The submonoid and rational subset membership problems for graph groups

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Abstract. We show that the membership problem in a finitely generated submonoid of a graph group (also called a right-angled Artin group or a free partially commutative group) is decidable if and only if the independence graph (commutation graph) is a transitive forest. As a consequence we obtain the first example of a finitely presented group with a decidable generalized word problem that does not have a decidable membership problem for finitely generated submonoids. We also show that the rational subset membership problem is decidable for a graph group if and only if the independence graph is a transitive forest, answering a question of Kambites, Silva, and the second author [24]. Finally we prove that for certain amalgamated free products and HNN-extensions the rational subset and submonoid membership problems are recursively equivalent. In particular, this applies to finitely generated groups with two or more ends that are either torsion-free or residually finite.

1 Introduction

Algorithmic problems concerning groups are a classical topic in algebra and theoretical computer science. Since the pioneering work of Dehn from 1910 [8], decision problems like the word problem or the generalized word problem (which is also known as the subgroup membership problem since it asks whether one can decide if a given group element belongs to a given finitely generated subgroup) have been intensively studied for various classes of groups. A first natural generalization of these classical decision problems is the submonoid membership problem: given a finite set S of elements of G and an element $g \in G$, does g belong to the submonoid generated by S? Notice that g has finite order if and only if g^{-1} is in the submonoid generated by g and so decidability of the submonoid membership problem lets one determine algorithmically the order of an element of the group G. A recent paper on the submonoid membership problem is Margolis, Meakin, and Šunić [27].

A further generalization is the rational subset membership problem: for a given rational subset L of a group G and an element $g \in G$ it is asked whether $g \in L$. The class of rational subsets of a group G is the smallest class that contains all finite subsets of G, and which is closed under union, product, and the Kleene hull (or Kleene star; it associates to a subset $L \subseteq G$ the submonoid L^* generated by L). Equivalently,

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it consists of the all subsets of G recognizable by finite automata. Rational subsets in arbitrary groups and monoids are an important research topic in language theory, see, e.g., [3, 24, 31]. The rational subset membership problem generalizes the submonoid membership problem and the the generalized word problem for a group, because every finitely generated submonoid (and hence subgroup) of a group is rational.

It is easy to see that decidability of the rational subset membership problem transfers to finitely generated subgroups. Grunschlag has shown that the property of having a decidable rational subset membership problem is preserved under finite extensions, i.e., if G has a decidable rational subset membership problem and $G \leq H$, where the index of G in H is finite, then H also has a decidable rational subset membership problem [20]. Kambites, Silva, and the second author [24] proved that the fundamental group of a finite graph of groups [36] with finite edge groups has a decidable rational subset membership problem provided all vertex groups have a decidable rational subset membership problem. In particular, this implies that decidability of the rational subset membership problem is preserved by free products, see also [31].

The main result of this paper is to characterize the decidability of the submonoid membership problem and the rational subset membership problem for graph groups. In particular we provide the first example, as far as we know, of a group with a decidable generalized word problem that does not have a decidable submonoid (and hence rational subset) membership problem.

A *graph group* [13] $\mathbb{G}(\Sigma, I)$ is specified by a finite undirected graph (Σ, I), which is also called an *independence alphabet* (or *commutation graph*). The graph group $\mathbb{G}(\Sigma, I)$ is formally defined as the quotient group of the free group generated by Σ modulo the set of all relations $ab = ba$, where $(a, b) \in I$. Graph groups are a group analogue to trace monoids (free partially commutative monoids), which play a prominent role in concurrency theory [12]. Graph groups are also called *free partially commutative groups* [10, 40], *right-angled Artin groups* [6, 7], and *semifree groups* [2]. They are currently a hot topic of interest in group theory, in particular because of the richness of the class of groups embeddable in graph groups. For instance, the Bestvina-Brady groups, which were used to distinguish the finiteness properties $\mathcal{F}_n$ and FP_n [4] (and were also essential for distinguishing the finiteness properties FDT and FHT for string rewriting systems [33]), are subgroups of graph groups. Crisp and Wiest show that the fundamental group of any orientable surface (and of most non-orientable surfaces) embeds in a graph group [7]. Another class of groups that embed into graph groups are fundamental groups of finite state complexes [16].

Algorithmic problems concerning graph groups have been intensively studied in the past, see, e.g., [10, 11, 15, 24, 25, 40]. In [10, 40] it was shown that the word problem for a graph group can be decided in linear time (on a random access machine). A recent result of Kapovich, Weidmann, and Myasnikov [25] shows that if (Σ, I) is a chordal graph (i.e., if (Σ, I) does not have an induced cycle of length at least 4), then the generalized word problem for $\mathbb{G}(\Sigma, I)$ is decidable. On the other hand, a classical result of Mihailova [30] states that already the generalized word problem for the direct product of two free groups of rank 2 is undecidable. Note that this group is the graph group $\mathbb{G}(\Sigma, I)$, where the graph (Σ, I) is a cycle on 4 nodes (also called $\mathsf{C4}$). In fact, Mihailova proves a stronger result: she constructs a *fixed* subgroup H of $\mathbb{G}(\mathsf{C4})$ such that it is undecidable, whether a given element of $\mathbb{G}(\mathsf{C4})$ belongs to H. Recently, it was shown by Kambites that a graph group $\mathbb{G}(\Sigma, I)$ contains a direct product of two free groups of rank 2 if and only if (Σ, I) contains an induced $\mathsf{C4}$ [23]. This leaves a gap between the decidability result of [25] and the undecidability result of Mihailova [30].

In [24] it is shown that the rational subset membership problem is decidable for a free product of direct products of a free group with a free Abelian group. Such a group is a graph group $\mathbb{G}(\Sigma, I)$, where every connected component of (Σ, I) results from connecting all nodes of a clique with all nodes from an edge-free graph. On the other hand, the only undecidability result for the rational subset membership problem for graph groups that was known so far is Mihailova's result for independence alphabets containing an induced $\mathsf{C4}$.

In this paper, we shall characterize those graph groups for which the rational subset membership problem is decidable: we prove that these are exactly those graph groups $\mathbb{G}(\Sigma, I)$, where (Σ, I) is a transitive forest (Theorem 2). The graph (Σ, I) is a transitive forest if it is the disjoint union of comparability graphs of rooted trees. An alternative characterization of transitive forests was presented in [39]: (Σ, I) is a transitive forest if and only if it neither contains an induced $\mathsf{C4}$ nor an induced path on 4 nodes (also called $\mathsf{P4}$). Graph groups $\mathbb{G}(\Sigma, I)$, where (Σ, I) is a transitive forest, have also appeared in [28]: they are exactly those graph groups which are subgroup separable (the case of $\mathsf{P4}$ appears in [32]). Recall that a group G is called subgroup separable if, for every finitely generated subgroup $H \leq G$ and every $g \in G \setminus H$ there exists a normal subgroup $N \leq G$ having finite index such that $g \notin NH$. Subgroup separability implies decidability of the generalized word problem.

One half of Theorem 2 can be easily obtained from a result of Aalbersberg and Hoogeboom [1]: The problem of deciding whether the intersection of two rational subsets of the trace monoid (free partially commutative monoid) $\mathbb{M}(\Sigma, I)$ is nonempty is decidable if and only if (Σ, I) is a transitive forest. Now, $L \cap K \neq \emptyset$ for two given rational subsets $L, K \subseteq \mathbb{M}(\Sigma, I)$ if and only if $1 \in LK^{-1}$ in the graph group $\mathbb{G}(\Sigma, I)$. Hence, if (Σ, I) is not a transitive forest, then the rational subset membership problem for $\mathbb{G}(\Sigma, I)$ is undecidable. In fact, we construct a fixed rational subset $L \subseteq \mathbb{G}(\Sigma, I)$ such that it is undecidable whether $g \in L$ for a given group element $g \in \mathbb{G}(\Sigma, I)$.

The converse direction in Theorem 2 is an immediate corollary of our Theorem 1, which is one of the main group theoretic results of this paper. It states that the rational subset membership problem is decidable for every group that can be built up from the trivial group using the following four operations: (i) taking finitely generated subgroups, (ii) finite extensions, (iii) direct products with $\mathbb{Z}$, and (iv) finite graphs of groups with finite edge groups. Note that the only operation that is not covered by the results cited earlier is the direct product with $\mathbb{Z}$. In fact, it seems to be an open question whether decidability of the rational subset membership problem is preserved under direct products with $\mathbb{Z}$. Hence, we have to follow another strategy. We will introduce a property of groups that implies the decidability of the rational subset membership problem, and which has all the desired closure properties. Our proof of Theorem 1 uses mainly techniques from formal language theory (e.g., semilinear sets, Parikh's theorem) and is inspired by the methods from [1, 5].

It should be noted that due to the above reduction from the intersection problem for rational trace languages to the rational subset membership problem for the corresponding graph group, we also obtain an alternative to the quite difficult proof from [1] for the implication "(Σ, I) is a transitive forest $\Rightarrow$ intersection problem for rational subsets of $\mathbb{M}(\Sigma, I)$ is decidable".

In Section 4 we consider the *submonoid membership problem* for groups. We prove that for an amalgamated free product $G *_A H$ such that A is a finite proper subgroup of G and H and there exist $g \in G$, $h \in H$ with $g^{-1}Ag \cap A = 1 = h^{-1}Ah \cap A$, the rational subset membership problem is recursively equivalent to the submonoid membership problem (Theorem 4). An analogous result is proved for certain HNN extensions with finite associated subgroups. As a consequence we obtain that the rational subset membership problem is recursively equivalent to the submonoid membership problem for a group with two or more ends that is either torsion-free or residually finite (Corollary 2). Using similar techniques, we are also able to prove that the submonoid membership problem is undecidable for the graph group $\mathbb{G}(\Sigma, I)$, where (Σ, I) is $\mathsf{P4}$ (Theorem 7). The result of [25] shows that this graph group does have a decidable generalized word problem, thereby giving our example of a group with a decidable generalized word problem but an undecidable submonoid membership problem. Together with Mihailova's undecidability result for $\mathsf{C4}$ and our decidability result for transitive forests (Theorem 2) it also follows that the submonoid membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if (Σ, I) is a transitive forest (Corollary 3).

Another consequence of our results is that the rational subset membership problem for groups is recursively equivalent to the submonoid membership problem if and only if a free product of groups with decidable submonoid membership problems has a decidable submonoid membership problem.

2 Preliminaries

We assume that the reader has some basic knowledge in formal language theory (see, e.g., [3, 22]) and group theory (see, e.g., [26, 35]).

2.1 Formal languages

Let Σ be a finite alphabet. We use $\Sigma^{-1} = \{a^{-1} \mid a \in \Sigma\}$ to denote a disjoint copy of Σ. Let $\Sigma^{\pm 1} = \Sigma \cup \Sigma^{-1}$. Define $(a^{-1})^{-1} = a$; this defines an involution ${}^{-1} : \Sigma^{\pm 1} \to \Sigma^{\pm 1}$, which can be extended to the free monoid $(\Sigma^{\pm 1})^*$ by setting $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$. For a word $w \in \Sigma^*$ and $a \in \Sigma$ we denote by $|w|_a$ the number of occurrences of a in w. For a subset $\Gamma \subseteq \Sigma$, we denote by $\pi_\Gamma(w)$ the projection of the word w to the alphabet Γ, i.e., we erase in w all symbols from $\Sigma \setminus \Gamma$.

Let $\mathbb{N}^\Sigma$ be the set of all mappings from Σ to $\mathbb{N}$. By fixing an arbitrary linear order on the alphabet Σ, we may identify a mapping $f \in \mathbb{N}^\Sigma$ with a tuple from $\mathbb{N}^{|\Sigma|}$. For a word $w \in \Sigma^*$, the *Parikh image* $\Psi(w)$ is defined as the mapping $\Psi(w) : \Sigma \to \mathbb{N}$ such that $[\Psi(w)](a) = |w|_a$ for all $a \in \Sigma$. For a language $L \subseteq \Sigma^*$, the Parikh image is $\Psi(L) = \{\Psi(w) \mid w \in L\}$. For a set $K \subseteq \mathbb{N}^\Sigma$ and $\Gamma \subseteq \Sigma$ let $\overline{\pi}_\Gamma(K) = \{f{\restriction}_\Gamma \in \mathbb{N}^\Gamma \mid f \in K\}$, where $f{\restriction}_\Gamma$ denotes the restriction of f to Γ. We also need a notation for the composition of erasing letters and taking the Parikh image. So, for $L \subseteq \Sigma^*$ and $\Gamma \subseteq \Sigma$, let $\Psi_\Gamma(L) = \overline{\pi}_\Gamma(\Psi(L))(= \Psi(\pi_\Gamma(L)))$; it may be viewed as a subset of $\mathbb{N}^{|\Gamma|}$. A special case occurs when $\Gamma = \emptyset$. Then either $\Psi_\emptyset(L) = \emptyset$ (if $L = \emptyset$) or $\Psi_\emptyset(L)$ is the singleton set consisting of the unique mapping from $\emptyset$ to $\mathbb{N}$.

A subset $K \subseteq \mathbb{N}^k$ is said to be *linear* if there are $x, x_1, \ldots, x_\ell \in \mathbb{N}^k$ such that $K = \{x + \lambda_1 x_1 + \cdots + \lambda_\ell x_\ell \mid \lambda_1, \ldots, \lambda_\ell \in \mathbb{N}\}$, i.e. K is a translate of a finitely generated submonoid of $\mathbb{N}^k$. A *semilinear* set is a finite union of linear sets.

Let $\mathsf{G} = (N, \Gamma, S, P)$ be a context-free grammar, where N is the set of nonterminals, Γ is the terminal alphabet, $S \in N$ is the start nonterminal, and $P \subseteq N \times (N \cup \Gamma)^*$ is the finite set of productions. For $u, v \in (N \cup \Gamma)^*$ we write $u \Rightarrow_{\mathsf{G}} v$ if v can be derived from u by applying a production from P. For $A \in N$, we define $L(\mathsf{G}, A) = \{w \in \Gamma^* \mid A \overset{*}{\Rightarrow}_{\mathsf{G}} w\}$ and $L(\mathsf{G}) = L(\mathsf{G}, S)$. Parikh's theorem states that the Parikh image of a context-free language is semilinear [34].

We will allow a more general form of productions in context-free grammars, where the right-hand side of a production is a regular language over the alphabet $N \cup \Gamma$. Such a production $A \to L$ represents the (possibly infinite) set of productions $\{A \to s \mid s \in L\}$. Clearly, such an extended context-free grammar can be transformed effectively into an equivalent context-free grammar with only finitely many productions.

Let M be a monoid. The set $\mathrm{RAT}(M)$ of all *rational subsets* of M is the smallest subset of 2^M, which contains all finite subsets of M, and which is closed under union, product, and Kleene hull (the Kleene hull L^* of a subset $L \subseteq M$ is the submonoid of M generated by L). By Kleene's theorem, a subset $L \subseteq \Sigma^*$ is rational if and only if L can be recognized by a finite automaton. If M is generated by the finite set Σ and $h : \Sigma^* \to M$ is the corresponding canonical monoid homomorphism, then $L \in \mathrm{RAT}(M)$ if and only if $L = h(K)$ for some $K \in \mathrm{RAT}(\Sigma^*)$. In this case, L can be specified by a finite automaton over the alphabet Σ. The rational subsets of the free commutative monoid $\mathbb{N}^k$ are exactly the semilinear subsets of $\mathbb{N}^k$ [14].

2.2 Groups

Let G be a finitely generated group and let Σ be a finite group generating set for G. Hence, $\Sigma^{\pm 1}$ is a finite monoid generating set for G and there exists a canonical monoid homomorphism $h : (\Sigma^{\pm 1})^* \to G$. The language

$$\mathrm{WP}_\Sigma(G) = h^{-1}(1)$$

is called the *word problem* of G with respect to Σ, i.e., $\mathrm{WP}_\Sigma(G)$ consists of all words over the alphabet $\Sigma^{\pm 1}$ which are equal to 1 in the group G. It is well known and easy to see that if Γ is another finite generating set for G, then $\mathrm{WP}_\Sigma(G)$ is decidable if and only if $\mathrm{WP}_\Gamma(G)$ is decidable.

The *submonoid membership problem* for G is the following decision problem:

INPUT: A finite set of words $\Delta \subseteq (\Sigma^{\pm 1})^*$ and a word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in h(\Delta^*)$?

Note that the subset $h(\Delta^*) \subseteq G$ is the submonoid of G generated by $h(\Delta) \subseteq G$. If we replace in the submonoid membership problem the finitely generated submonoid

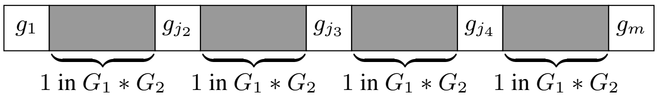

Fig. 1. Case (3) in Lemma 1: we have $g_1 g_{j_2} g_{j_3} g_{j_4} g_m = 1$ in either G_1 or G_2

$h(\Delta^*)$ by the finitely generated subgroup $h((\Delta \cup \Delta^{-1})^*)$, then we obtain the *subgroup membership problem*, which is also known as the *generalized word problem* for G. This term is justified, since the word problem is a particular instance, namely with $\Delta = \emptyset$. A generalization of the submonoid membership problem for G is the *rational subset membership problem*:

INPUT: A finite automaton A over the alphabet $\Sigma^{\pm 1}$ and a word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in h(L(A))$?

Note that $h(w) \in h(L(A))$ if and only if $1 \in h(w^{-1}L(A))$. Since $w^{-1}L(A)$ is again a rational language, the rational subset membership problem for G is recursively equivalent to the decision problem of asking whether $1 \in h(L(A))$ for a given finite automaton A over the alphabet $\Sigma^{\pm 1}$. It should be noted that for all the computational problems introduced above the decidability is independent of the chosen generating set for G.

In the rational subset (resp. submonoid) membership problem, the rational subset (resp. submonoid) is part of the input. Non-uniform variants of these problems, where the rational subset (resp. submonoid) is fixed, have been studied as well. More generally, we can define for a subset $S \subseteq G$ the *membership problem for S within G*:

INPUT: A word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in S$?

The *free group $F(\Sigma)$* generated by Σ can be defined as the quotient monoid

$$F(\Sigma) = (\Sigma^{\pm 1})^* / \{aa^{-1} = \varepsilon \mid a \in \Sigma^{\pm 1}\}.$$

As usual, the *free product* of two groups G_1 and G_2 is denoted by $G_1 * G_2$. We will always assume that $G_1 \cap G_2 = \emptyset$. An *alternating word* in $G_1 * G_2$ is a sequence $g_1 g_2 \cdots g_m$ with $m \geq 0$, $g_i \in G_1 \cup G_2$, and $g_i \in G_1 \Leftrightarrow g_{i+1} \in G_2$. Its length is m. The alternating word $g_1 g_2 \cdots g_m$ is *irreducible* if $g_i \neq 1$ for every $1 \leq i \leq m$. Every element of $G_1 * G_2$ can be written uniquely as an alternating irreducible word. We will need the following simple fact about free products:

Lemma 1. *Let $g_1 g_2 \cdots g_m$ be an alternating word in $G_1 * G_2$. If $g_1 g_2 \cdots g_m = 1$ in $G_1 * G_2$, then one of the following three cases holds:*

(1) $m \leq 1$
(2) there exists $1 \leq i < m$ such that $g_1 g_2 \cdots g_i = g_{i+1} \cdots g_m = 1$ in $G_1 * G_2$
(3) there exist $i \in \{1, 2\}$, $k \geq 2$, and $1 = j_1 < j_2 < \cdots < j_k = m$ such that $g_{j_1}, g_{j_2}, \ldots, g_{j_k} \in G_i$, $g_{j_1} g_{j_2} \cdots g_{j_k} = 1$ in G_i, and $g_{j_\ell+1} g_{j_\ell+2} \cdots g_{j_{\ell+1}-1} = 1$ in $G_1 * G_2$ for all $1 \leq \ell < k$.

Proof. Case (3) from the Lemma is visualized in Figure 1 for $k = 5$. Shaded areas represent alternating sequences, which are equal to 1 in $G_1 * G_2$. The non-shaded blocks are either all from G_1 or from G_2, and their product equals 1 in G_1 or G_2, respectively.

We prove the lemma by induction on m, the case $m \leq 1$ being trivial. So assume that $m \geq 2$. Since $g_1 g_2 \cdots g_m = 1$ in $G_1 * G_2$, there must exist $1 \leq j \leq m$ with $g_j = 1$. If $j = 1$ or $j = m$, then we are in case (2) from the lemma. Hence, we may assume that $m \geq 3$ and that $2 \leq j \leq m - 1$. It follows

$$g_1 \cdots g_{j-2}(g_{j-1}g_{j+1})g_{j+2} \cdots g_m = 1$$

in $G_1 * G_2$. Since the alternating word $g_1 \cdots g_{j-2}(g_{j-1}g_{j+1})g_{j+2} \cdots g_m$ has length $m - 2$, we can apply the induction hypothesis to it. If $m - 2 = 1$, i.e., $m = 3$, then we obtain case (3) from the lemma (with $k = 2$, $j_1 = 1$, and $j_2 = 3$). If a non-empty and proper prefix of $g_1 \cdots g_{j-2}(g_{j-1}g_{j+1})g_{j+2} \cdots g_m$ equals 1 in the group $G_1 * G_2$, then the same is true for $g_1 g_2 \cdots g_m$. Finally, if case (3) from the lemma applies to the alternating word $g_1 \cdots g_{j-2}(g_{j-1}g_{j+1})g_{j+2} \cdots g_m$, then again the same is true for $g_1 g_2 \cdots g_m$. □

Notice that (3) in Lemma 1 can only occur when m is odd.

Assume that $A \leq G$ and $B \leq H$ are groups and $\varphi : A \to B$ is an isomorphism. The *amalgamated free product* $G *_\varphi H$ is the quotient

$$(G * H)/\{a = \varphi(a) \mid a \in A\}.$$

Without loss of generality we may assume that $A = G \cap H$ and that φ is the identity map on A; in this situation we briefly write $G *_A H$ for $G *_\varphi H$. Every element of $G *_A H$ can be written as a word $c_1 \cdots c_n$, where $n \geq 0$, $c_1, \ldots, c_n \in G \cup H$, if $n > 1$ then $c_1, \ldots, c_n \in (G \cup H) \setminus A$, if $n = 1$ then $c_1 \neq 1$, and $c_i \in G \setminus A \Leftrightarrow c_{i+1} \in H \setminus A$ for all $1 \leq i < n$. Such a word is called a *reduced sequence*. The normal form theorem for amalgamated free products states that every nonempty reduced sequence represents a nontrivial element of $G *_A H$ [26, Chapter IV, Theorem 2.6].

If G is a group and $\varphi : A \to B$ is an isomorphism between subgroups A, B of G, then the *HNN extension* $*_\varphi G$, with base G, stable letter t, and associated subgroups A, B is the quotient group

$$G * \langle t \rangle / \{t^{-1}at = \varphi(a) \mid a \in A\}$$

where t is the generator of an infinite cyclic group. Every element of $*_\varphi G$ can be written as a word $g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$, where $n \geq 0$, $g_0, \ldots, g_n \in G$, and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$. Such a word is referred to as a *reduced sequence* if it contains no factor of the form $t^{-1}at$ or tbt^{-1} with $a \in A$, respectively $b \in B$. Britton's Lemma [26, Chapter IV] says that if $w = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$ is a reduced sequence with $n \geq 1$, then w represents a nontrivial element of $*_\varphi G$.

We will also consider fundamental groups of finite graphs of groups, which is a group theoretic construction generalizing free products, free products with amalgamation, and HNN-extensions, see e.g. [36]. We omit the quite technical definition. In order to deal with the rational subset membership problem for graph groups, free products suffice.

2.3 Trace monoids and graph groups

In the following we introduce some notions from trace theory, see [9, 12] for more details. An *independence alphabet* is just a finite undirected graph (Σ, I) without loops. Hence, $I \subseteq \Sigma \times \Sigma$ is an irreflexive and symmetric relation. The *trace monoid* $\mathbb{M}(\Sigma, I)$ is defined as the quotient

$$\mathbb{M}(\Sigma, I) = \Sigma^*/\{ab = ba \mid (a, b) \in I\}.$$

Elements of $\mathbb{M}(\Sigma, I)$ are called *traces*. Note that $\mathbb{M}(\Gamma, J)$ is a submonoid of $\mathbb{M}(\Sigma, I)$ in case (Γ, J) is an *induced subgraph* of (Σ, I). The latter means that $\Gamma \subseteq \Sigma$ and $J = I \cap (\Gamma \times \Gamma)$.

Traces can be represented conveniently by *dependence graphs*, which are node-labelled directed acyclic graphs. Let $u = a_1 \cdots a_n$ be a word, where $a_i \in \Sigma$. The vertex set of the dependence graph of u is $\{1, \ldots, n\}$ and vertex i is labelled with $a_i \in \Sigma$. There is an edge from vertex i to j if and only if $i < j$ and $(a_i, a_j) \notin I$. Then, two words define the same trace in $\mathbb{M}(\Sigma, I)$ if and only if their dependence graphs are isomorphic. The set of minimal (resp. maximal) elements of a trace $t \in \mathbb{M}(\Sigma, I)$ is $\min(t) = \{a \in \Sigma \mid \exists u \in \mathbb{M}(\Sigma, I) : t = au\}$ (resp. $\max(t) = \{a \in \Sigma \mid \exists u \in \mathbb{M}(\Sigma, I) : t = ua\}$). A *trace rewriting system* R over $\mathbb{M}(\Sigma, I)$ is just a finite subset of $\mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ [9]. We can define the *one-step rewrite relation* $\rightarrow_R \subseteq \mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ by: $x \rightarrow_R y$ if and only if there are $u, v \in \mathbb{M}(\Sigma, I)$ and $(\ell, r) \in R$ such that $x = u\ell v$ and $y = urv$ in $\mathbb{M}(\Sigma, I)$. A trace t is *irreducible* with respect to R if there does not exist a trace u with $t \rightarrow_R u$.

The *graph group* $\mathbb{G}(\Sigma, I)$ is defined as the quotient

$$\mathbb{G}(\Sigma, I) = F(\Sigma)/\{ab = ba \mid (a, b) \in I\}.$$

If (Σ, I) is the empty graph, i.e., $\Sigma = \emptyset$, then we set $\mathbb{M}(\Sigma, I) = \mathbb{G}(\Sigma, I) = 1$ (the trivial group). Note that $(a, b) \in I$ implies $a^{-1}b = ba^{-1}$ in $\mathbb{G}(\Sigma, I)$. Thus, the graph group $\mathbb{G}(\Sigma, I)$ can be also defined as the quotient

$$\mathbb{G}(\Sigma, I) = \mathbb{M}(\Sigma^{\pm 1}, I)/\{aa^{-1} = \varepsilon \mid a \in \Sigma^{\pm 1}\}.$$

Here, we implicitly extend $I \subseteq \Sigma \times \Sigma$ to $I \subseteq \Sigma^{\pm 1} \times \Sigma^{\pm 1}$ by setting $(a^\alpha, b^\beta) \in I$ if and only if $(a, b) \in I$ for $a, b \in \Sigma$ and $\alpha, \beta \in \{1, -1\}$. Note that $\mathbb{M}(\Sigma, I)$ is a rational subset of $\mathbb{G}(\Sigma, I)$.

Define a trace rewriting system R over $\mathbb{M}(\Sigma^{\pm 1}, I)$ as follows:

$$R = \{(aa^{-1}, \varepsilon) \mid a \in \Sigma^{\pm 1}\}. \tag{1}$$

One can show that for every trace $t \in \mathbb{M}(\Sigma^{\pm 1}, I)$, there exists a unique *normal form* $\mathrm{NF}_R(t)$ such that $t \xrightarrow{*}_R \mathrm{NF}_R(t)$ and $\mathrm{NF}_R(t)$ is irreducible with respect to R. Moreover, for all $u, v \in \mathbb{M}(\Sigma^{\pm 1}, I)$, $u = v$ in $\mathbb{G}(\Sigma, I)$ if and only if $\mathrm{NF}_R(u) = \mathrm{NF}_R(v)$ (in $\mathbb{M}(\Sigma^{\pm 1}, I)$) [10]. This leads to a linear time solution for the word problem of $\mathbb{G}(\Sigma, I)$ [10, 40].

If the graph (Σ, I) is the disjoint union of two graphs (Σ_1, I_1) and (Σ_2, I_2), then $\mathbb{G}(\Sigma, I) = \mathbb{G}(\Sigma_1, I_1) * \mathbb{G}(\Sigma_2, I_2)$. If (Σ, I) is obtained from (Σ_1, I_1) and (Σ_2, I_2) by

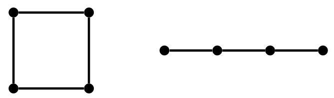

Fig. 2. The graphs C4 and P4

connecting each element of Σ_1 to each element of Σ_2, then $\mathbb{G}(\Sigma, I) = \mathbb{G}(\Sigma_1, I_1) \times \mathbb{G}(\Sigma_2, I_2)$. Graph groups were studied e.g. in [13]; they are also known as *free partially commutative groups* [10, 40], *right-angled Artin groups* [6, 7], and *semifree groups* [2].

A *transitive forest* is an independence alphabet (Σ, I) such that there exists a forest F of rooted trees (i.e., a disjoint union of rooted trees) with node set Σ and such that for all $a, b \in \Sigma$ with $a \neq b$: $(a, b) \in I$ if and only if a and b are comparable in F (i.e., either a is a proper descendant of b or b is a proper descendant of a). It can be shown that (Σ, I) is a transitive forest if and only if (Σ, I) does not contain an induced subgraph, which is a cycle on 4 nodes (also called C4, see Figure 2 on the left) or a simple path on 4 nodes (also called P4, see Figure 2 on the right) [39]. The next lemma follows easily by induction. We sketch the proof.

Lemma 2. *The class C of all groups, which are of the form $\mathbb{G}(\Sigma, I)$ for a transitive forest (Σ, I), is the smallest class such that:*

(1) $1 \in C$
(2) if $G_1, G_2 \in C$*, then also* $G_1 * G_2 \in C$
(3) if $G \in C$ *then* $G \times \mathbb{Z} \in C$

Proof. First we verify that graphs groups associated to transitive forests satisfy (1)-(3). Case (1) results from the empty graph. It is immediate that transitive forests are closed under disjoint union, which implies (2). If F is a forest of rooted trees, then one can obtain a rooted tree by adding a new root whose children are the roots of the trees from F. On the group level this corresponds to (3).

For the converse, we proceed by induction on the number of vertices. If the forest (Σ, I) consists of more than one rooted tree, then $\mathbb{G}(\Sigma, I)$ is the free product of the graph groups associated to the various rooted trees in (Σ, I), all of which have a smaller number of vertices. If there is a single tree, then in (Σ, I) the root is connected to every other vertex. Thus $\mathbb{G}(\Sigma, I) = G \times \mathbb{Z}$ where G is the graph group corresponding to the transitive forest obtained by removing the vertex corresponding to the root and making its children the roots of the trees in the forest so obtained. $\square$

Of course, a similar statement is true for trace monoids of the form $\mathbb{M}(\Sigma, I)$ with (Σ, I) a transitive forest; one just has to replace in (3) the group $\mathbb{Z}$ by the monoid $\mathbb{N}$.

3 The rational subset membership problem

Let $\mathcal{C}$ be the smallest class of groups such that:

– the trivial group 1 belongs to $\mathcal{C}$

– if $G \in \mathcal{C}$ and $H \leq G$ is finitely generated, then also $H \in \mathcal{C}$
– if $G \in \mathcal{C}$ and $G \leq H$ such that G has finite index in H (i.e., H is a finite extension of G), then also $H \in \mathcal{C}$
– if $G \in \mathcal{C}$, then also $G \times \mathbb{Z} \in \mathcal{C}$
– if $\mathbb{A}$ is a finite graph of groups [36] whose edge groups are finite and whose vertex groups belong to $\mathcal{C}$, then the fundamental group of $\mathbb{A}$ belongs to $\mathcal{C}$ (in particular, the class $\mathcal{C}$ is closed under free products).

This last property is equivalent to saying that $\mathcal{C}$ is closed under taking amalgamated products over finite groups and HNN-extensions with finite associated subgroups [36]. The main result in this section is:

Theorem 1. *For every group $G \in \mathcal{C}$, the rational subset membership problem is decidable.*

It is well known that decidability of the rational subset membership problem is preserved under taking finitely generated subgroups and finite extensions [20]. Moreover, the decidability of the rational subset membership problem is preserved by graph of group constructions with finite edge groups [24]. Hence, in order to prove Theorem 1, it would suffice to show that the decidability of the rational subset membership problem is preserved under direct products by $\mathbb{Z}$. But currently we can neither prove nor disprove this. This forces us to adopt an alternate strategy: we will introduce an abstract property of groups that implies the decidability of the rational subset membership problem, and which has the desired closure properties.

Let $\mathcal{L}$ be a class of formal languages closed under inverse homomorphism. A finitely generated group G is said to be an *$\mathcal{L}$-group* if $\mathrm{WP}_{\Sigma}(G)$ belongs to $\mathcal{L}$ for some finite generating set Σ. This notion is independent of the choice of generating set [17, 21, 24].

A language $L_0 \subseteq \Sigma^*$ belongs to the class *RID* (rational intersection decidable) if there is an algorithm that, given a finite automaton over Σ recognizing a rational language L, can determine whether $L_0 \cap L \neq \emptyset$. It was shown in [24] that the class RID is closed under inverse homomorphism and that a group G has a decidable rational subset membership problem if and only if it is an RID-group. This follows from the fact that if L is a rational subset of a group G, then $g \in L$ if and only if $1 \in g^{-1}L$ and that $g^{-1}L$ is again a rational subset.

Let $K \subseteq \Theta^*$ be a language over an alphabet Θ. Then K belongs to the class SLI (semilinear intersection) if, for every finite alphabet Γ (disjoint from Θ) and every rational language $L \subseteq (\Theta \cup \Gamma)^*$, the set

$$\Psi_{\Gamma}(\{w \in L \mid \pi_{\Theta}(w) \in K\}) = \Psi_{\Gamma}(L \cap \pi_{\Theta}^{-1}(K)) \tag{2}$$

is semilinear, and the tuples in a semilinear representation of this set can be effectively computed from Γ and a finite automaton for L. This latter effectiveness statement will be always satisfied throughout the paper, and we shall not explicitly check it. In words, the set (2) is obtained by first taking those words from L that project into K when Γ-letters are erased, and then erasing the Θ-letters, followed by taking the Parikh image.

In a moment, we shall see that the class SLI is closed under inverse homomorphism, hence the class of SLI-groups is well defined. In fact, we show more generally that the class SLI is closed under inverse images by sequential functions [3]. This will imply, moreover, that the class of SLI-groups is closed under taking finite extensions [17, 21, 24].

A *sequential transducer* A with input alphabet Σ and output alphabet Ω can be defined as a finite state automaton with transitions labeled by elements from the set $\Sigma \times \Omega^*$ such that the following restriction is satisfied: If there are states q, q_1, q_2 and a transition from q to q_i (for $i \in \{1, 2\}$) with label $(a, w_i) \in \Sigma \times \Omega^*$ then $q_1 = q_2$ and $w_1 = w_2$. This is not the standard definition of a sequential transducer (see e.g. [3]), but it is easily seen to be equivalent. The language defined by A is a relation $R \subseteq \Sigma^* \times \Omega^*$, and it is easy to see that R is the graph of a partial function $f : \Sigma^* \to \Omega^*$. A *sequential function* is a partial function, which is computed by a sequential transducer.

Lemma 3. *Let $K \subseteq \Theta^*$ belong to SLI and let $f : \Sigma^* \to \Theta^*$ be a sequential function. Then $f^{-1}(K)$ belongs to SLI. In particular, the class of SLI-groups is well defined and is closed under taking finite extensions.*

Proof. Let Γ be an alphabet disjoint from Σ and let L be a rational subset of $(\Gamma \cup \Sigma)^*$. Let A be a sequential transducer computing the sequential function $f : \Sigma^* \to \Omega^*$. Define a transducer A' by adding to each state of A a loop with label (a, a) for each $a \in \Gamma$. Clearly, A' is a sequential transducer, which computes a sequential function $F : (\Gamma \cup \Sigma)^* \to (\Gamma \cup \Theta)^*$.

The following two observations are immediate from the fact that the only transitions of A' involving letters from Γ are loops with labels of the form (a, a):

(a) $\Psi_\Gamma F$ coincides with Ψ_Γ on the domain of F (we read the composition of functions from right to left, i.e., in $\Psi_\Gamma F$ we first apply F, followed by Ψ_Γ)
(b) $\pi_\Theta F = f \pi_\Sigma$.

We now claim that the following equality holds:

$$F(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = F(L) \cap \pi_\Theta^{-1}(K). \tag{3}$$

First note that $L \cap \pi_\Sigma^{-1}(f^{-1}(K)) = L \cap F^{-1}(\pi_\Theta^{-1}(K))$ by (b). So if w belongs to the left hand side of (3), then $w = F(u)$ with $u \in L \cap F^{-1}(\pi_\Theta^{-1}(K))$. Thus $w \in F(L) \cap \pi_\Theta^{-1}(K)$. Conversely, if $u \in F(L) \cap \pi_\Theta^{-1}(K)$, then there exists $w \in L$ such that $F(w) = u$. But then $w \in L \cap F^{-1}(\pi_\Theta^{-1}(K)) = L \cap \pi_\Sigma^{-1}(f^{-1}(K))$ and so u belongs to the left hand side of (3).

Now, since $L \cap \pi_\Sigma^{-1}(f^{-1}(K)) = L \cap F^{-1}(\pi_\Theta^{-1}(K))$ is contained in the domain of F, we may conclude from (a) and (3) that

$$\Psi_\Gamma(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = \Psi_\Gamma F(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = \Psi_\Gamma(F(L) \cap \pi_\Theta^{-1}(K)). \tag{4}$$

But $F(L)$ is rational since the class of rational languages is closed under images via sequential functions [3]. Therefore, since K belongs to SLI, we may deduce that the Parikh-image $\Psi_\Gamma(F(L) \cap \pi_\Theta^{-1}(K))$ is semilinear. This completes the proof of the first statement from the lemma in light on (4).

Since a homomorphism is a sequential function, the language class SLI is closed under inverse homomorphism. Hence, the class of SLI-groups is well defined. Finally, let us assume that G is an SLI-group and that G is a finite index subgroup of H. Let Σ (resp. Δ) be a finite generating set for G (resp. H). Then in [24, Lemma 3.3] it is shown that there exists a sequential function $f : \Delta^* \to \Sigma^*$ such that $\mathrm{WP}_\Delta(H) = f^{-1}(\mathrm{WP}_\Sigma(G))$. Hence, H is an SLI-group. $\square$

Let us quickly dispense with the decidability of the rational subset membership problem for SLI-groups.

Lemma 4. *The class of languages SLI is contained in the class of languages RID. In particular, every SLI-group has a decidable rational subset membership problem.*

Proof. Let $K \subseteq \Theta^*$ belong to SLI. Let A be a finite automaton over the alphabet Θ. We have to decide whether $L(A) \cap K \neq \emptyset$. Since K belongs to SLI, the set

$$\Psi_\emptyset(\{w \in L(A) \mid \pi_\Theta(w) \in K\}) = \Psi_\emptyset(L(A) \cap K)$$

is effectively semilinear and so has a decidable membership problem (c.f. [24]). As mentioned earlier, $\Psi_\emptyset(L(A) \cap K)$ consists of the unique function $\emptyset \to \mathbb{N}$ if $L(A) \cap K$ is non-empty and is empty otherwise. Thus we can test emptiness for $L(A) \cap K$. $\square$

Having already taken care of finite extensions by Lemma 3, let's turn to finitely generated subgroups. We show that the language class SLI is closed under intersection with rational subsets. This guarantees that the class of SLI-groups is closed under taking finitely generated subgroups [21].

Lemma 5. *Let $K \subseteq \Theta^*$ belong to SLI and let $R \subseteq \Theta^*$ be rational. Then $R \cap K$ belongs to SLI. In particular, every finitely generated subgroup of an SLI-group is an SLI-group.*

Proof. Let $L \subseteq (\Gamma \cup \Theta)^*$ be rational, where Γ is a finite alphabet disjoint from Θ. We have

$$L \cap \pi_\Theta^{-1}(R \cap K) = L \cap \pi_\Theta^{-1}(R) \cap \pi_\Theta^{-1}(K).$$

But rational languages are closed under inverse homomorphism and intersection, so $\Psi_\Gamma(L \cap \pi_\Theta^{-1}(R) \cap \pi_\Theta^{-1}(K))$ is semilinear as K belongs to SLI. This establishes the lemma. $\square$

Next, we show that the class of SLI-groups is closed under direct products with $\mathbb{Z}$:

Lemma 6. *If G is an SLI-group, then $G \times \mathbb{Z}$ is also an SLI-group.*

Proof. Let Σ be a finite generating set for G. Choose a generator $a \notin \Sigma$ of $\mathbb{Z}$. Then $G \times \mathbb{Z}$ is generated by $\Sigma \cup \{a\}$. Let Γ be a finite alphabet ($\Gamma \cap (\Sigma^{\pm 1} \cup \{a, a^{-1}\}) = \emptyset$) and let L be a rational subset of $(\Sigma^{\pm 1} \cup \{a, a^{-1}\} \cup \Gamma)^*$. We have

$$\begin{aligned}
&\Psi_\Gamma\Big(\{w \in L \mid \pi_{\Sigma^{\pm 1} \cup \{a, a^{-1}\}}(w) \in \mathrm{WP}_{\Sigma \cup \{a\}}(G \times \mathbb{Z})\}\Big) = \\
&\qquad \overline{\pi}_\Gamma\Big(\, \Psi_{\Gamma \cup \{a, a^{-1}\}}(\{w \in L \mid \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G)\}) \cap \\
&\qquad\qquad \{f \in \mathbb{N}^{\Gamma \cup \{a, a^{-1}\}} \mid f(a) = f(a^{-1})\}\Big).
\end{aligned}$$

This set is semilinear, since $\{f \in \mathbb{N}^{\Gamma \cup \{a, a^{-1}\}} \mid f(a) = f(a^{-1})\}$ is semilinear and semilinear sets are closed under intersection and projection [18]. $\square$

By Lemmas 3–6, Theorem 1 would be established, if we could prove the closure of $\mathcal{C}$ under graph of groups constructions with finite edge groups. Unfortunately we are only able to prove this closure under the restriction that every vertex group of the graph of groups is residually finite (which is the case for groups in $\mathcal{C}$). In general we can just prove closure under free product. This, in fact, constitutes the most difficult part of the proof of Theorem 1.

Lemma 7. *If G_1 and G_2 are SLI-groups, then $G_1 * G_2$ is also an SLI-group.*

Proof. Assume that Σ_i is a finite generating set for G_i. Thus, $\Sigma = \Sigma_1 \cup \Sigma_2$ is a generating set for the free product $G_1 * G_2$. Let Γ be a finite alphabet ($\Gamma \cap \Sigma^{\pm 1} = \emptyset$) and let $\Theta = \Sigma^{\pm 1} \cup \Gamma$. Let $L \subseteq \Theta^*$ be rational and let $A = (Q, \Theta, \delta, q_0, F)$ be a finite automaton with $L = L(A)$, where Q is the set of states, $\delta \subseteq Q \times \Theta \times Q$ is the transition relation, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states. For $p, q \in Q$ and $w \in \Theta^*$ we write $p \xrightarrow{w}_A q$ if there exists a path in A from p to q, labelled by the word w.

For every pair of states $(p, q) \in Q \times Q$ let us define the language

$$L[p, q] \subseteq (\Sigma_1^{\pm 1} \cup \Gamma \cup (Q \times Q))^* \cup (\Sigma_2^{\pm 1} \cup \Gamma \cup (Q \times Q))^* \subseteq (\Theta \cup (Q \times Q))^*$$

as follows:

$$\begin{aligned} L[p, q] = \bigcup_{i \in \{1,2\}} \{ & w_0(p_1, q_1)w_1(p_2, q_2) \cdots w_{k-1}(p_k, q_k)w_k \mid \\ & k \geq 1 \ \wedge \ (p_1, q_1), \ldots, (p_k, q_k) \in Q \times Q \ \wedge \\ & w_0, \ldots, w_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^* \ \wedge \ \pi_{\Sigma_i^{\pm 1}}(w_0 \cdots w_k) \in \mathrm{WP}_{\Sigma_i}(G_i) \ \wedge \\ & p \xrightarrow{w_0}_A p_1 \ \wedge \ q_1 \xrightarrow{w_1}_A p_2 \ \wedge \cdots \wedge \ q_{k-1} \xrightarrow{w_{k-1}}_A p_k \ \wedge \ q_k \xrightarrow{w_k}_A q \} \end{aligned}$$

Since the language

$$\begin{aligned} \{ & w_0(p_1, q_1)w_1(p_2, q_2) \cdots w_{k-1}(p_k, q_k)w_k \mid \\ & k \geq 1 \ \wedge \ (p_1, q_1), \ldots, (p_k, q_k) \in Q \times Q \ \wedge \ w_0, \ldots, w_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^* \ \wedge \\ & p \xrightarrow{w_0}_A p_1 \ \wedge \ q_1 \xrightarrow{w_1}_A p_2 \ \wedge \cdots \wedge \ q_{k-1} \xrightarrow{w_{k-1}}_A p_k \ \wedge \ q_k \xrightarrow{w_k}_A q \} \end{aligned}$$

is a rational language over the alphabet $\Sigma_i^{\pm 1} \cup \Gamma \cup (Q \times Q)$ for $i \in \{1, 2\}$ and G_i is an SLI-group, it follows that the Parikh image $\Psi_{\Gamma \cup (Q \times Q)}(L[p, q]) \subseteq \mathbb{N}^{\Gamma \cup (Q \times Q)}$ is semilinear. Let $K[p, q] \subseteq (\Gamma \cup (Q \times Q))^*$ be some rational language such that

$$\Psi(K[p, q]) = \Psi_{\Gamma \cup (Q \times Q)}(L[p, q]). \tag{5}$$

Next, we define a context-free grammar $\mathsf{G} = (N, \Gamma, S, P)$ as follows:

- the set of nonterminals is $N = \{S\} \cup (Q \times Q)$, where S is a new symbol not contained in $Q \times Q$.
- S is the start nonterminal.

– P consists of the following productions:

$$\begin{aligned} S &\to (q_0, q_f) && \text{for all } q_f \in F \\ (p,q) &\to K[p,q] && \text{for all } p,q \in Q \\ (q,q) &\to \varepsilon && \text{for all } q \in Q \end{aligned}$$

By Parikh's theorem, the Parikh image $\Psi(L(\mathsf{G})) \subseteq \mathbb{N}^\Gamma$ is semilinear. Thus, the following claim proves the lemma:

Claim 1. $\Psi(L(\mathsf{G})) = \Psi_\Gamma(\{w \in L(A) \mid \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)\})$

Proof of Claim 1. We prove the following more general identity for all $(p,q) \in Q \times Q$:

$$\Psi(L(\mathsf{G},(p,q))) = \Psi_\Gamma(\{w \in \Theta^* \mid p \xrightarrow{w}_A q \ \wedge\ \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)\})$$

For the inclusion

$$\Psi(L(\mathsf{G},(p,q))) \subseteq \Psi_\Gamma(\{w \in \Theta^* \mid p \xrightarrow{w}_A q \ \wedge\ \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)\}) \tag{6}$$

assume that $(p,q) \overset{*}{\Rightarrow}_{\mathsf{G}} u \in \Gamma^*$. We show by induction on the length of the G-derivation $(p,q) \overset{*}{\Rightarrow}_{\mathsf{G}} u$ that there exists a word $w \in \Theta^*$ such that $p \xrightarrow{w}_A q$, $\pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)$, and $\Psi(u) = \Psi_\Gamma(w)$.

Case 1. $p = q$ and $u = \varepsilon$: We can choose $w = \varepsilon$.

Case 2. $(p,q) \Rightarrow_{\mathsf{G}} u' \overset{*}{\Rightarrow}_{\mathsf{G}} u$ for some $u' \in K[p,q]$. By (5), there exists a word $v \in L[p,q]$ such that $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$. Since $v \in L[p,q]$, there exist $k \geq 1$, $(p_1,q_1),\dots,(p_k,q_k) \in Q \times Q$, $i \in \{1,2\}$, and $v_0,\dots,v_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^*$ such that

– $p \xrightarrow{v_0}_A p_1,\ q_1 \xrightarrow{v_1}_A p_2, \dots, q_{k-1} \xrightarrow{v_{k-1}}_A p_k,\ q_k \xrightarrow{v_k}_A q$,
– $v = v_0(p_1,q_1)v_1(p_2,q_2)\cdots v_{k-1}(p_k,q_k)v_k$, and
– $\pi_{\Sigma_i^{\pm 1}}(v_0 \cdots v_k) \in \mathrm{WP}_{\Sigma_i}(G_i)$.

Since $u' \overset{*}{\Rightarrow}_{\mathsf{G}} u \in \Gamma^*$ and $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$, there must exist $u_1,\dots,u_k \in \Gamma^*$ such that

$$(p_i,q_i) \overset{*}{\Rightarrow}_{\mathsf{G}} u_i \quad \text{and} \quad \Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi(u_1) + \cdots + \Psi(u_k)$$

for all $1 \leq i \leq k$. By induction, we obtain words $w_1,\dots,w_k \in \Theta^*$ such that for all $1 \leq i \leq k$:

– $p_i \xrightarrow{w_i}_A q_i$
– $\pi_{\Sigma^{\pm 1}}(w_i) \in \mathrm{WP}_\Sigma(G_1 * G_2)$, and
– $\Psi(u_i) = \Psi_\Gamma(w_i)$.

Let us set $w = v_0 w_1 v_1 \cdots w_k v_k \in \Theta^*$. We have:

– $p \xrightarrow{v_0}_A p_1 \xrightarrow{w_1}_A q_1 \xrightarrow{v_1}_A p_2 \cdots p_k \xrightarrow{w_k}_A q_k \xrightarrow{v_k}_A q$, i.e., $p \xrightarrow{w}_A q$,
– $\pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)$, and
– $\Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi(u_1) + \cdots + \Psi(u_k) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi_\Gamma(w_1) + \cdots + \Psi_\Gamma(w_k) = \Psi_\Gamma(w)$.

This concludes the proof of inclusion (6). For the other inclusion, assume that

$$p \xrightarrow{w}_A q \quad \text{and} \quad \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G_1 * G_2)$$

for a word $w \in \Theta^*$. By induction over the length of the word w we show that $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G}, (p,q)))$.

We will make a case distinction according to the three cases in Lemma 1. Note that we either have $w \in \Gamma^*$ or the word $w \in \Theta^*$ can be (not necessarily uniquely) written as $w = w_1 \cdots w_n$ with $n \geq 1$ such that $w_i \in ((\Gamma \cup \Sigma_1^{\pm 1})^* \cup (\Gamma \cup \Sigma_2^{\pm 1})^*) \setminus \Gamma^*$ and $w_i \in (\Gamma \cup \Sigma_1^{\pm 1})^* \Leftrightarrow w_{i+1} \in (\Gamma \cup \Sigma_2^{\pm 1})^*$.

Case 1. $w \in (\Gamma \cup \Sigma_1^{\pm 1})^*$ (the case $w \in (\Gamma \cup \Sigma_2^{\pm 1})^*$ is analogous): Then $\pi_{\Sigma_1^{\pm 1}}(w) \in \mathrm{WP}_{\Sigma_1}(G_1)$. Together with $p \xrightarrow{w}_A q$, we obtain $w(q,q) = w(q,q)\varepsilon \in L[p,q]$. Since $(p,q) \to K[p,q]$ and $(q,q) \to \varepsilon$ are productions of G, there exists a word $u \in \Gamma^*$ such that $(p,q) \xRightarrow{*}_{\mathsf{G}} u$ and $\Psi(u) = \Psi_\Gamma(w)$, i.e., $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G}, (p,q)))$.

Case 2. $w = w_1 w_2$ with $w_1 \neq \varepsilon \neq w_2$ and $\pi_{\Sigma^{\pm 1}}(w_1), \pi_{\Sigma^{\pm 1}}(w_2) \in \mathrm{WP}_\Sigma(G_1 * G_2)$. Then there exists a state $r \in Q$ such that

$$p \xrightarrow{w_1}_A r \xrightarrow{w_2}_A q.$$

By induction, we obtain

$$\Psi_\Gamma(w_1) \in \Psi(L(\mathsf{G}, (p,r))) \text{ and}$$
$$\Psi_\Gamma(w_2) \in \Psi(L(\mathsf{G}, (r,q))).$$

Hence, we get

$$\begin{aligned} \Psi_\Gamma(w) &= \Psi_\Gamma(w_1) + \Psi_\Gamma(w_2) \\ &\in \Psi(L(\mathsf{G}, (p,r))) + \Psi(L(\mathsf{G}, (r,q))) \\ &\subseteq \Psi(L(\mathsf{G}, (p,q))), \end{aligned}$$

where the last inclusion holds, since $(p,r)(r,q) \in L[p,q]$, and so either $(p,q) \to (p,r)(r,q)$ or $(p,q) \to (r,q)(p,r)$ is a production of G.

Case 3. $w = v_0 w_1 v_1 \cdots w_k v_k$ such that $k \geq 1$,

- $\pi_{\Sigma^{\pm 1}}(w_i) \in \mathrm{WP}_\Sigma(G_1 * G_2)$ for all $i \in \{1, \ldots, k\}$, and
- for some $i \in \{1,2\}$: $v_0, \ldots, v_k \in (\Gamma \cup \Sigma_i^{\pm 1})^* \setminus \Gamma^*$ and $\pi_{\Sigma_i^{\pm 1}}(v_0 \cdots v_k) \in \mathrm{WP}_{\Sigma_i}(G_i)$.

There exist states $p_1, q_1, \ldots, p_k, q_k \in Q$ such that

$$p \xrightarrow{v_0}_A p_1 \xrightarrow{w_1}_A q_1 \xrightarrow{v_1}_A p_2 \cdots p_k \xrightarrow{w_k}_A q_k \xrightarrow{v_k}_A q.$$

By induction, we obtain

$$\Psi_\Gamma(w_i) \in \Psi(L(\mathsf{G}, (p_i, q_i))) \tag{7}$$

for all $1 \leq i \leq k$. Moreover, from the definition of the language $L[p,q]$ we obtain

$$v = v_0(p_1, q_1)v_1(p_2, q_2) \cdots v_{k-1}(p_k, q_k)v_k \in L[p,q].$$

Hence, by (5) there is a word $u' \in K[p,q]$ such that $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$ and $(p,q) \to u'$ is a production of G. With (7) we obtain

$$(p,q) \Rightarrow_{\mathsf{G}} u' \stackrel{*}{\Rightarrow}_{\mathsf{G}} u$$

for a word $u \in \Gamma^*$ such that

$$\Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi_\Gamma(w_1) + \cdots + \Psi_\Gamma(w_k) = \Psi_\Gamma(w),$$

i.e., $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G},(p,q)))$. This concludes the proof of Claim 1 and hence of the lemma. $\square$

If we were to weaken the definition of the class $\mathcal{C}$ by only requiring closure under free products instead of closure under finite graphs of groups with finite edge groups, then Lemmas 4–7 would already imply Theorem 1. In fact, this weaker result suffices in order to deal with graph groups, and readers only interested in graph groups can skip the following considerations concerning graphs of groups.

To obtain the more general closure result for the class $\mathcal{C}$ concerning graph of group constructions, we reduce to the case of free products. Recall that a group G is *residually finite* if, for each $g \in G \setminus \{1\}$, there is a finite index normal subgroup N of G with $g \notin N$. Now we use a standard trick for graphs of residually finite groups with finite edge groups.

Lemma 8. *Let $\mathbb{A}$ be a finite graph of groups such that the vertex groups are residually finite SLI-groups and the edge groups are finite. Then the fundamental group of $\mathbb{A}$ is an SLI-group.*

Proof. Let G be the fundamental group of $\mathbb{A}$. Then G is residually finite [36, II.2.6 Proposition 12]. Since there are only finitely many edge groups and each edge group is finite, there is a finite index normal subgroup $N \leq G$ intersecting trivially each edge group, and hence each conjugate of an edge group. Thus the finitely generated subgroup $N \leq G$ acts on the Bass-Serre tree for G [36] with trivial edge stabilizers, forcing N to be a free product of conjugates of subgroups of the vertex groups of G and a free group [36]. Since N is finitely generated, these free factors must also be finitely generated. Since every finitely generated subgroup of an SLI-group is an SLI-group (Lemma 5) and $\mathbb{Z}$ is an SLI-group (Lemma 6), we may deduce that N is a free product of SLI-groups and hence is an SLI-group by Lemma 7. Since G contains N as a finite index subgroup, Lemma 3 implies that G is an SLI-group, as required. $\square$

Clearly, the trivial group 1 is an SLI-group. Also all the defining properties of $\mathcal{C}$ preserve residual finiteness (the only non-trivial case being the graph of group constructions [36]). Hence, Lemmas 4–6 and Lemma 8 immediately yield Theorem 1.

Our main application of Theorem 1 concerns graph groups:

Theorem 2. *The rational subset membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if (Σ, I) is a transitive forest. Moreover, if (Σ, I) is not a transitive forest, then there exists a fixed rational subset L of $\mathbb{G}(\Sigma, I)$ such that the membership problem for L within $\mathbb{G}(\Sigma, I)$ is undecidable.*

Proof. The decidability part follows immediately from Theorem 1: Lemma 2 implies that every graph group $\mathbb{G}(\Sigma, I)$ with (Σ, I) a transitive forest belongs to the class $\mathcal{C}$.

Now assume that (Σ, I) is not a transitive forest. By [39] it suffices to consider the case that (Σ, I) is either a C4 or a P4. For the case of a C4 we can use Mihailova's result [30]. Now assume that (Σ, I) is a P4. We will reuse a construction by Aalbersberg and Hoogeboom [1], which is based on *2-counter machines*. A 2-counter machine is a tuple $C = (Q, \text{Ins}, q_0, q_f)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, $q_f \in Q$ is the final state, and $\text{Ins} \subseteq Q \times \{i1, i2, d1, d2, z1, z2, p1, p2\} \times Q$ is the set of instructions. The set of configurations of C is $Q \times \mathbb{N} \times \mathbb{N}$. For two configurations $(q, n_1, n_2), (q', m_1, m_2)$ we write $(q, n_1, n_2) \Rightarrow_C (q', m_1, m_2)$ if there exists an instruction $(q, \alpha k, q') \in \text{Ins}$, so $\alpha \in \{i, d, z, p\}, k \in \{1, 2\}$, such that $m_{3-k} = n_{3-k}$ and one of the following three cases holds:

- $\alpha = i$ and $m_k = n_k + 1$
- $\alpha = d$ and $m_k = n_k - 1$
- $\alpha = z$ and $m_k = n_k = 0$
- $\alpha = p$ and $m_k = n_k > 0$

Since Turing machines can be simulated by 2-counter machines [29], it is undecidable whether for a given 2-counter machine $C = (Q, \text{Ins}, q_0, q_f)$ there exist $m, n \in \mathbb{N}$ with $(q_0, 0, 0) \Rightarrow_C^* (q_f, m, n)$. In [1], this problem is reduced to the question, whether $L \cap K = \emptyset$ for given rational trace languages $L, K \subseteq \mathbb{M}(\Sigma, I)$, where $\Sigma = \{a, b, c, d\}$ and $I = \{(a, b), (b, c), (c, d)\}$. In fact, the language K is fixed, more precisely

$$\begin{aligned} K &= ba(d(cb)^+a)^*dc^* \\ &= \{[ab^{j_0}c^{j_1}dab^{j_1}c^{j_2}d \cdots ab^{j_{\ell-1}}c^{j_\ell}d]_I \mid \ell \geq 1, j_0 = 1, j_1, \ldots, j_\ell \geq 1\}. \end{aligned}$$

The problem is that in the construction of [1] the language L is not fixed since it depends on the 2-counter machine C. Aalbersberg and Hoogeboom encode the pair of counter values $(m, n) \in \mathbb{N} \times \mathbb{N}$ by the single number $2^m 3^n$. The language L is constructed in such a way that $K \cap L$ contains exactly those traces of the form $[ab^{j_0}c^{j_1}dab^{j_1}c^{j_2}d \cdots ab^{j_{\ell-1}}c^{j_\ell}d]_I$, such that $\ell \geq 1$, $j_0 = 1$, and there exist states $q_1, \ldots, q_\ell$ and $m_1, n_1, \ldots, m_\ell, n_\ell \in \mathbb{N}$ with $q_\ell = q_f$, $2^{m_i}3^{n_i} = j_i$, and $(q_0, 0, 0) \Rightarrow_C (q_1, m_1, n_1) \Rightarrow_C (q_2, m_2, n_2) \Rightarrow_C \cdots \Rightarrow_C (q_\ell, m_\ell, n_\ell)$ (note that $j_0 = 1$ encodes the initial counter values $(0, 0)$).

In order to construct a fixed rational subset of $\mathbb{G}(\Sigma, I)$ with an undecidable membership problem, we start with a fixed (universal) 2-counter machine $C = (Q, \text{Ins}, q_0, q_f)$ such that it is undecidable whether $\exists m', n' \in \mathbb{N} : (q_0, m, n) \Rightarrow_C^* (q_f, m', n')$ for given natural numbers m, n. Such a machine C can be obtained by simulating a universal Turing machine. Let $L \subseteq \mathbb{M}(\Sigma, I)$ be the *fixed* rational trace language constructed by Aalbersberg and Hoogeboom from C, and let us replace the fixed trace language $K = ba(d(cb)^+a)^*dc^*$ by the (non-fixed) language

$$\begin{aligned} K_{m,n} &= b^{2^m 3^n} a(d(cb)^+a)^*dc^* \\ &= \{[ab^{j_0}c^{j_1}dab^{j_1}c^{j_2}d \cdots ab^{j_{\ell-1}}c^{j_\ell}d]_I \mid \ell \geq 1, j_0 = 2^m 3^n, j_1, \ldots, j_\ell \geq 1\}. \end{aligned}$$

Then it is undecidable, whether $K_{m,n} \cap L \neq \emptyset$ for given $m, n \in \mathbb{N}$. Hence, it is undecidable, whether $b^{-2^m 3^n} \in a(d(cb)^+a)^*dc^*L^{-1}$ in the graph group $\mathbb{G}(\Sigma, I)$. Clearly, $a(d(cb)^+a)^*dc^*L^{-1}$ is a fixed rational subset of the graph group $\mathbb{G}(\Sigma, I)$. ☐

We conclude this section with a further application of Theorem 1 to *graph products* (which should not be confused with graphs of groups). A graph product is given by a tuple $(\Sigma, I, (G_v)_{v\in\Sigma})$, where (Σ, I) is an independence alphabet and G_v is a group, which is associated with the node $v \in \Sigma$. The group $\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma})$ defined by this tuple is the quotient

$$\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma}) = \mathop{\ast}_{v\in\Sigma} G_v / \{xy = yx \mid x \in G_u, y \in G_v, (u, v) \in I\},$$

i.e., we take the free product $\ast_{v\in\Sigma} G_v$ of the groups G_v ($v \in \Sigma$), but let elements from adjacent groups commute. Note that $\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma})$ is the graph group $\mathbb{G}(\Sigma, I)$ in the case every G_v is isomorphic to $\mathbb{Z}$. Graph products were first studied by Green [19].

Theorem 3. *If (Σ, I) is a transitive forest and every group G_v ($v \in V$) is finitely generated and virtually Abelian (i.e., has an Abelian subgroup of finite index), then the rational subset membership problem for $\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma})$ is decidable.*

Proof. Assume that the assumptions from the theorem are satisfied. We show that $\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma})$ belongs to the class $\mathcal{C}$. Since (Σ, I) is a transitive forest, the group $\mathbb{G}(\Sigma, I, (G_v)_{v\in\Sigma})$ can be built up from trivial groups using the following two operations: (i) free products and (ii) direct products with finitely generated virtually Abelian groups. Since the class $\mathcal{C}$ is closed under free products, it suffices to prove that if G belongs to the class $\mathcal{C}$ and H is finitely generated virtually Abelian, then $G \times H$ also belongs to the class $\mathcal{C}$. As a finitely generated virtually Abelian group, H is a finite extension of a finite rank free Abelian group $\mathbb{Z}^n$. By the closure of the class $\mathcal{C}$ under direct products with $\mathbb{Z}$, $G \times \mathbb{Z}^n$ belongs to the class $\mathcal{C}$. Now, $G \times H$ is a finite extension of $G \times \mathbb{Z}^n$, proving the theorem, since $\mathcal{C}$ is closed under finite extensions. $\square$

4 The submonoid membership problem

Recall that the submonoid membership problem for a group G asks whether a given element of G belongs to a given finitely generated submonoid of G. Hence, there is a trivial reduction from the submonoid membership problem for G to the rational subset membership problem for G. We will show that for every amalgamated free product $G \ast_A H$ such that:

1. $A = G \cap H$ is a finite, proper subgroup of G and H;
2. there exist $g \in G$, $h \in H$ with $g^{-1}Ag \cap A = 1 = h^{-1}Ah \cap A$,

there is in fact also a reduction in the opposite direction. Similarly, if $\ast_\varphi G$ is an HNN extension with $\varphi : A \to B$ with

1. A is a finite subgroup of G;
2. there exists $g \in G$ such that $g^{-1}Ag \cap A = 1$ or $g^{-1}Ag \cap B = 1$

then the rational subset problem reduces to the submonoid membership problem for $\ast_\varphi G$. We remark that in 2, one could by symmetry switch the roles of B and A.

Using the following lemma, it will suffice to consider a free product $G \ast F_2$, where F_2 is a free group of rank two.

Lemma 9. *Let $G *_A H$ be an amalgamated free product such that $H \neq A$, $[G : A] \geq 5$, and there exists $h \in H$ with $h^{-1}Ah \cap A = 1$. Then $G *_A H$ contains as a subgroup the free product $G * F_2$ of G with a free group of rank two.*

Proof. Since $[G : A] \geq 5$, we can choose elements $g_1, g_2, g_3, g_4 \in G \setminus A$ which belong to pairwise distinct left A-cosets. Moreover, choose an element $h \in H \setminus A$ with $h^{-1}Ah \cap A = 1$. First we claim that $x = g_1 h g_2^{-1}$ and $y = g_3 h g_4^{-1}$ freely generate a free subgroup of $G *_A H$. For this, note that $g_i^{-1} g_j \in G \setminus A$ if $i \neq j$. Thus, every word over $\{x, x^{-1}, y, y^{-1}\}$ which does not contain a factor from $\{xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y\}$ yields a reduced sequence for the amalgamated product. The normal form theorem for amalgamated free products [26, Chapter IV, Theorem 2.6] then implies that $\{x, y\}$ is the base of a free subgroup of $G *_A H$. Hence, the conjugates $u = hxh^{-1}$ and $v = hyh^{-1}$ also form a base for a free subgroup of $G *_A H$. Since $h^{-1}Ah \cap A = 1$ (and hence if $a \in A$, then $h^{-1}ah \in H \setminus A$) a word over $G \setminus \{1\} \cup \{u, u^{-1}, v, v^{-1}\}$, which does not contain a factor from $(G \setminus \{1\})(G \setminus \{1\}) \cup \{uu^{-1}, u^{-1}u, vv^{-1}, v^{-1}v\}$, yields a reduced sequence for the amalgamated product. Again, the normal form theorem for amalgamated free products implies that the subgroup of $G *_A H$ generated by $G \cup \{u, v\}$ is isomorphic to $G * F_2$. $\square$

We now prove the analogous result for HNN extensions.

Lemma 10. *Let $*_\varphi G$ be an HNN extension with stable letter t and finite associated subgroups A, B (so $\varphi : A \to B$) such that $[G : B] \geq 3$ and there exists $g \in G$ with $g^{-1}Ag \cap A = 1$ or $g^{-1}Ag \cap B = 1$. Then $*_\varphi G$ contains as a subgroup the free product $G * F_2$ of G with a free group of rank two.*

Proof. By Lemma 9, it suffices to show that $*_\varphi G$ contains a subgroup $G * \mathbb{Z}$. We may assume that $A \neq 1 \neq B$, because otherwise $*_\varphi G \simeq G * \mathbb{Z}$. Choose $g_1, g_2 \in G \setminus B$ so that g_1, g_2 are in different left cosets of B. Suppose first there exists $g \in G$ with $g^{-1}Ag \cap A = 1$ and set $x = g_1 t^{-1} g t g_2^{-1}$. Since $g \notin A$ (because otherwise $A = 1$) and $g_2^{-1} g_1 \notin B$, one easily deduces that x^n is a reduced sequence for the HNN extension for all $n > 0$ and hence x is of infinite order by Britton's lemma. Set $y = t^{-1} g t x t^{-1} g^{-1} t$. Then y is of infinite order, being a conjugate of x. We claim that G and $\langle y \rangle$ generate their free product inside of $*_\varphi G$. We need to show that a word over $G \setminus \{1\} \cup \{y, y^{-1}\}$ with no factor from $(G \setminus \{1\})(G \setminus \{1\}) \cup \{yy^{-1}, y^{-1}y\}$ results in a reduced sequence for the HNN extension. The key point is that if $h \in G \setminus B$, then $t^{-1} g^{-1} t h t^{-1} g t$ is reduced. On the other hand, if $b \in B \setminus \{1\}$, then $t^{-1} g^{-1} t b t^{-1} g t = t^{-1} g^{-1} \varphi^{-1}(b) g t^{-1}$, which is reduced since $g^{-1}Ag \cap A = 1$.

Now assume that there exists $g \in G$ with $g^{-1}Ag \cap B = 1$. The group A must be a proper subgroup of G, because otherwise we have $1 = g^{-1}Ag \cap B = G \cap B = B$. So choose $g_0 \in G \setminus A$ and set $x = g_1 t^{-1} g_0 t g_2^{-1}$. The same argument as above shows that x has infinite order. Set $y = t^{-1} g t^{-1} x t g^{-1} t$; again y has infinite order, being a conjugate of x. Again, we claim that G and $\langle y \rangle$ generate their free product in $*_\varphi G$. Once more, we must prove that a word over $G \setminus \{1\} \cup \{y, y^{-1}\}$ with no factor from $(G \setminus \{1\})(G \setminus \{1\}) \cup \{yy^{-1}, y^{-1}y\}$ yields a reduced sequence for the HNN extension. The key point is that if $h \in G \setminus B$, then $t g^{-1} t h t^{-1} g t^{-1}$ is reduced. On the other hand, if $b \in B \setminus \{1\}$, then $t g^{-1} t b t^{-1} g t^{-1} = t g^{-1} \varphi^{-1}(b) g t^{-1}$, which is reduced since $g^{-1}Ag \cap B = 1$. $\square$

The following lemma is crucial for us:

Lemma 11.

(1) Let G and H be finitely generated groups such that the finite group A is a proper subgroup of both G and H and there exists $h \in H$ with $h^{-1}Ah \cap A = 1$. Then the rational subset membership problem for G can be reduced to the submonoid membership problem for $G *_A H$.

(2) If $\varphi : A \to B$ is an isomorphism between finite subgroups of a finitely generated group G and there exists $g \in G$ with $g^{-1}Ag \cap A = 1$ or $g^{-1}Ag \cap B = 1$, then the rational subset membership problem for G can be reduced to the submonoid membership problem for $*_\varphi G$.

Remark 1. In our proof of Lemma 11 we will implicitly construct Turing machines that carry out the reductions in (1) and (2). These machines will depend on the element g (and h) in (1), respectively (2). Here one might argue that these elements are not known. But this is not a real problem, since g and h are fixed elements which do not depend on the input for the reduction. So there exists a Turing machine that can do the reduction, although we don't know which Turing machine if we don't know the elements g and h.

Proof of Lemma 11. If G is finite, then the rational subset membership problem for G is decidable, so we may assume without loss of generality that G is infinite. Since A is finite, we have $[G : A] \geq 5$ in (1), respectively $[G : B] \geq 3$ in (2). Then Lemmas 9 and 10 imply that $G * F_2$ is a subgroup of $G *_A H$, respectively $*_\varphi G$. Since the submonoid membership problem for a finitely generated subgroup of a group K reduces to the submonoid membership problem for K itself, it suffices to prove the following: the rational subset membership problem for G can be reduced to the submonoid membership problem for $G * F_2$. Let Σ be a finite generating set for G and use $h : (\Sigma^{\pm 1} \cup \Gamma^{\pm})^* \to G * F_2$ for the canonical morphism. Let $A = (Q, \Sigma^{\pm 1}, \delta, q_0, F)$ be a finite automaton and let $t \in (\Sigma^{\pm 1})^*$. By introducing ε-transitions, we may assume that the set of final states F consists of a single state $q_f \neq q_0$. One can effectively find a subset $\widetilde{Q} \subseteq F_2$ in bijection with Q via $q \mapsto \widetilde{q}$ such that $\widetilde{Q}$ freely generates a free subgroup of F_2.

We construct a finite subset $\Delta \subseteq (\Sigma^{\pm 1} \cup \Gamma^{\pm 1})^*$ and an element $u \in (\Sigma^{\pm 1} \cup \Gamma^{\pm 1})^*$ such that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$. Let

$$\Delta = \{\widetilde{q}\, c\, \widetilde{p}^{-1} \mid (q, c, p) \in \delta\} \quad \text{and} \quad u = \widetilde{q_0}\, t\, \widetilde{q_f}^{-1}. \tag{8}$$

Note that in (8), we have $c \in \Sigma^{\pm 1} \cup \{\varepsilon\}$, since we introduced ε-transitions. Recall $(q, c, p) \in \delta$ means $q \xrightarrow{c} p$ in A. We begin with a critical claim.

Claim 1. Suppose that in $G * F_2$, we have

$$\widetilde{q_0}\, t\, \widetilde{q_f}^{-1} = (\widetilde{p_1} v_1 \widetilde{q_1}^{-1}) \cdots (\widetilde{p_n} v_n \widetilde{q_n}^{-1}) \tag{9}$$

where $p_i \xrightarrow{v_i} q_i$ in A, for $i \in \{1, \ldots, n\}$. Then $h(t) \in h(L(A))$.

The claim is proved by induction on n. If $n = 1$, then since $q_0 \neq q_f$, the normal form theorem for free products easily implies $q_0 = p_1$, $q_f = q_1$ and $t = v_1$ in G. Thus

$q_0 \xrightarrow{v_1} q_f$ in A, whence $v_1 \in L(A)$, and so $h(t) \in h(L(A))$. Next suppose the claim holds for $n - 1 \geq 1$ and consider the claim for $n > 1$.

First suppose that $q_i = p_{i+1}$ for some i. Then

$$\widetilde{q}_0 \, t \, \widetilde{q_f}^{-1} = (\widetilde{p}_1 v_1 \widetilde{q}_1^{-1}) \cdots (\widetilde{p}_i v_i v_{i+1} \widetilde{q}_{i+1}^{-1}) \cdots (\widetilde{p}_n v_n \widetilde{q}_n^{-1})$$

in $G * F_2$ and $p_i \xrightarrow{v_i v_{i+1}} q_{i+1}$ in A. Induction now gives the desired conclusion.

Next suppose that for some i, we have $p_i = q_i$ and $v_i = 1$ in G. Then

$$\widetilde{q}_0 \, t \, \widetilde{q_f}^{-1} = (\widetilde{p}_1 v_1 \widetilde{q}_1^{-1}) \cdots (\widetilde{p}_{i-1} v_{i-1} \widetilde{q}_{i-1}^{-1})(\widetilde{p}_{i+1} v_{i+1} \widetilde{q}_{i+1}^{-1}) \cdots (\widetilde{p}_n v_n \widetilde{q}_n^{-1})$$

in $G * F_2$ and we can again apply the induction hypothesis.

Finally, suppose $p_i = q_i$ implies $v_i \neq 1$ in G and suppose $q_i \neq p_{i+1}$, all i. Then we claim that the right hand side of (9) is already in normal form. Consider a typical window $\widetilde{q}_{i-1}^{-1} \widetilde{p}_i v_i \widetilde{q}_i^{-1} \widetilde{p}_{i+1}$ (where we take $\widetilde{q}_0 = 1 = \widetilde{p}_{n+1}$). Then no two neighbouring elements belong to the same factor of the free product $G * \langle \widetilde{Q} \rangle = G * \langle \widetilde{s}_1 \rangle * \cdots * \langle \widetilde{s}_m \rangle$, where $Q = \{s_1, \ldots, s_m\}$, since $q_j \neq p_{j+1}$ for $j = i-1, i$ and $p_i \neq q_i$ when $v_i = 1$ in G. Since such windows cover the right hand side of (9) we may conclude that it is in normal form in $G * F_2$. Comparison with the left hand side then shows that $n = 1$, contradicting $n > 1$. So this case does not arise and the proof of the claim is complete.

Now we may prove that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$. Suppose first that $h(t) = h(t')$ with $t' \in L(A)$. Write $t' = a_1 \cdots a_n$ with $a_i \in \Sigma^{\pm 1} \cup \{\varepsilon\}$ and such that $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \longrightarrow \cdots \longrightarrow q_{n-1} \xrightarrow{a_n} q_f$. Then, as $h(t) = h(t')$, clearly we have

$$u = \widetilde{q}_0 t' \widetilde{q_f}^{-1} = (\widetilde{q}_0 a_1 \widetilde{q}_1^{-1})(\widetilde{q}_1 a_2 \widetilde{q}_2^{-1}) \cdots (\widetilde{q}_{n-1} a_n \widetilde{q_f}^{-1}) \in \Delta^*$$

in $G * F_2$. Conversely, suppose $h(u) \in h(\Delta^*)$. Then we can write

$$u = \widetilde{q}_0 \, t \, \widetilde{q_f}^{-1} = (\widetilde{p}_1 a_1 \widetilde{q}_1^{-1}) \cdots (\widetilde{p}_n a_n \widetilde{q}_n^{-1})$$

in $G * F_2$, where $p_i \xrightarrow{a_i} q_i$ are certain transitions of A. Claim 1 then implies $h(t) \in h(L(A))$. ☐

Theorem 4. *Let G and H be finitely generated groups such that the finite group A is a proper subgroup of both G and H and there exist $g \in G$, $h \in H$ with $g^{-1}Ag \cap A = 1 = h^{-1}Ah \cap A$. Then, for the amalgamated free product $G *_A H$ the rational subset membership problem and the submonoid membership problem are recursively equivalent.*

Proof. It suffices to show that the rational subset membership problem for $G *_A H$ can be reduced to the submonoid membership problem for $G *_A H$. The rational subset membership problem for $G *_A H$ can be reduced to the rational subset membership problems for G and H [24]. By Lemma 11 both these problems can be reduced to the submonoid membership problem for $G *_A H$. ☐

Note that the assumptions in Theorem 4 are satisfied for every free product $G * H$ of nontrivial finitely generated groups G and H.

A similar result holds for HNN extensions:

Theorem 5. *Let G be a finitely generated group and let $\varphi : A \to B$ be an isomorphism between finite subgroups of G. Suppose there exists $g \in G$, with $g^{-1}Ag \cap A = 1$ or $g^{-1}Ag \cap B = 1$. Then the rational subset membership problem and the submonoid membership problem are recursively equivalent for the HNN extension $*_\varphi G$.*

Proof. We just need to establish that the rational subset membership problem for $*_\varphi G$ can be reduced to the submonoid membership problem. The rational subset membership problem for $*_\varphi G$ can be reduced to the rational subset membership problem for G by the results of [24]. By Lemma 11 this problem can be reduced to the submonoid membership problem for $*_\varphi G$. This completes the proof. $\square$

Let us say that a group G is *virtually a free product* if it has a finite index subgroup H that splits nontrivially as a free product $H = G_1 * G_2$.

Corollary 1. *Let G be a finitely generated group that is virtually a free product. Then the rational subset and submonoid membership problems are recursively equivalent.*

Proof. Suppose G has decidable submonoid membership problem. We need to show that G has decidable rational subset problem. Let H be a finite index subgroup of G that splits nontrivially as a free product. Clearly H has decidable submonoid membership problem and hence has decidable rational subset membership problem by Theorem 4. It then follows G has decidable rational subset membership problem by [20, 24]. $\square$

In order for a finitely generated group to be virtually a free product, it must have two or more ends. On the other hand, a group with two or more ends that is either virtually torsion-free or residually finite is easily seen, via Stallings ends theorem [38], to be virtually a free product, as we now show. First we recall the notion of ends of a locally finite graph.

Let Γ be a locally finite graph, i.e., every node of Γ has only finitely many neighbours. Consider the inverse system $\Gamma \setminus C$ where C runs over the finite subgraphs of Γ. Then the sets of connected components $\pi_0(\Gamma \setminus C)$ form an inverse system of sets; the projective limit $\mathrm{Ends}(\Gamma) = \varprojlim \pi_0(\Gamma \setminus C)$ is known as the set of *ends* of Γ. The number of ends of Γ is the cardinality of $\mathrm{Ends}(\Gamma)$. The number of ends of a finitely generated group G is the number of ends of the Cayley-graph of G with respect to any finite set of generators; this number is independent of the finite generating set we choose for G and it is either 0, 1, 2 or ∞ [38]. Here are some examples: (i) every finite group has 0 ends, (ii) $\mathbb{Z} \times \mathbb{Z}$ has one end, (iii) $\mathbb{Z}$ has two ends, and (iv) F_2 has infinitely many ends. Stallings' famous ends theorem [38] says that if G is a finitely generated group with two or more ends, then G splits nontrivially as an amalgamated product or an HNN-extension over a finite subgroup. This can be reformulated in terms of actions on trees via Bass-Serre theory [36].

A group acts *nontrivially* on a tree if it has no global fixed-point, i.e., there is no node v in the tree with $Gv = \{v\}$. A group G is said to *split* over a subgroup H if there is a nontrivial action of G on a tree T such that H is the stabilizer of an edge e and the orbit Ge consists of all edges of T. This is equivalent to G splitting as an amalgamated product or HNN-extension with H as the amalgamation base, respectively the associated subgroup [36]. We shall need the following simple lemma.

Lemma 12. *Let G be a finitely generated group with a nontrivial action on a tree T and let $H \leq G$ be a finite index subgroup. Then H acts nontrivially on T.*

Proof. Recall that if g is an automorphism of a tree T, then g is said to be *elliptic* if g fixes some point of T. It is well known (this follows immediately from [36, I.6.4, Proposition 25], for instance) that if g^n ($n \geq 1$) is elliptic, then g is elliptic. Now if H has a global fixed point, then H consists entirely of elliptic automorphisms of T. Let $[G : H] = n$ and $g \in G$. Then $g^n \in H$ and hence is elliptic. It follows that every element of G is elliptic. But it is well known [36, I.6.5, Corollary 3] that any finitely generated group of elliptic automorphisms of a tree has a global fixed point, contradicting that the action of G is nontrivial. It follows that the action of H is nontrivial. □

Theorem 6. *Let G be a finitely generated group with two or more ends such that the intersection of all the finite index subgroups of G is torsion-free. Then G is virtually a free product and hence the rational subset membership and submonoid membership problems for G are recursively equivalent.*

Proof. By Stallings ends theorem [38], G splits nontrivially over a finite subgroup. So by Bass-Serre theory [36] G acts nontrivially on a tree T so that there is one orbit of edges and the stabilizer of an edge is finite. Let H be an edge stabilizer; since H is a finite group, by hypothesis there is a normal subgroup $N \lhd G$ of finite index such $H \cap N = \{1\}$. By Lemma 12 the action of N on T is nontrivial. Since each edge stabilizer in G is a conjugate of H, it follows no element of $N \setminus \{1\}$ fixes an edge. Therefore, N splits nontrivially as a free product [36]. This completes the proof. □

Corollary 2. *Let G be a finitely generated group with two or more ends which is either virtually torsion-free or residually finite. Then the rational subset membership and submonoid membership problems for G are recursively equivalent.*

Proof. Clearly Theorem 6 applies under either of these hypotheses. □

Let us now come back to graph groups. Theorems 2 and 4 imply that the submonoid membership problem is undecidable for every graph group of the form

$$\mathbb{G}(\Sigma \cup \{a\}, I) \simeq \mathbb{G}(\Sigma, I) * \mathbb{Z},$$

where $a \notin \Sigma$ and (Σ, I) is not a transitive forest. In the rest of the paper, we will sharpen this result. We show that for a graph group the submonoid membership problem is decidable if and only if the rational subset membership problem is decidable, i.e., if and only if the independence alphabet is a transitive forest. In fact, by our previous results, it suffices to consider a P4:

Theorem 7. *Let $\Sigma = \{a, b, c, d\}$ and $I = \{(a, b), (b, c), (c, d)\}$, i.e, (Σ, I) is a* P4. *Then there exists a fixed submonoid M of $\mathbb{G}(\Sigma, I)$ such that the membership problem of M within $\mathbb{G}(\Sigma, I)$ is undecidable.*

Proof. We follow the strategy of the proof of Lemma 11, but working in the graph group $\mathbb{G}(\Sigma, I)$ makes the encoding more complicated. Let R denote the trace rewriting system over the trace monoid $\mathbb{M}(\Sigma^{\pm 1}, I)$ defined in (1), Section 2.3. As usual denote by $h : (\Sigma^{\pm 1})^* \to \mathbb{G}(\Sigma, I)$ denote the canonical morphism, which will be identified with the canonical morphism $h : \mathbb{M}(\Sigma^{\pm 1}, I) \to \mathbb{G}(\Sigma, I)$. Let us fix a finite automaton A over the alphabet $\Sigma^{\pm 1}$ such that the membership problem for $h(L(A))$ within $\mathbb{G}(\Sigma, I)$ is undecidable; such an automaton exists by Theorem 2. Without loss of generality assume that

$$A = (\{1, \ldots, n\}, \Sigma^{\pm 1}, \delta, q_0, \{q_f\}),$$

where $\delta \subseteq \{1, \ldots, n\} \times (\Sigma^{\pm 1} \cup \{\varepsilon\}) \times \{1, \ldots, n\}$ and $q_0 \neq q_f$ (since we allow ε-transitions, we may assume that there is only a single final state q_f, which is different from the initial state q_0). For a state $q \in \{1, \ldots, n\}$, define the trace $\widetilde{q} \in \mathbb{M}(\Sigma^{\pm 1}, I)$ by

$$\widetilde{q} = (ada)^q d(ada)^{-q} = (ada)^q d(a^{-1}d^{-1}a^{-1})^q.$$

Note that the dependence graph of $\widetilde{q}$ is a linear chain. Moreover, every symbol from $\Sigma^{\pm 1}$ is dependent on ad, i.e., does not commute with ad. The following statement is straightforward to prove.

Claim 2. Let $q_1, \ldots, q_k \in \{1, \ldots, n\}$, $\varepsilon_1, \ldots, \varepsilon_k \in \{1, -1\}$ such that $q_i \neq q_{i+1}$ for all $1 \leq i \leq k-1$. Then

$$\mathrm{NF}_R(\widetilde{q}_1^{\varepsilon_1} \widetilde{q}_2^{\varepsilon_2} \cdots \widetilde{q}_k^{\varepsilon_k}) = (ada)^{q_1} d^{\varepsilon_1} (ada)^{q_2 - q_1} \cdots d^{\varepsilon_{k-1}} (ada)^{q_k - q_{k-1}} d^{\varepsilon_k} (ada)^{-q_k}.$$

Note that this trace starts (resp. ends) with a copy of ada (resp. $a^{-1}d^{-1}a^{-1}$).

Let $\varphi : (\Sigma^{\pm 1})^* \to (\Sigma^{\pm 1})^*$ be the injective morphism defined by $\varphi(x) = xx$ for $x \in \Sigma^{\pm 1}$. Thus, $w \in L(A)$ if and only if $\varphi(w) \in \varphi(L(A))$. Since $(x, y) \in I$ implies that $\varphi(x)$ and $\varphi(y)$ commute, φ can be lifted to an injective morphism $\varphi : \mathbb{M}(\Sigma^{\pm 1}, I) \to \mathbb{M}(\Sigma^{\pm 1}, I)$. The reader can easily verify that, for every trace $t \in \mathbb{M}(\Sigma^{\pm 1}, I)$, the equality $\mathrm{NF}_R(\varphi(t)) = \varphi(\mathrm{NF}_R(t))$ holds. In particular, $\varphi(t)$ is irreducible if and only if t is irreducible and $h(t) = h(u)$ if and only if $h(\varphi(t)) = h(\varphi(u))$.

Let us fix a trace $t \in \mathbb{M}(\Sigma^{\pm 1}, I)$ and define

$$\Delta = \{\widetilde{q}\varphi(x)\widetilde{p}^{-1} \mid (q, x, p) \in \delta\} \subseteq \mathbb{M}(\Sigma^{\pm 1}, I) \text{ and } u = \widetilde{q_0}\varphi(t)\widetilde{q_f}^{-1} \in \mathbb{M}(\Sigma^{\pm 1}, I).$$

We will show that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$.

Let us define a 1-*cycle* to be a word in $(\Sigma^{\pm 1})^*$of the form

$$\widetilde{q}_1\varphi(v_1)\widetilde{q}_2^{-1}\, \widetilde{q}_2\varphi(v_2)\widetilde{q}_3^{-1} \cdots \widetilde{q}_{k-1}\varphi(v_{k-1})\widetilde{q}_k^{-1}\, \widetilde{q}_k\varphi(v_k)\widetilde{q}_1^{-1}$$

such that $k \geq 1$, $q_1, \ldots, q_k \in \{1, \ldots, n\}$, $v_1, \ldots, v_k \in (\Sigma^{\pm 1})^*$, and $v_1 \cdots v_k = 1$ in $\mathbb{G}(\Sigma, I)$ (hence, also $\varphi(v_1) \cdots \varphi(v_k) = 1$ in $\mathbb{G}(\Sigma, I)$). Note that a 1-cycle equals 1 in $\mathbb{G}(\Sigma, I)$. We say that a word of the form $\widetilde{q}_1\varphi(v_1)\widetilde{p}_1^{-1}\, \widetilde{q}_2\varphi(v_2)\widetilde{p}_2^{-1} \cdots \widetilde{q}_m\varphi(v_m)\widetilde{p}_m^{-1}$, where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$ and $v_1, \ldots, v_m \in (\Sigma^{\pm 1})^*$, contains a 1-cycle, if there are positions $1 \leq i \leq j \leq m$ such that $\widetilde{q}_i\varphi(v_i)\widetilde{p}_i^{-1} \cdots \widetilde{q}_j\varphi(v_j)\widetilde{p}_j^{-1}$ is a 1-cycle. If a word does not contain a 1-cycle, then it is called 1-*cycle-free*.

Claim 3. Let $m \geq 1$ and

$$v = \widetilde{q}_1\varphi(v_1)\widetilde{p}_1^{-1}\, \widetilde{q}_2\varphi(v_2)\widetilde{p}_2^{-1} \cdots \widetilde{q}_m\varphi(v_m)\widetilde{p}_m^{-1},$$

where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$ and $v_1, \ldots, v_m \in (\Sigma^{\pm 1})^*$. If $v = 1$ in $\mathbb{G}(\Sigma, I)$, then v contains a 1-cycle.

Proof of Claim 3. We prove Claim 3 by induction over m. Assume that $v = 1$ in $\mathbb{G}(\Sigma, I)$. If $m = 1$, then we obtain the identity

$$\widetilde{q_1}\varphi(v_1)\widetilde{p_1}^{-1} = (ada)^{q_1} d (ada)^{-q_1} \varphi(v_1)(ada)^{p_1} d^{-1} (ada)^{-p_1} = 1 \tag{10}$$

in $\mathbb{G}(\Sigma, I)$. Assume without loss of generality that v_1, viewed as a trace, is irreducible with respect to R. Then also $\varphi(v_1)$ is irreducible. If $\varphi(v_1) = \varepsilon$ and $p_1 = q_1$, then v is a 1-cycle. If $\varphi(v_1) = \varepsilon$, and $p_1 \neq q_1$, then we obtain a contradiction, since $\mathrm{NF}_R(\widetilde{q_1}\widetilde{p_1}^{-1})$ is nonempty by Claim 2. Now assume that $\varphi(v_1) \neq \varepsilon$. In the trace

$$(ada)^{q_1} d (ada)^{-q_1} \varphi(v_1)(ada)^{p_1} d^{-1} (ada)^{-p_1}$$

only the last a^{-1} of the factor $(a^{-1}d^{-1}a^{-1})^{q_1}$ may cancel against the first a of $\varphi(v_1)$ (in case $a \in \min(v_1)$) and the first a of the factor $(ada)^{p_1}$ may cancel against the last a^{-1} of $\varphi(v_1)$ (in case $a^{-1} \in \max(v_1)$). To see this, note that if $a \notin \min(v_1)$, then $(ada)^{-q_1}\varphi(v_1)$ is irreducible with respect to R. If $a \in \min(v_1)$ then $\varphi(v_1) = aa\varphi(t)$ for some trace t. Then

$$(a^{-1}d^{-1}a^{-1})^{q_1}\varphi(v_1) = (a^{-1}d^{-1}a^{-1})^{q_1} aa\varphi(t) \to_R (a^{-1}d^{-1}a^{-1})^{q_1-1} a^{-1}d^{-1}a\varphi(t).$$

Since a and d do not commute, we cannot have $d \in \min(a\varphi(t))$, hence cancellation stops and $\mathrm{NF}_R((a^{-1}d^{-1}a^{-1})^{q_1}\varphi(v_1)) = (a^{-1}d^{-1}a^{-1})^k a^{-1}d^{-1}a\varphi(t)$ where $k = q_1 - 1 \geq 0$. Moreover, if a^{-1} is a maximal symbol of t, then $\varphi(t) = \varphi(t')a^{-1}a^{-1}$ for some trace t'. Hence, by making a possible cancellation with the first a in $(ada)^{p_1}$, it follows finally that

$$\mathrm{NF}_R(\widetilde{q_1}\varphi(v_1)\widetilde{p_1}^{-1}) = (ada)^{q_1} d (ada)^{-k} a^{-1} d^{-1} x da (ada)^{\ell} d^{-1} (ada)^{-p_1} \neq \varepsilon$$

for some trace x, where $\ell = p_1 - 1 \geq 0$. This contradicts again (10) and proves the inductive base case $m = 1$ in Claim 3.

Now assume that $m \geq 2$.

Case 1. There is $1 \leq i < m$ such that $p_i = q_{i+1}$. Then $v = 1$ in $\mathbb{G}(\Sigma, I)$ implies

$$\widetilde{q_1}\varphi(v_1)\widetilde{p_1}^{-1} \cdots \widetilde{q_{i-1}}\varphi(v_{i-1})\widetilde{p_{i-1}}^{-1}\, \widetilde{q_i}\varphi(v_i v_{i+1})\widetilde{p_{i+1}}^{-1}\, \widetilde{q_{i+2}}\varphi(v_{i+2})\widetilde{p_{i+2}}^{-1} \cdots \widetilde{q_m}\varphi(v_m)\widetilde{p_m}^{-1}$$

is 1 in $\mathbb{G}(\Sigma, I)$. By induction, we can conclude that above word contains a 1-cycle. But then also the word v must contain a 1-cycle.

Case 2. $p_i \neq q_{i+1}$ for all $1 \leq i < m$. If there is $1 \leq i \leq m$ such that $v_i = 1$ in $\mathbb{G}(\Sigma, I)$ and $q_i = p_i$ then v contains the 1-cycle $\widetilde{q_i}\varphi(v_i)\widetilde{p_i}^{-1}$. Now assume that $q_i \neq p_i$ whenever $v_i = 1$ in $\mathbb{G}(\Sigma, I)$. Let v' be the word that results from v by deleting all factors $\varphi(v_i)$, which are equal 1 in $\mathbb{G}(\Sigma, I)$. In the following, we consider v' as a trace. Consider a maximal factor of v' of the form

$$\widetilde{p_i}^{-1}\widetilde{q_{i+1}}\widetilde{p_{i+1}}^{-1}\widetilde{q_{i+2}} \cdots \widetilde{p_{j-1}}^{-1}\widetilde{q_j} \tag{11}$$

where $j \geq i+1$ and $\varphi(v_{i+1}) = \cdots = \varphi(v_{j-1}) = 1$, $\varphi(v_i) \neq 1 \neq \varphi(v_j)$ in $\mathbb{G}(\Sigma, I)$. Claim 2 show that the R-normal form of this trace starts (resp. ends) with a copy of ada (resp. $a^{-1}d^{-1}a^{-1}$), and similarly for maximal prefixes (resp. suffixes) of the form

$$\widetilde{q}_1\widetilde{p}_1^{-1}\cdots\widetilde{q}_{i-1}\widetilde{p}_{i-1}^{-1}\widetilde{q}_i \qquad \text{(resp. } \widetilde{p}_i^{-1}\widetilde{q}_{i+1}\widetilde{p}_{i+1}^{-1}\cdots\widetilde{p}_m^{-1}\widetilde{q}_m). \tag{12}$$

In v', factors of the form (11) and (12) are separated by traces $\varphi(v_i)$, where $\varphi(v_i) \neq 1$ in $\mathbb{G}(\Sigma, I)$. Without loss of generality assume that each such trace $\varphi(v_i)$ is irreducible and hence non-empty. As for the base case $m = 1$, one can show that in such a concatenation, only a single minimal a and a single maximal a^{-1} of a trace $\varphi(v_i) \neq \varepsilon$ may be cancelled. It follows that $\mathrm{NF}_R(v) \neq \varepsilon$, which contradicts $v = 1$ in $\mathbb{G}(\Sigma, I)$. This concludes the proof of Claim 3.

Now we can prove $h(t) \in h(L(A))$ if and only if $h(u) = h(\widetilde{q}_0\varphi(t)\widetilde{q}_f^{-1}) \in h(\Delta^*)$. First assume that $h(t) \in h(L(A))$. Let $a_1\cdots a_m \in L(A)$ such that $(q_{i-1}, a_i, q_i) \in \delta$ for $1 \leq i \leq m$, $q_m = q_f$, and $a_1\cdots a_m = t$ in $\mathbb{G}(\Sigma, I)$. Then

$$h(\widetilde{q}_0\varphi(t)\widetilde{q}_f^{-1}) = h(\widetilde{q}_0\varphi(a_1)\widetilde{q}_1^{-1}\widetilde{q}_1\varphi(a_2)\widetilde{q}_2^{-1}\cdots\widetilde{q}_{m-1}\varphi(a_m)\widetilde{q}_m^{-1}) \in h(\Delta^*).$$

Now assume that $h(\widetilde{q}_0\varphi(t)\widetilde{q}_f^{-1}) \in h(\Delta^*)$. Thus,

$$\widetilde{q}_0\varphi(t)\widetilde{q}_f^{-1} = \widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\widetilde{q}_2\varphi(a_2)\widetilde{p}_2^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}$$

in $\mathbb{G}(\Sigma, I)$, where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$, $a_1, \ldots, a_m \in \Sigma^{\pm 1} \cup \{\varepsilon\}$, and $(q_i, a_i, p_i) \in \delta$ for $1 \leq i \leq m$. Without loss of generality we may assume that the word $\widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\widetilde{q}_2\varphi(a_2)\widetilde{p}_2^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}$ is 1-cycle-free (otherwise we can remove all 1-cycles from this word; note that a 1-cycle equals 1 in the group $\mathbb{G}(\Sigma, I)$). Let

$$v = \widetilde{q}_f\varphi(t^{-1})\widetilde{q}_0^{-1}\,\widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\,\widetilde{q}_2\varphi(a_2)\widetilde{p}_2^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}.$$

Since $v = 1$ in $\mathbb{G}(\Sigma, I)$, Claim 3 implies that v contains a 1-cycle. We claim that this 1-cycle must be the whole word v: first of all, the suffix $\widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}$ of v is 1-cycle-free. If a prefix $\widetilde{q}_f\varphi(t^{-1})\widetilde{q}_0^{-1}\,\widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\cdots\widetilde{q}_i\varphi(a_i)\widetilde{p}_i^{-1}$ for $i < m$ is a 1-cycle, then $\widetilde{q}_{i+1}\varphi(a_{i+1})\widetilde{p}_{i+1}^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1} = 1$ in $\mathbb{G}(\Sigma, I)$. Hence, Claim 3 implies that the word $\widetilde{q}_{i+1}\varphi(a_{i+1})\widetilde{p}_{i+1}^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}$ contains a 1-cycle, contradicting the fact that the word $\widetilde{q}_1\varphi(a_1)\widetilde{p}_1^{-1}\cdots\widetilde{q}_m\varphi(a_m)\widetilde{p}_m^{-1}$ is 1-cycle-free. Thus, indeed, v is a 1-cycle. Hence, $q_0 = q_1$, $q_f = p_m$, $p_i = q_{i+1}$ for $1 \leq i < m$, and $t^{-1}a_1\cdots a_m = 1$ in $\mathbb{G}(\Sigma, I)$, i.e., $h(t) = h(a_1\cdots a_m) \in h(L(A))$. This shows that the membership problem for the submonoid $h(\Delta^*)$ within $\mathbb{G}(\Sigma, I)$ is indeed undecidable. $\square$

Recall that a graph is not a transitive forest if and only if it either contains an induced $\mathsf{C4}$ or $\mathsf{P4}$ [39]. Together with Mihailova's result for the generalized word problem of $F(\{a,b\}) \times F(\{c,d\})$, Theorems 2 and 7 imply:

Corollary 3. *The submonoid membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if (Σ, I) is a transitive forest. Moreover, if (Σ, I) is not a transitive forest, then there exists a fixed submonoid M of $\mathbb{G}(\Sigma, I)$ such that the membership problem for M within $\mathbb{G}(\Sigma, I)$ is undecidable.*

Since $\mathsf{P4}$ is a chordal graph, the generalized word problem for $\mathbb{G}(\mathsf{P4})$ is decidable [25]. Hence, $\mathbb{G}(\mathsf{P4})$ is an example of a group for which the generalized word problem is decidable but the submonoid membership problem is undecidable.

5 Open problems

The definition of the class $\mathcal{C}$ at the beginning of Section 3 leads to the question whether decidability of the rational subset membership problem is preserved under direct products with $\mathbb{Z}$. An affirmative answer would lead in combination with the results from [24, 31] to a more direct proof of Theorem 1.

Concerning graph groups, the precise borderline for the decidability of the generalized word problem remains open. By [25], the generalized word problem is decidable if the independence alphabet is chordal. Since every transitive forest is chordal, Theorem 2 does not add any new decidable cases. On the other hand, if the independence alphabet contains an induced $\mathsf{C4}$, then the generalized word problem is undecidable [30]. But it is open for instance, whether for a cycle of length 5 the corresponding graph group has a decidable generalized word problem.

Another open problem concerns the complexity of the rational subset membership problem for graph groups, where the independence alphabet is a transitive forest. If the independence alphabet is part of the input, then our decision procedure does not yield an elementary algorithm, i.e., an algorithm where the running time is bounded by an exponent tower of fixed height. This is due to the fact that each calculation of the Parikh image of a context-free language leads to an exponential blow-up in the size of the semilinear sets in the proof of Lemma 7. An NP lower bound follows from the NP-completeness of integer programming.

Theorem 4 and 5 lead to various research directions. One might try to get rid of the restriction that $g^{-1}Ag \cap A = 1 = h^{-1}Ah \cap A$ for some $g \in G$, $h \in H$ and the analogous restrictions for HNN extensions. These two results together would imply that Corollary 2 holds for all groups with two or more ends.

In fact it is natural to ask whether, for every finitely generated group G, the submonoid membership and rational subset membership problems are recursively equivalent. By Theorem 4, this is equivalent to the preservation of the decidability of the submonoid membership problem under free products (which is again not known to hold): simply choose for H in Theorem 4 any nontrivial group with a decidable rational subset membership problem. Recall that the decidability of the generalized word problem as well as the rational subset membership problem is preserved under free products. Notice that for a torsion group, the submonoid membership problem is equivalent to the generalized word problem, while the rational subset membership problem reduces to membership in products $H_1 \cdots H_n$ of finitely generated subgroups.

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