha,\beta}) \equiv_{\sigma_{\alpha,\beta}} typ(V_{\alpha,\beta})$. This can be deduced as follows:

$$\begin{aligned} typ(T_{\alpha,\beta}) &= typ(T(\ldots S_{i,\alpha,\beta}/x_i \ldots)) \\ &= typ(T)(\ldots (z_{i,1} \cdots z_{i,n})/x_i \ldots) && (typ(S_{i,\alpha,\beta}) = z_{i,1} \cdots z_{i,n}) \\ &\equiv_{\sigma_{\alpha,\beta}} typ(S)(\ldots (z_{i,1} \cdots z_{i,n})/x_i \ldots) && (typ(T) \equiv_\sigma typ(S) \text{ and} \\ & && \ z_{i,j}\, I(\sigma_{\alpha,\beta})\, z_{k,l} \text{ if } x_i\, I(\sigma)\, x_k) \\ &= (z_{1,1} \cdots z_{1,n}) \cdots (z_{m,1} \cdots z_{m,n}) && (typ(S) = x_1 \cdots x_m) \\ &\equiv_{\sigma_{\alpha,\beta}} (z_{1,1} \cdots z_{m,1}) \cdots (z_{1,n} \cdots z_{m,n}) && (z_{i,j}\, I(\sigma_{\alpha,\beta})\, z_{k,l} \text{ if } i < k \text{ and } l < j) \\ &= typ(U_\alpha)(\ldots (z_{1,j} \cdots z_{m,j})/y_j \ldots) && (typ(U\alpha) = y_1 \cdots y_n) \\ &\equiv_{\sigma_{\alpha,\beta}} typ(V_\alpha)(\ldots (z_{1,j} \cdots z_{m,j})/y_j \ldots) && (typ(U_\alpha) \equiv_{\sigma_\alpha} typ(V_\alpha) \text{ and} \\ & && \ z_{i,j}\, I(\sigma_{\alpha,\beta})\, z_{k,l} \text{ if } y_j\, I(\sigma_\alpha)\, y_l) \\ &= typ(V_\alpha(\ldots T_{j,\alpha,\beta}/y_j \ldots)) = typ(V_{\alpha,\beta}). \end{aligned}$$

Hence we may apply Lemma 4.4 to the subformula $T_{\alpha,\beta} = V_{\alpha,\beta} \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} alph(z_{i,j}) \subseteq \sigma_{\alpha,\beta}(z_{i,j})$ of (17). We obtain an equivalent sentence

$$\begin{gathered}\forall x_1 \in L_1, \ldots, x_m \in L_m : \bigvee_{\alpha,\beta,\gamma} \exists z_{i,j} (1 \le i \le m, 1 \le j \le n) : \\ \bigwedge_{i=1}^{m} (x_i = S_{i,\alpha,\beta}(z_{i,1} \ldots z_{i,n}) \wedge \bigwedge_{j=1}^{n} z_{i,j} \in L_{\alpha,\beta,\gamma}(z_{i,j})),\end{gathered}$$

where $L_{\alpha,\beta,\gamma}(z_{i,j}) \in REG(\mathbb{M}(\Sigma, I))$ for all α, β, γ. Let

$$L_{\alpha,\beta,\gamma}(x_i) = \{S_{i,\alpha,\beta}(w_1, \ldots, w_n) \mid \forall j \in \{1, \ldots, n\} : w_j \in L_{\alpha,\beta,\gamma}(z_{i,j})\}.$$

Since $REG(\mathbb{M}(\Sigma, I))$ is closed under concatenation by Fact 2.6(2) also $L_{\alpha,\beta,\gamma}(x_i) \in REG(\mathbb{M}(\Sigma, I))$ (here it is important that $S_{i,\alpha,\beta}$ is linear. Since $Var(S_{i,\alpha,\beta}) \cap Var(S_{j,\alpha,\beta}) = \emptyset$ for $i \ne j$ and all α, β the above sentence is equivalent to

$$\forall x_1 \in L_1, \ldots, x_m \in L_m : \bigvee_{\alpha,\beta,\gamma} \bigwedge_{i=1}^{m} x_i \in L_{\alpha,\beta,\gamma}(x_i).$$

By Lemma 4.7 it can be decided whether this sentence is valid in $\mathbb{M}(\Sigma, I)$, which completes the proof. □

Now we are able to prove Theorem 4.1. Let $\mathcal{R}$ be a TRS over M. For $c = (l \to r) \in \mathcal{R}$, we denote the inverse rule $r \to l$ by c^{-1}. Thus, $u \to_{c^{-1}} v$ if and only if $v \to_c u$. We define $\mathcal{R}^{-1} = \{c^{-1} \mid c \in \mathcal{R}\}$.

Proof of Theorem 4.1. Fix values $\alpha, \beta \ge 1$ and fix a trace monoid $M = \mathbb{M}(\Sigma, I)$. Let $\mathcal{R}$ be an arbitrary TRS over M. Fix rules $c_1, \ldots c_k \in \mathcal{R}$, $d_1, \ldots, d_l \in \mathcal{R}^{-1}$ where $k, l \le \alpha$ and let $\underline{c} = (c_1, \ldots c_k, d_1, \ldots, d_l)$. It suffices to prove that it is decidable whether the sentence

$$\forall x, y : x \to_{\underline{c}} y \text{ implies } \exists z : x \to_{\mathcal{R}}^{\le \beta} z \ {}_{\mathcal{R}}\!\!\leftarrow^{\le \beta}\ y \tag{18}$$

is true in $\mathbb{M}(\Sigma, I)$. By Lemma 4.6 there exists a finite disjunction $\varphi(x,y)$ of the form

$$\bigvee_{\gamma} \exists x_1, \ldots, x_m : x = S_\gamma(x_1, \ldots, x_m) \wedge y = T_\gamma(x_1, \ldots, x_m) \wedge \bigwedge_{i=1}^{m} \mathrm{alph}(x_i) \subseteq \sigma_\gamma(x_i), \tag{19}$$

where $m = 2^{k+l}$ and $\sigma_\gamma : \{x_1, \ldots, x_m\} \mapsto 2^\Sigma$ for all γ such that $\mathit{typ}(S_\gamma) \equiv_{\sigma_\gamma} \mathit{typ}(T_\gamma)$ and for every ground substitution ϑ it holds $(M, \vartheta) \models \varphi(x,y)$ if and only if $[\vartheta(x)]_I \rightarrow_{\underline{c}} [\vartheta(y)]_I$. Fix a γ and let $S = S_\gamma$, $T = T_\gamma$, and $\sigma = \sigma_\gamma$. For $i \in \{1, \ldots, m\}$ let $L_i = \{u \in M \mid \mathit{alph}(u) \subseteq \sigma(x_i)\}$ which is recognizable. It suffices to prove that it is decidable whether the sentence

$$\forall x_1 \in L_1, \ldots, x_m \in L_m : \exists y : S(x_1, \ldots, x_m) \rightarrow_{\mathcal{R}}^{\leq \beta} y \;{}_{\mathcal{R}}{\leftarrow}^{\leq \beta}\; T(x_1, \ldots, x_m) \tag{20}$$

is valid in M. Let Δ be the set of all sequences that consist of at most β many rules from $\mathcal{R}$ followed by at most β many rules from $\mathcal{R}^{-1}$. Then (20) is equivalent to the following sentence

$$\forall x_1 \in L_1, \ldots, x_m \in L_m : \bigvee_{\underline{d} \in \Delta} S(x_1, \ldots, x_m) \rightarrow_{\underline{d}} T(x_1, \ldots, x_m).$$

By Lemma 4.8 it is decidable whether the last sentence is valid in M. □

5 Applications of (α, β)–confluence

In this section we will present some applications of Theorem 4.1. An immediate corollary of this result is

Corollary 5.1. Strong confluence is decidable for TRSs.

Another important application results from Lemma 2.9. The fact that for every $\alpha \geq 1$, all TRS in $\mathrm{CONFL}(\alpha, \alpha, M)$ are confluent, gives us a formal method for proving that a given (possibly non terminating) TRS is confluent.

In [Die90] a notion of a critical trace similar to definition 2.13 was introduced. Furthermore it is proven that the set of critical traces for a specific TRS is recognizable. Thus it can be decided whether this set is finite and if this is the case, confluence can be decided. Furthermore to the knowledge of the author all known criteria for terminating TRSs that imply the decidability of the confluence problem also imply the existence of a finite set of critical traces, see for instance [Die90], pp 134 for such a criteria. In contrast to this, in this section we will present a method that may also handle infinite sets of critical traces. The idea is to give sufficient conditions for a TRS that imply the equivalence of confluence and $(1, \alpha)$–confluence for a TRS. This allows to apply Theorem 4.1.

Let $M = \mathbb{M}(\Sigma, I)$ be a trace monoid. Given two traces $u_1, u_2 \in M$ we define the set $\mathit{overlap}(u_1, u_2)$ as the set of all traces $s \in M \backslash \{1\}$ such that there exist factorizations $u_1 = v_1 s w_1$, $u_2 = v_2 s w_2$ with $v_1 \, I \, v_2$ and $w_1 \, I \, w_2$. Note that $\mathit{overlap}(u_1, u_2)$ is always finite. Furthermore, every $s \in \mathit{overlap}(u_1, u_2)$ appears as the intersection of occurrences of u_1 and u_2 in in the trace $v_1 v_2 s w_2 w_1 = v_2 v_1 s w_1 w_2$. For a TRS $\mathcal{R}$ we define $\mathit{overlap}(\mathcal{R}) = \bigcup \{\mathit{overlap}(l_1, l_2) \mid l_1, l_2 \in \mathit{dom}(\mathcal{R})\}$. Note that $\mathit{dom}(\mathcal{R}) \subseteq \mathit{overlap}(\mathcal{R})$ and thus $\mathit{overlap}(\mathcal{R}) \neq \emptyset$. For a trace $u \in M$ we define $\mathit{dep}(u) = \{a \in \Sigma \mid \exists b \in \mathit{alph}(u) : (a,b) \in I^c\}$.

Theorem 5.1. The following problem is decidable.
INPUT: A TRS $\mathcal{R}$ over a trace monoid $M = \mathbb{M}(\Sigma, I)$ such that

(1) $\mathcal{R}$ satisfies condition (A) and

(2) for all $u \in \mathit{overlap}(\mathcal{R})$, the TRS $\pi_{\mathit{dep}(u)}(\mathcal{R})$ is terminating.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. Let $\mathcal{R}$ be a TRS with the properties stated in the theorem. We will prove that there exists an $\alpha \geq 1$ such that $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is α–confluent and that this constant α can be calculated effectively. By Theorem 4.1 this proves the theorem. First assume that $\mathcal{R}$ is α–confluent. Then $\mathcal{R}$ is locally confluent. Since $\mathit{overlap}(\mathcal{R}) \neq \emptyset$, by property (2) $\pi_\Gamma(\mathcal{R})$ must be terminating for some $\Gamma \neq \emptyset$. Thus, $\mathcal{R}$ must be terminating. Since $\mathcal{R}$ is locally confluent it follows that $\mathcal{R}$ is confluent.

Now assume that $\mathcal{R}$ is confluent. Consider a critical situation $(t_1, t, t_2) \in \mathit{Crit}(\mathcal{R})$. Thus, there exist rules $(l_1 \to r_1), (l_2 \to r_2) \in \mathcal{R}$ and traces p_i, q_i, w_i ($i \in \{1, 2\}$), and $s \neq 1$ such that $l_1 = p_1 s q_1$, $l_2 = p_2 s q_2$, $t = p_1 w_1 l_2 w_2 q_1 = p_2 w_2 l_1 w_1 q_2$, $t_1 = p_2 w_2 r_1 w_1 q_2$, $t_2 = p_1 w_1 r_2 w_2 q_1$, and $s \, I \, w_1 w_2$ (plus some other independencies that are not important in the following). The only factors that are unbounded in t are w_1 and w_2. Note that $s \in \mathit{overlap}(l_1, l_2)$.

First we claim that it is possible to construct effectively an $\alpha \geq 1$ that depends only one $p_1, p_2, q_1, q_2, r_1, r_2$, s (and that is thus independent of w_1 and w_2) such that the pair

$$(t_1, t_2) = (p_2 w_2 r_1 w_1 q_2, p_1 w_1 r_2 w_2 q_1) \in \mathit{CP}(\mathcal{R})$$

is α–confluent. We will denote this α by $\alpha(p_1, p_2, q_1, q_2, r_1, r_2, s)$. Since $\mathcal{R}$ is confluent there exist $\alpha_1, \alpha_2 \geq 0$ and a $u \in M$ such that

$$p_2 w_2 r_1 w_1 q_2 \to_{\mathcal{R}}^{\alpha_1} u \quad \text{and} \quad p_1 w_1 r_2 w_2 q_1 \to_{\mathcal{R}}^{\alpha_2} u. \tag{21}$$

We claim that an upper bound for α_1 and α_2, which is only dependent from p_1, p_2, q_1, q_2, r_1, and r_2, can be determined effectively. Let $\Gamma = \mathit{dep}(s)$. From $s \, I \, w_1 w_2$ it follows $\pi_\Gamma(w_1 w_2) = 1$. Thus (21) implies

$$\pi_\Gamma(p_2 r_1 q_2) \to_{\pi_\Gamma(\mathcal{R})}^{\alpha_1} \pi_\Gamma(u) \quad \text{and} \quad \pi_\Gamma(p_1 r_2 q_1) \to_{\pi_\Gamma(\mathcal{R})}^{\alpha_2} .$$

Since $\pi_\Gamma(\mathcal{R})$ is a terminating TRS, it is possible to construct all derivations in $\pi_\Gamma(\mathcal{R})$ that are emanating from $\pi_\Gamma(p_2 r_1 q_2)$ and $\pi_\Gamma(p_1 r_2 q_1)$, respectively. The length of the longest of these derivations is an upper bound for α_1 and α_2 which can be determined effectively. Let $\alpha(p_1, p_2, q_1, q_2, r_1, r_2, s)$ this number.

Now let

$$\alpha = \{\alpha(p_1, p_2, q_1, q_2, r_1, r_2, s) \mid (p_1 s q_1, r_1), (p_2 s q_2, r_2) \in \mathcal{R},\ p_1 \, I \, p_2,\ q_1 \, I \, q_2,\ s \neq 1\}.$$

From the previous discussion and Lemma 2.14 it follows that $\mathcal{R}$ is α–confluent. Furthermore, α can be determined effectively. ☐

Corollary 5.2. The following problem is decidable.
INPUT: A TRS $\mathcal{R}$ over a trace monoid $M = \mathbb{M}(\Sigma, I)$ such that there exists a clique covering $(\Sigma_1, \ldots, \Sigma_n)$ of (Σ, I^c) such that (as usual, $\pi_i = \pi_{\Sigma_i}$)

(1) for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and for all rules $(l \to r) \in \mathcal{R}$, if $\pi_i(\pi_j(l)) = 1$ then $\pi_i(\pi_j(r)) = 1$ and

(2) for all $i \in \{1, \ldots, n\}$, the semi–Thue system $\pi_i(\mathcal{R})$ is terminating.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. Let $\mathcal{R}$ be a TRS that satisfies conditions (1) and (2) of the corollary and let $(\Sigma_1, \ldots, \Sigma_n)$ be a clique covering of (Σ, I^c) with the stated properties. We will prove that $\mathcal{R}$ satisfies the input–conditions of Theorem 5.1.
Claim 1: $\mathcal{R}$ satisfies condition (A): Let $l \in \mathit{dom}(\mathcal{R})$. Condition (2) implies that for all $i \in \{1, \ldots, n\}$ it holds $\pi_i(l) \neq 1$ (otherwise $\pi_i(\mathcal{R})$ would contain a rule $1 \to \pi_i(r)$ and thus $\pi_i(\mathcal{R})$ would not be terminating). It follows $a \, I^c \, l$ for all $a \in \Sigma$. Thus, $\mathcal{R}$ satisfies condition (A1). It remains to verify condition (A2). Let $(l_1 \to r_1), (l_2 \to r_2) \in \mathcal{R}$ and let $l_1 = p_1 q_1$, $l_2 = p_2 q_2$ be factorizations with $p_j \neq 1 \neq q_j$ for $j \in \{1, 2\}$, $p_1 \, I \, p_2$, and $q_1 \, I \, q_2$. We have to construct factorizations $r_1 = s_1 t_1$, $r_2 = s_2 t_2$ such that $a \, I \, p_j$ implies $a \, I \, s_j$ and $a \, I \, q_j$ implies $a \, I \, t_j$ for all $a \in \Sigma$, $j \in \{1, 2\}$. Since $u \, I \, v$ implies $\pi_i(u) = 1$ or $\pi_i(v) = 1$ for all $i \in \{1, \ldots, n\}$, exactly one of the following two cases has to hold for every $i \in \{1, \ldots, n\}$ (note that $\pi_i(p_1 q_1) = \pi_i(l_1) \neq 1 \neq \pi_i(l_2) = \pi_i(p_2 q_2)$):

- Case 1: $\pi_i(p_1) \neq 1 \neq \pi_i(q_2)$ and $\pi_i(p_2) = 1 = \pi_i(q_1)$
- Case 2: $\pi_i(p_2) \neq 1 \neq \pi_i(q_1)$ and $\pi_i(p_1) = 1 = \pi_i(q_2)$

We define a factorization $\pi_i(r_1) = s^{(i)}t^{(i)}$ for every $i \in \{1, \dots, n\}$ as follows:

(1) $s^{(i)} = \pi_i(r_1)$ and $t^{(i)} = 1$ if case 1 holds for i.

(2) $s^{(i)} = 1$ and $t^{(i)} = \pi_i(r_1)$ if case 2 holds for i.

We claim that $(s^{(1)}, \dots, s^{(n)})$ and $(t^{(1)}, \dots, t^{(n)})$ are reconstructible. By Lemma 2.2 it suffices to prove $\pi_i(s^{(i)}) = \pi_j(s^{(i)})$ for all $i, j \in \{1, \dots, n\}$. If both i and j satisfy case 1 or both i and j satisfy case 2 this obviously holds. Thus, w.l.o.g assume that i satisfies case 1 and j satisfies case 2, i.e.,

$$\pi_i(p_1) \neq 1 \neq \pi_i(q_2),\ \pi_i(p_2) = 1 = \pi_i(q_1), \text{ and } \pi_j(p_2) \neq 1 \neq \pi_j(q_1),\ \pi_j(p_1) = 1 = \pi_j(q_2).$$

It follows $\pi_i(\pi_j(l_1)) = \pi_i(\pi_j(p_1q_1)) = 1$ and $\pi_i(\pi_j(l_2)) = \pi_i(\pi_j(p_2q_2)) = 1$. Thus, the second condition on $\mathcal{R}$ implies $\pi_i(\pi_j(r_1)) = 1 = \pi_i(\pi_j(r_2))$. Thus $\pi_i(s^j)) = \pi_i(1) = 1 = \pi_j(\pi_i(r_1)) = \pi_j(s^{(i)})$ also holds in this case and $(s^{(1)}, \dots, s^{(n)})$ and $(t^{(1)}, \dots, t^{(n)})$ are both reconstructible. Therefore there exist unique traces s_1, t_1 with $\pi_i(s_1) = s^{(i)}$ and $\pi_i(t_1) = t^{(i)}$ for all $i \in \{1, \dots, n\}$. It follows $r_1 = s_1t_1$ by Fact 2.1.

Now let $a\,I\,p_1$. Thus, for each $j \in \{1, \dots, n\}$ with $a \in \Sigma_j$ it holds $\pi_j(p_1) = 1$. The construction of $(s^{(1)}, \dots, s^{(n)})$ implies $s^{(j)} = 1$ for all $j \in \{1, \dots, n\}$ with $a \in \Sigma_j$. Thus $a\,I\,s_1$ (otherwise there would exist a clique Σ_j with $a \in \Sigma_j$ and $\pi_j(s_1) = s^{(j)} \neq 1$). Analogously it can be proven that $a\,I\,q_1$ implies $a\,I\,t_1$. Therefore we have constructed the desired factorization of r_1. The desired factorization of r_2 can be constructed analogously. The claim is now proven and we can proceed with the proof of the corollary.

Claim 2: $\pi_{dep(u)}(\mathcal{R})$ is terminating for every $u \in overlap(\mathcal{R})$: Let $u \in overlap(\mathcal{R})$ and assume that $\pi_{dep(u)}(\mathcal{R})$ is not terminating. Let Σ_i be a clique with $\Sigma_i \subseteq dep(u)$. Since $u \neq 1$ such a clique exists. But then $\pi_i(\mathcal{R})$ would be also non–terminating, which is a contradiction to condition (2). Thus $\pi_{dep(u)}(\mathcal{R})$ is terminating. □

Example 5.2. Let $\mathcal{R}$ be a TRS over a direct product $\prod_{i=1}^{n} \Sigma_i^*$ of free monoids. If the semi–Thue system $\pi_{\Sigma_i}(\mathcal{R})$ is terminating for every $i \in \{1, \dots, n\}$ then by Corollary 5.2 it can be decided whether $\mathcal{R}$ is confluent.

Let $\mathcal{R}$ be a special TRS over a trace monoid $\mathbb{M}(\Sigma, I)$ such that for some clique covering $(\Sigma_1, \dots, \Sigma_n)$ of (Σ, I^c) it holds $\pi_i(l) \neq 1$ for all $i \in \{1, \dots, n\}$ and all left–hand sides $l \in dom(\mathcal{R})$. Again by Corollary 5.2 it can be decided whether $\mathcal{R}$ is confluent.

A trace $u \in \mathbb{M}(\Sigma, I)$ is called connected if there does not exist a factorization $u = vw$ such that $v \neq 1 \neq w$ and $v\,I\,w$. The following statement is similar to Corollary 5.2.

Corollary 5.3. The following problem is decidable.
INPUT: A TRS $\mathcal{R}$ over a trace monoid $M = \mathbb{M}(\Sigma, I)$ such that

(1) all $l \in dom(\mathcal{R})$ are connected and

(2) for some clique covering $(\Sigma_1, \dots, \Sigma_n)$ of (Σ, I^c), the semi–Thue system $\pi_i(\mathcal{R})$ is terminating for all $i \in \{1, \dots, n\}$.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. The proof will follow the arguments of the proof of Corollary 5.2 Let $\mathcal{R}$ be a TRS that satisfies conditions (1) and (2) of the corollary and let $(\Sigma_1, \dots, \Sigma_n)$ be a clique covering of (Σ, I^c) with the stated properties. Again, for all $l \in dom(\mathcal{R})$ and all $i \in \{1, \dots, n\}$ it holds $\pi_i(l) \neq 1$ and $\mathcal{R}$ satisfies condition (A1). It suffices to prove that $\mathcal{R}$ satisfies condition (A2) (the rest of the proof is identical to the proof of claim 2 from the proof of Corollary 5.2). Assume that there exist rules $(l_1 \to r_1), (l_2 \to r_2) \in \mathcal{R}$ and factorizations $l_1 = p_1q_1$, $l_2 = p_2q_2$ with $p_i \neq 1 \neq q_i$ for

$i \in \{1,2\}$, $p_1 \, I \, p_2$, and $q_1 \, I \, q_2$. We will deduce a contradiction. Assume that $\pi_i(p_1) \neq 1$ for some $i \in \{1,\dots,n\}$. Since $p_1 \, I \, p_2$ it follows $\pi_i(p_2) = 1$. Because of $\pi_i(p_2q_2) = \pi_i(l_2) \neq 1$, we can deduce $\pi_i(q_2) \neq 1$, which implies $\pi_i(q_1) = 1$ because of $q_1 \, I \, q_2$. Thus, for all $i \in \{1,\dots,n\}$, if $\pi_i(p_1) \neq 1$ then $\pi_i(q_1) = 1$. Hence $p_1 \, I \, q_1$ (otherwise there would exist a clique Σ_j with $\pi_j(p_1) \neq 1 \neq \pi_j(q_1)$). Together with $p_1 \neq 1 \neq q_1$ this contradicts the fact that $l_1 = p_1q_1$ is connected. □

Example 5.3. Let M be the trace monoid $\mathbb{M}(\{a,b,c,d\}, \{(a,d),(d,a),(b,c),(c,b)\})$ with the unique clique covering $(\{a,b\},\{b,d\},\{c,d\},\{a,c\})$ and consider the TRS $\mathcal{R} = \{[b^2dc^2] \to [a]\}$. The trace $[b^2dc^2]$ is connected and the projections of $\mathcal{R}$ onto the four cliques are all length–reducing and thus terminating. Therefore by Corollary 5.3 it can be decided whether $\mathcal{R}$ is confluent. Note that Corollary 5.2 does not apply to $\mathcal{R}$ since we have $\pi_{\{a,b\}}(\pi_{\{a,c\}}([b^2dc^2])) = 1$ but $\pi_{\{a,b\}}(\pi_{\{a,c\}}([a])) \neq 1$.

6 Conclusion

In this paper we have shown that for the class of length–reducing trace rewriting systems over a given trace monoid M, confluence is decidable if and only if M is free or free commutative. Thus, we have located the borderline between decidability and undecidability for this problem in terms of the underlying trace monoid. In contrast to the proven undecidability results, the problem of being (α,β)–confluent ($\alpha,\beta \geq 1$) was shown to be decidable for trace rewriting systems. This result was used to present new sufficient criteria that imply the decidability of confluence for terminating trace rewriting systems.

The following points deserve further investigations. First, the complexity for deciding (α,β)–confluence should be analyzed. Interesting classes of trace rewriting system for which it is still an open question whether confluence is decidable and which should be investigated further include special systems, monadic systems (i.e. for all rules $l \to r$ it holds $|r| \leq 1$ and $|l| > |r|$, see [BO93] for the semi–Thue case) and one–rule systems. For the last class, confluence can be decided in almost all cases, see [WD95].

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