Application of verification techniques in inverse monoid theory

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September 17, 2005

Monoid presentations

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- Markov (1947), Post (1947): There is a finite presentation P such that Σ^*/P has an undecidable word problem.
- Novikov (1955), Boone (1957): There is a finite presentation P such that Σ^*/P is a group and has an undecidable word problem.

Inverse monoids

A monoid $\mathcal{M}$ is inverse if for every $x \in \mathcal{M}$ there exists a unique $x^{-1} \in \mathcal{M}$ with $x\,x^{-1}x = x$ and $x^{-1}x\,x^{-1} = x^{-1}$.

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For a finite set Γ let $\mathrm{FIM}(\Gamma)$ be the free inverse monoid generated by Γ (exists, since inverse monoids form a variety).

An explicite representation of $\mathrm{FIM}(\Gamma)$: Let

- $\Gamma^{-1} = \{a^{-1} \mid a \in \Gamma\}$ be a disjoint copy of Γ,
- $(a^{-1})^{-1} := a$,
- $(b_1 b_2 \cdots b_n)^{-1} := b_n^{-1} \cdots b_2^{-1} b_1^{-1}$ for $b_i \in \Gamma \cup \Gamma^{-1}$,
- $\mathcal{W} = \{(s, s\,s^{-1}s),\ (s\,s^{-1}t\,t^{-1},\ t\,t^{-1}s\,s^{-1}) \mid s, t \in (\Gamma \cup \Gamma^{-1})^*\}$ (Wagner equations)

Then $\mathrm{FIM}(\Gamma) \simeq (\Gamma \cup \Gamma^{-1})^*/\mathcal{W}$.

Finitely presented inverse monoids

For a presentation $P \subseteq (\Gamma \cup \Gamma^{-1})^* \times (\Gamma \cup \Gamma^{-1})^*$ let

$$\mathrm{FIM}(\Gamma)/P := (\Gamma \cup \Gamma^{-1})^*/(P \cup \mathcal{W}).$$

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A presentation $P \subseteq (\Gamma \cup \Gamma^{-1})^* \times (\Gamma \cup \Gamma^{-1})^*$ is idempotent if for every $(e, f) \in P$, e and f are idempotents of $\mathrm{FIM}(\Gamma)$ ($e^2 = e$ and $f^2 = f$ in $\mathrm{FIM}(\Gamma)$).

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Theorem (Margolis, Meakin)

The following problem is decidable:
INPUT: Idempotent presentation P over alphabet $\Gamma \cup \Gamma^{-1}$, words $u, v \in (\Gamma \cup \Gamma^{-1})^*$.
QUESTION: $u = v$ in $\mathrm{FIM}(\Gamma)/P$?

Free groups

The free group $\mathrm{FG}(\Gamma)$ generated by Γ:

$$\mathrm{FG}(\Gamma) = (\Gamma \cup \Gamma^{-1})^* / \{(bb^{-1}, \varepsilon) \mid b \in \Gamma \cup \Gamma^{-1}\}$$

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Remark: Let $u \in (\Gamma \cup \Gamma^{-1})^*$. Then $u^2 = u$ in $\mathrm{FIM}(\Gamma)$ if and only if $u = 1$ in $\mathrm{FG}(\Gamma)$.

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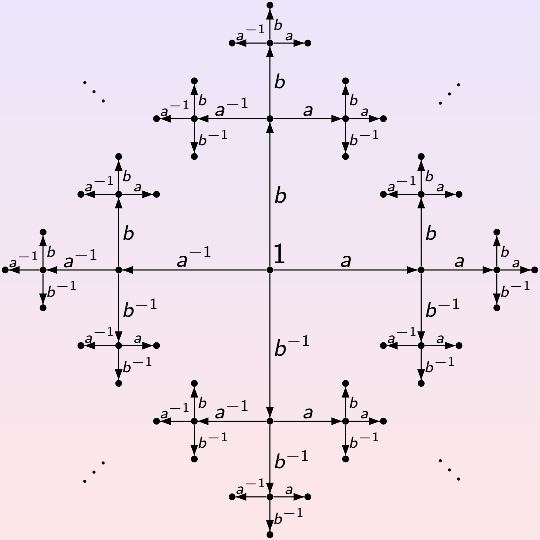
The Cayley graph of $\mathrm{FG}(\Gamma)$ is the following edge-labeled graph with node set $\mathrm{FG}(\Gamma)$:

$$(\mathrm{FG}(\Gamma), (E_a)_{a \in \Gamma \cup \Gamma^{-1}}),$$

where $E_a = \{(u, v) \mid u, v \in \mathrm{FG}(\Gamma), v = ua \text{ in } \mathrm{FG}(\Gamma)\}$.

Cayley graph of a free group:

The Cayley graph of the free group $\mathrm{FG}(\{a, b\})$:



Munn trees

Let $u = c_1 c_2 \cdots c_n \in (\Gamma \cup \Gamma^{-1})^*$ be a word.
The Munn tree of u is the finite subtree of the Cayley graph of $\mathrm{FG}(\Gamma)$, that are visited while traveling along the path $c_1 c_2 \cdots c_n$ in the Cayley graph of $\mathrm{FG}(\Gamma)$.

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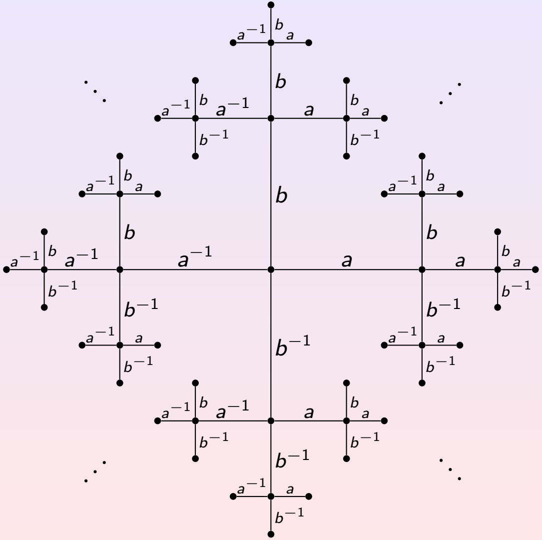
Theorem (Munn)

Let $u, v \in (\Gamma \cup \Gamma^{-1})^*$. Then $u = v$ in $\mathrm{FIM}(\Gamma)$ if and only if

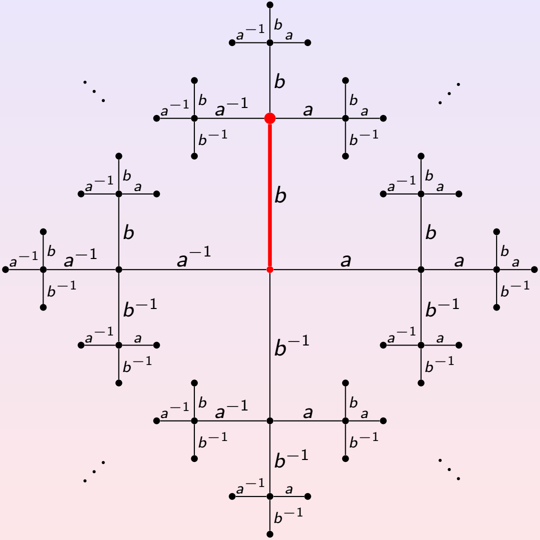
1. *$u = v$ in the free group $\mathrm{FG}(\Gamma)$*
2. *u and v have the same Munn trees.*

Munn trees

The Munn tree of $bb^{-1}abb^{-1}a$:



The Munn tree of $\color{red}{b}b^{-1}abb^{-1}a$:



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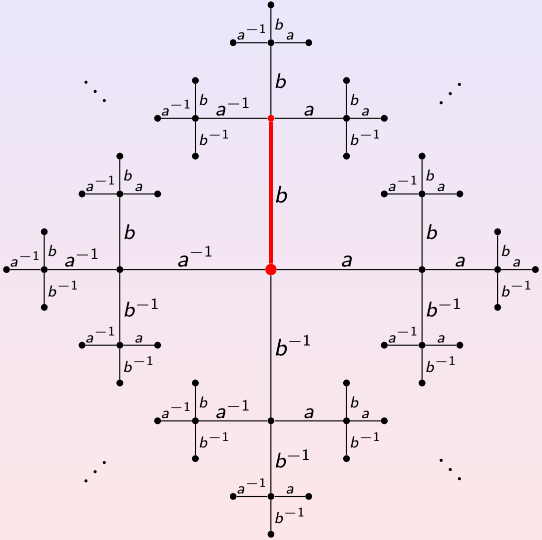

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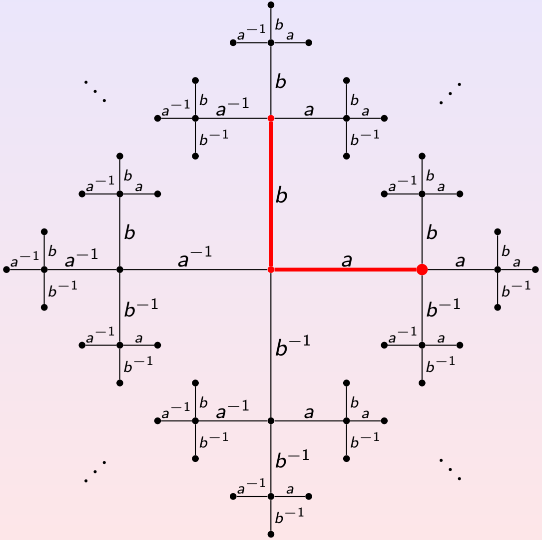

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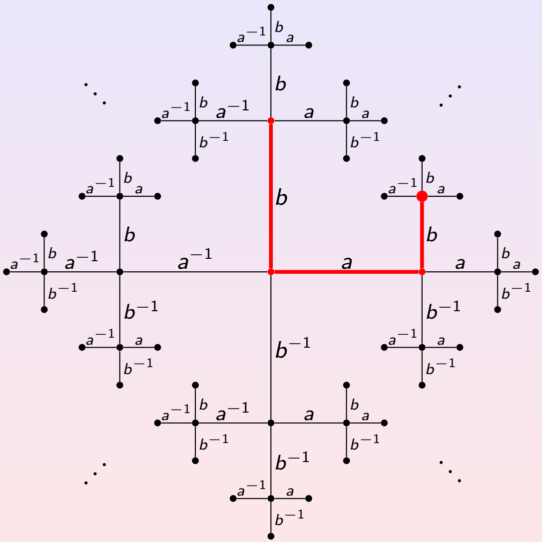

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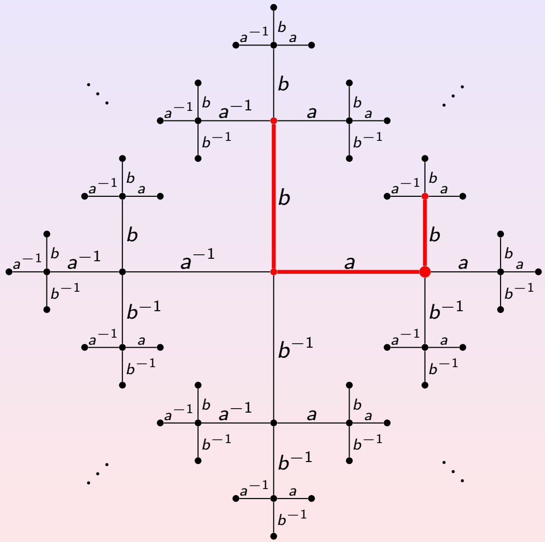

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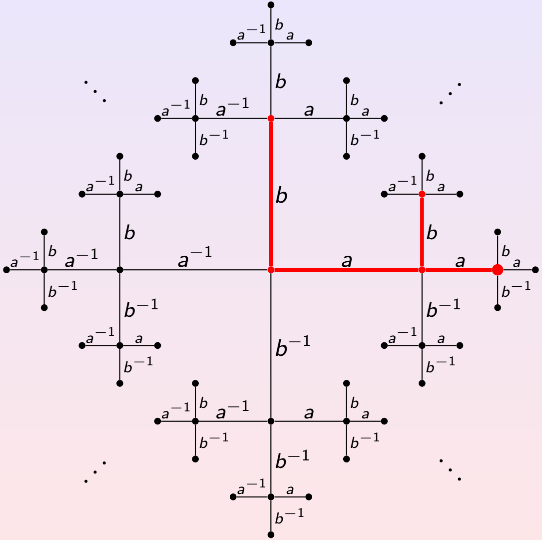

Closure of a word

Let $u \in (\Gamma \cup \Gamma^{-1})^*$ be a word and $P \subseteq (\Gamma \cup \Gamma^{-1})^* \times (\Gamma \cup \Gamma^{-1})^*$ be an idempotent presentation.
W.l.o.g. Munn tree of $e \subseteq$ Munn tree of f for all $(e, f) \in P$.

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The closure $\mathrm{cl}_P(u)$ of u wrt. P for an example:

$$u = aa^{-1}bb^{-2}, \quad P = \{(aa^{-1}, a^2a^{-2}), (aa^{-1}bb^{-1}, aa^{-1}baa^{-1}b^{-1})\}$$

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The relations in P seen as identities between Munn trees:

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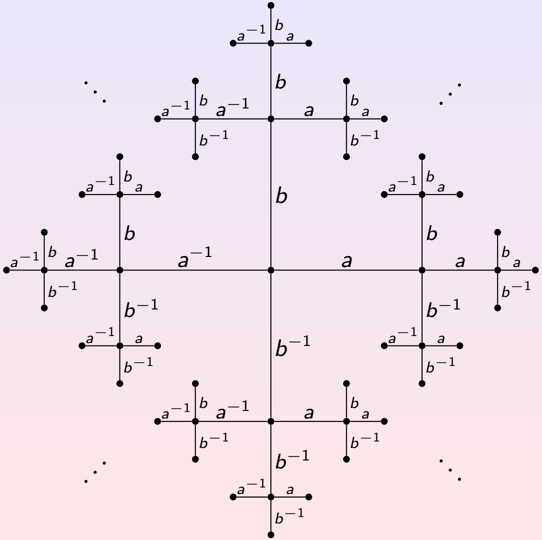
1. Construct the Munn tree of $u = aa^{-1}bb^{-2}$.



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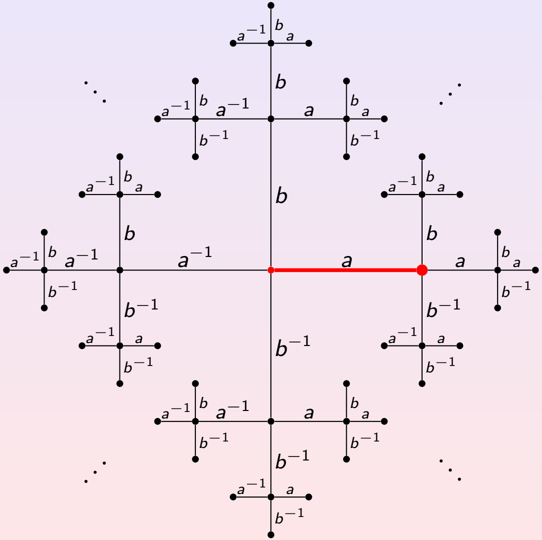

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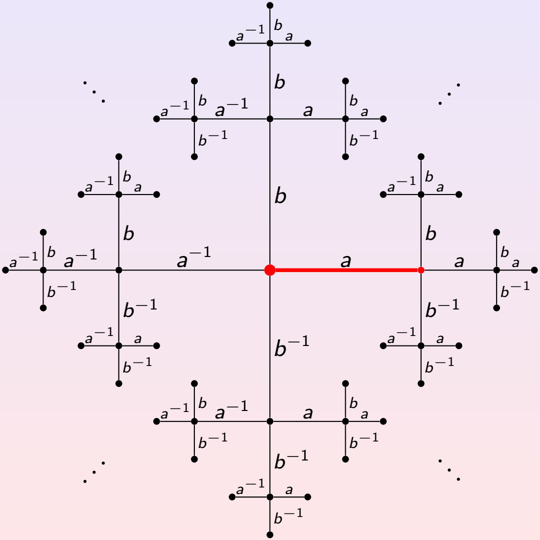

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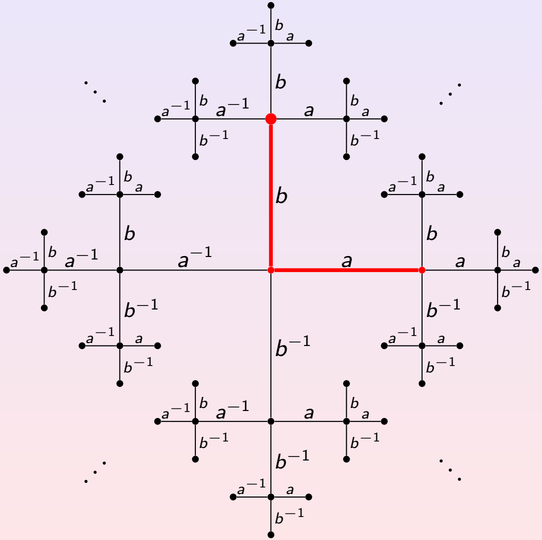

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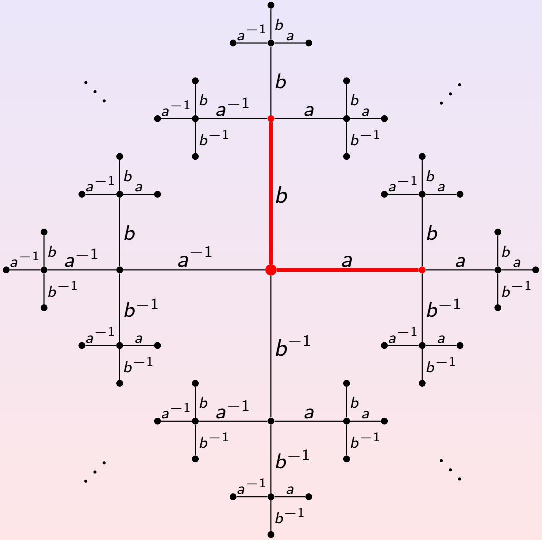

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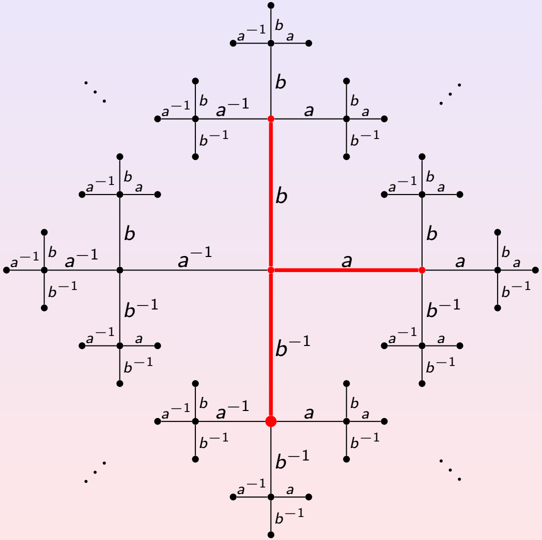
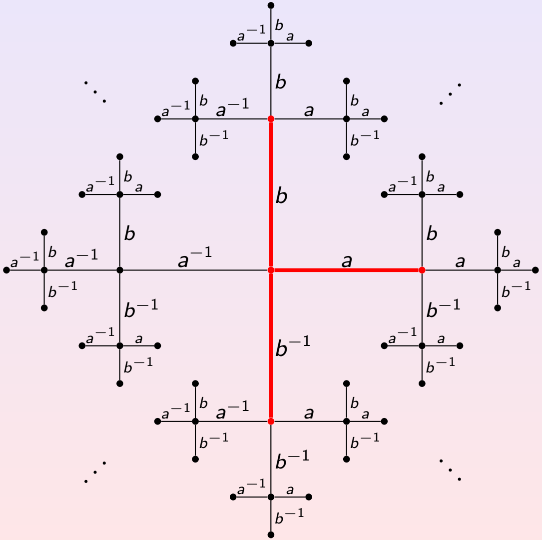


Closure of $u = aa^{-1}bb^{-2}$ w.r.t. P

2. Find Munn tree of a left-hand side of a relation (ℓ, r) in red part.



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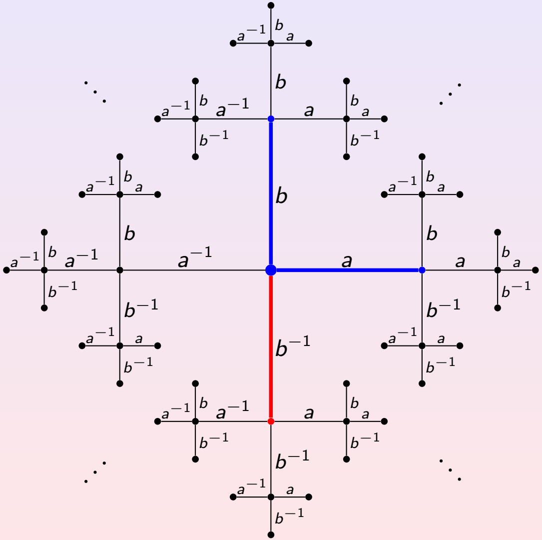
2. $(\color{blue}{aa^{-1}bb^{-1}}, aa^{-1}baa^{-1}b^{-1})$



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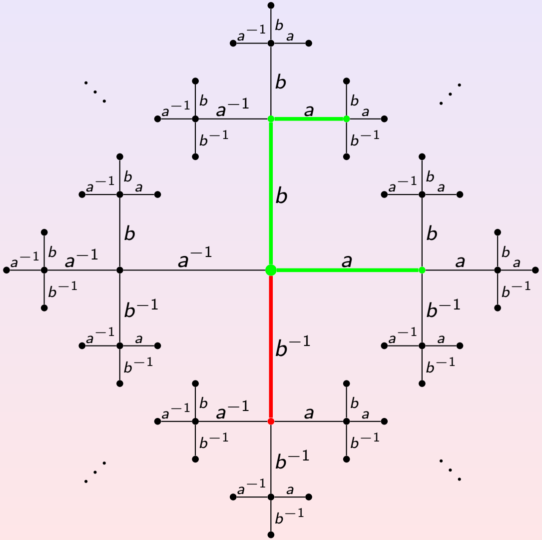
3. Extend by Munn tree of right-hand side $aa^{-1}baa^{-1}b^{-1}$.



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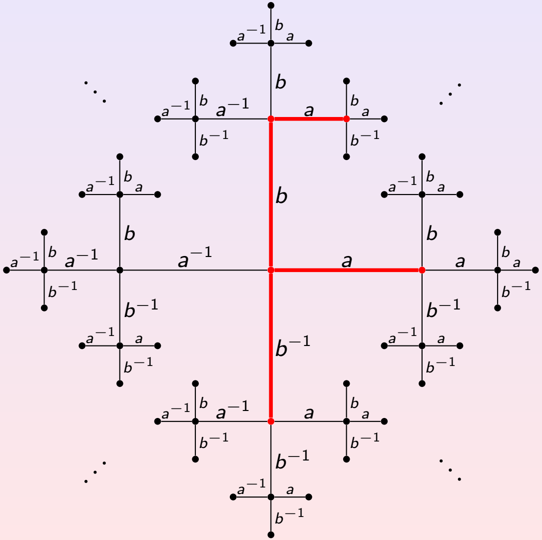

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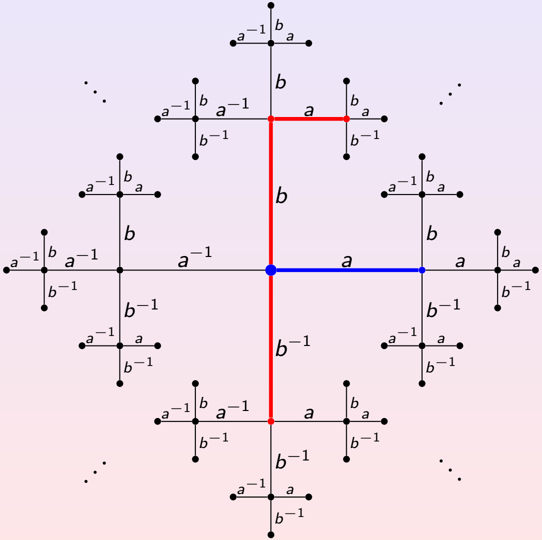

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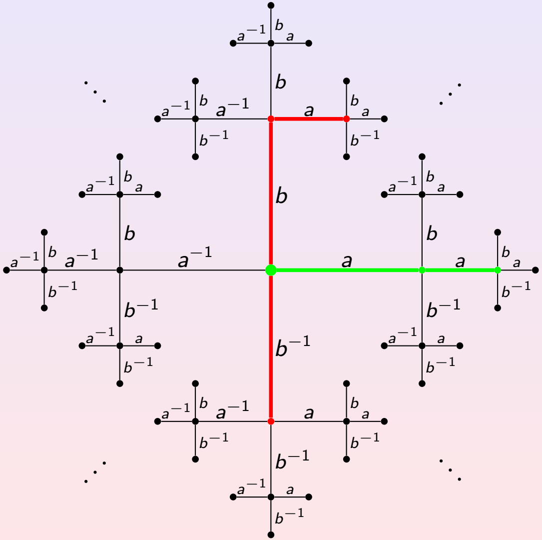
3. Extend by Munn tree of right-hand side a^2a^{-2}.



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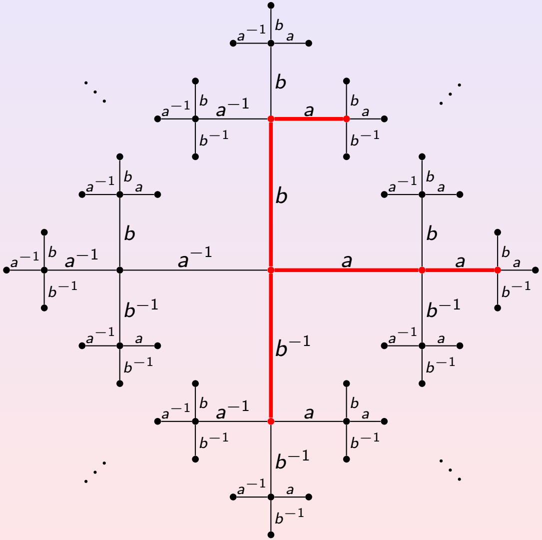

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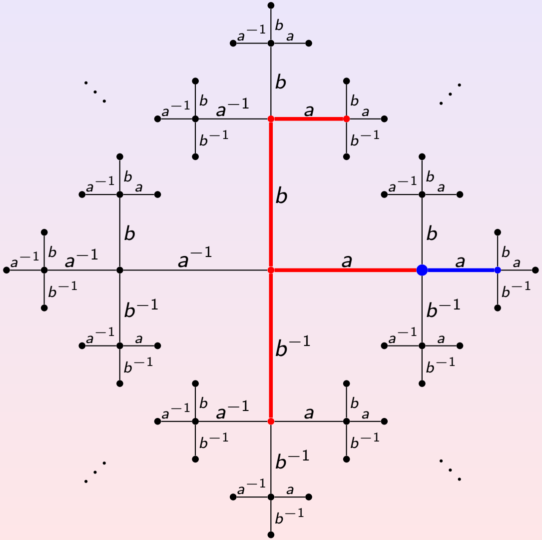

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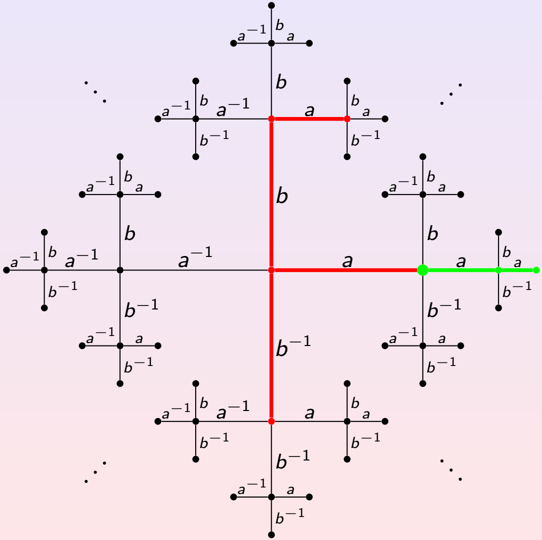

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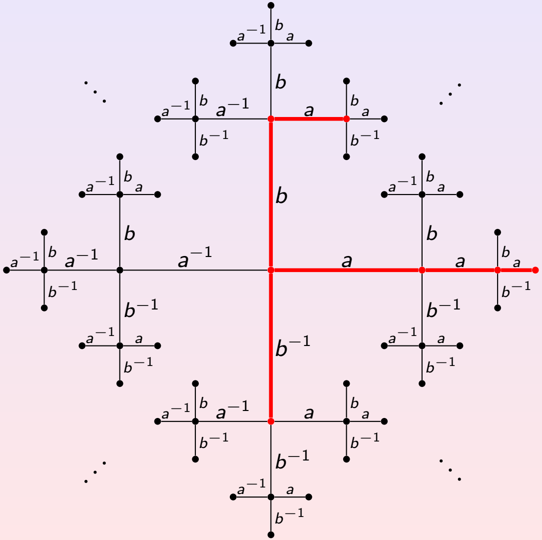

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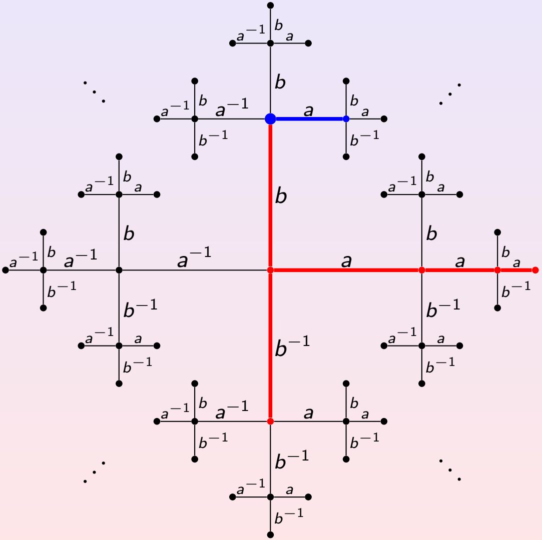

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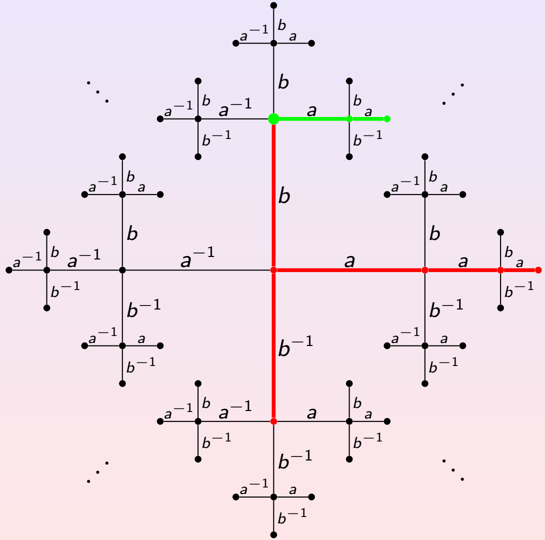

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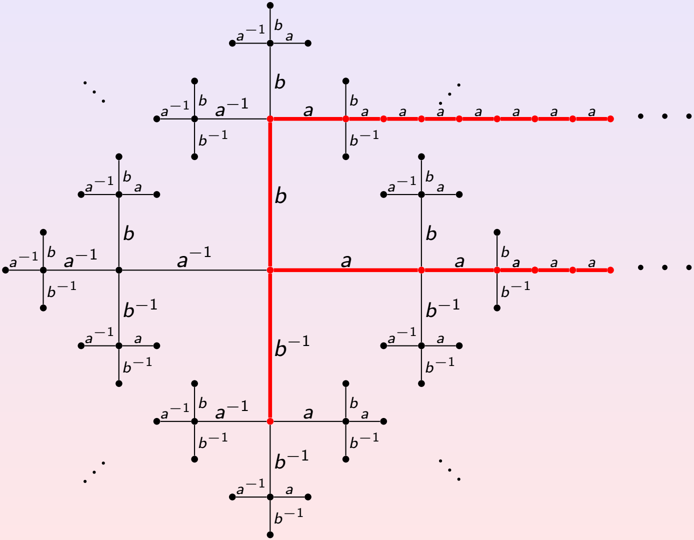
In the limit, we obtain the closure:



Closure of a word

Theorem (Stephen/Margolis,Meakin)

Let $u, v \in (\Gamma \cup \Gamma^{-1})^*$ and P be an idempotent presentation. Then $u = v$ in $\mathrm{FIM}(\Gamma)/P$ if and only if:

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This leads to a decision algorithm for "$u = v$ in $\mathrm{FIM}(\Gamma)/P$":

- "$u = v$ in the free group $\mathrm{FG}(\Gamma)$" can be checked in linear time.
- "$\mathrm{cl}_P(u) = \mathrm{cl}_P(v)$" can be expressed in monadic second-order logic over the Cayley graph of $\mathrm{FG}(\Gamma)$.
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But: The monadic second-order theory of the Cayley graph of $\mathrm{FG}(\Gamma)$ has no elementary decision algorithm.

Complexity of the word problem

Our first observation: "$\mathsf{cl}_P(u) = \mathsf{cl}_P(v)$" can be expressed in the modal μ-calculus over the Cayley graph of $\mathrm{FG}(\Gamma)$.

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Walukiewicz 96: It can be checked in exponential time, whether a formula of the modal μ-calculus holds in a context-free graph (pushdown system).

Theorem (L., Ondrusch)

The following problem is EXPTIME-complete:

INPUT: Idempotent presentation P over alphabet $\Gamma \cup \Gamma^{-1}$, words $u, v \in (\Gamma \cup \Gamma^{-1})^*$.

QUESTION: $u = v$ in $\mathrm{FIM}(\Gamma)/P$?

Moreover, EXPTIME-hardness already holds for a fixed alphabet Γ.

Complexity of the word problem

Note that:

$$\mathsf{cl}_P(u) = \mathsf{cl}_P(v) \quad \Leftrightarrow \quad \mathrm{MT}(u) \subseteq \mathsf{cl}_P(v) \ \wedge \ \mathrm{MT}(v) \subseteq \mathsf{cl}_P(u)$$

Complexity of the word problem

Note that:

$$\mathsf{cl}_P(u) = \mathsf{cl}_P(v) \quad \Leftrightarrow \quad \mathrm{MT}(u) \subseteq \mathsf{cl}_P(v) \ \wedge \ \mathrm{MT}(v) \subseteq \mathsf{cl}_P(u)$$

Let G be the Cayley graph of $\mathrm{FG}(\Gamma)$ with the node 1 marked.
Let $\varphi_{v,P}$ be the following sentence:

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Then $(G,x) \models \varphi_{v,P}$ if and only if $x \in \mathsf{cl}_P(v)$.

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Thus,

$$\mathrm{MT}(u) \subseteq \mathsf{cl}_P(v) \quad \Leftrightarrow \quad (G, 1) \models \bigwedge_{i=0}^{|u|} \langle u[1,i] \rangle \varphi_{v,P}.$$

Application to model-checking of pushdown systems

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Theorem (L., Ondrusch)

There exists a fixed alphabet Γ such that the following problem UWP-IP(Γ) is EXPTIME-complete:

INPUT: Idempotent presentation P over alphabet $\Gamma \cup \Gamma^{-1}$, words $u, v \in (\Gamma \cup \Gamma^{-1})^*$.

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Corollary

There exists a fixed pushdown system G *such that the following problem is EXPTIME-complete:*

INPUT: Formula φ *of the modal* μ*-calculus (of nesting depth 1)*

QUESTION: $G \models \varphi$*?*

Complexity of the word problem for a fixed presentation

Theorem

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Proof idea: Reduction to membership problem for a fixed tree automaton, which only depends on the presentation P.

More general decision problems

Let $\mathcal{M} = (M, \circ, 1)$ be a monoid with a finite generating set Σ.
Let $\mathbb{C}_{\mathcal{M}}^{\mathrm{reg}}$ be the following structure:

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Proof idea: Reduction to the monadic second-order theory of the Cayley graph of the free group $\mathrm{FG}(\Gamma)$.

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Let P be an idempotent presentation P over alphabet $\Gamma \cup \Gamma^{-1}$.

- *For two rational subsets $L, K \subseteq \mathrm{FIM}(\Gamma)/P$ we can decide, whether $L \cap K = \emptyset$.*
- *The generalized word problem of $\mathrm{FIM}(\Gamma)/P$ (Does u belongs to the submonoid generated by $v_1, \ldots, v_n$?) is decidable.*

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Proof idea: Find large grid-structures as minors in the Cayley graph of $\mathrm{FIM}(\Gamma)$.