Model-checking hierarchical graphs

Markus Lohrey
University of Stuttgart

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Hierarchical graph definitions (HGDs) I

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An HGD D

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Macros: $S \succ A \succ B \succ C$

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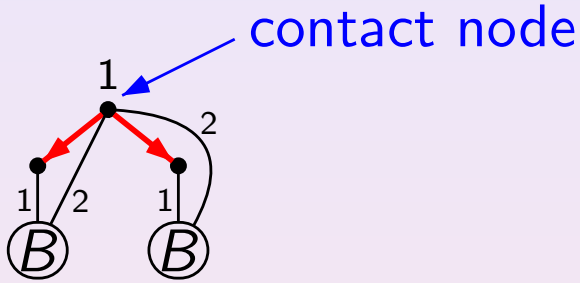
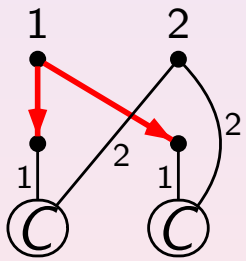
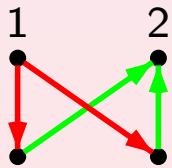
$A :=$ 

$B :=$ 

$C :=$ 

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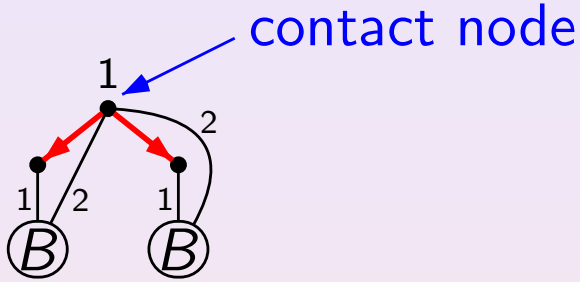
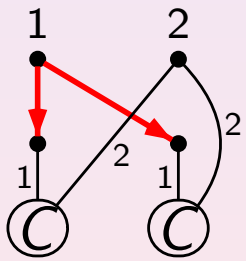
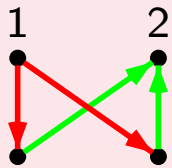
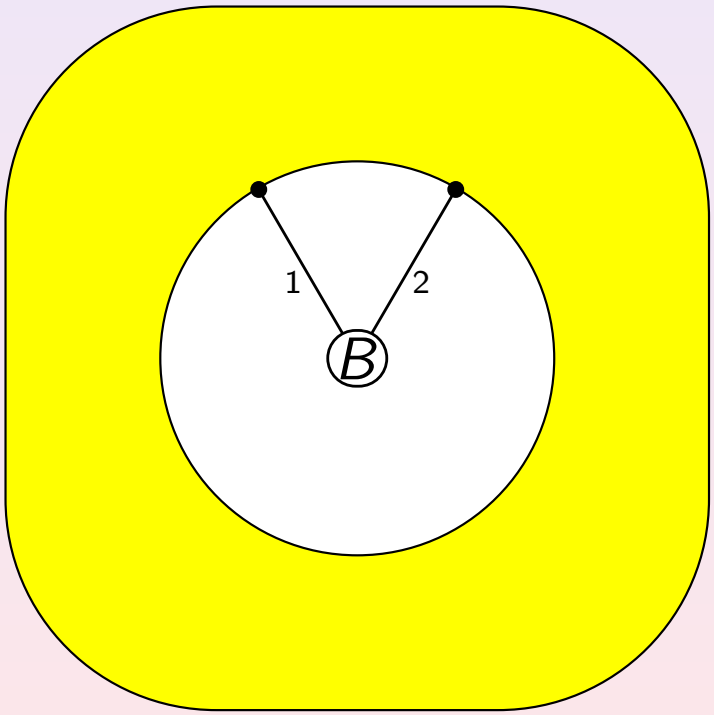
$A :=$ 

$B :=$ 

$C :=$ 



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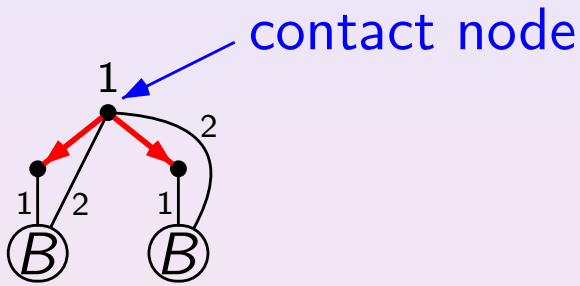
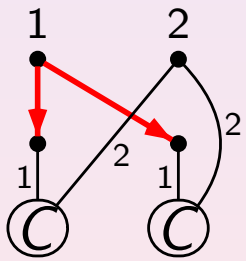
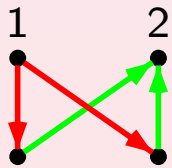
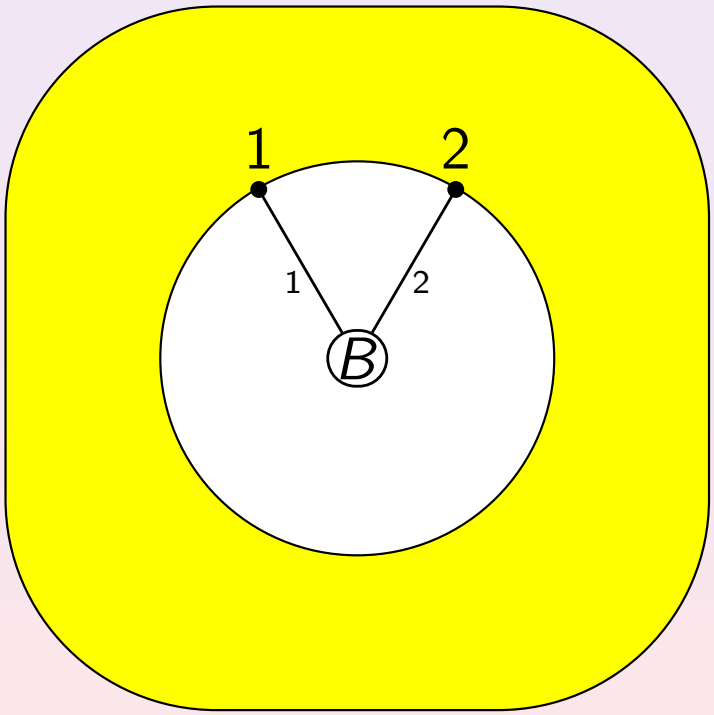
$A :=$ 

$B :=$ 

$C :=$ 



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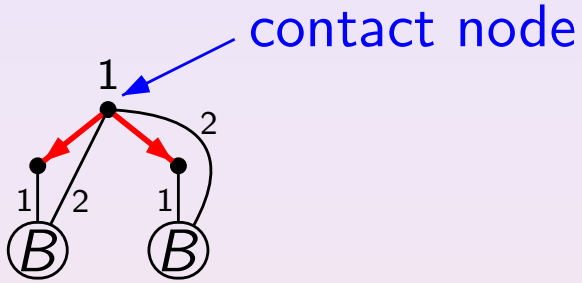
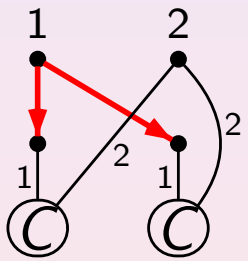
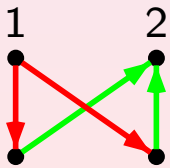
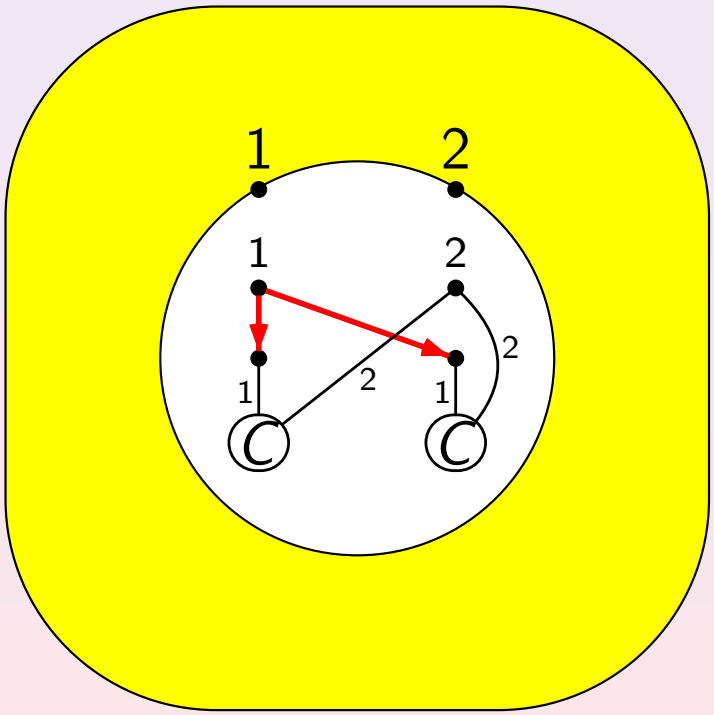
$A :=$ 

$B :=$ 

$C :=$ 



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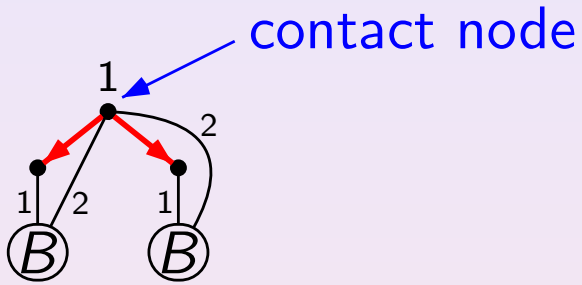
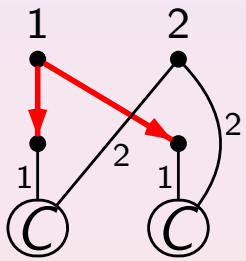
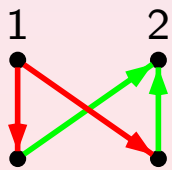
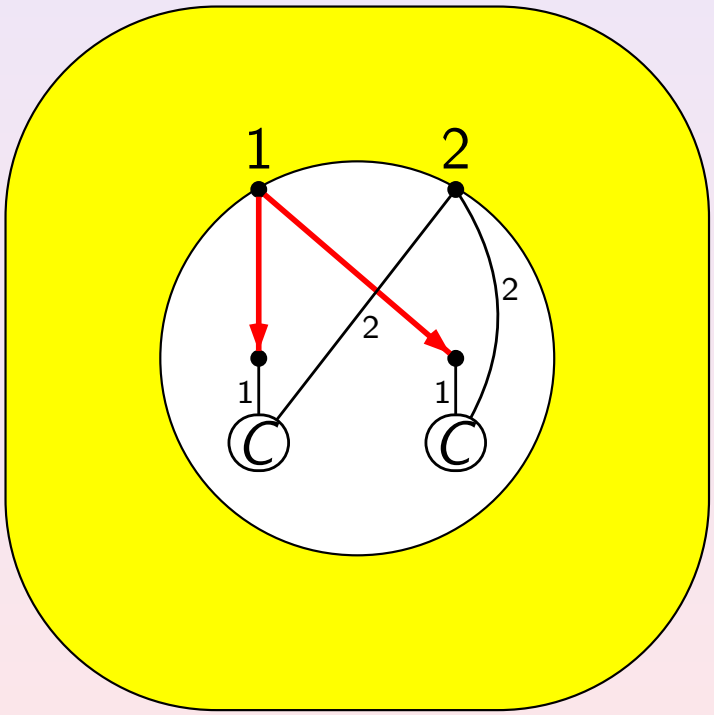
$A :=$ 

$B :=$ 

$C :=$ 



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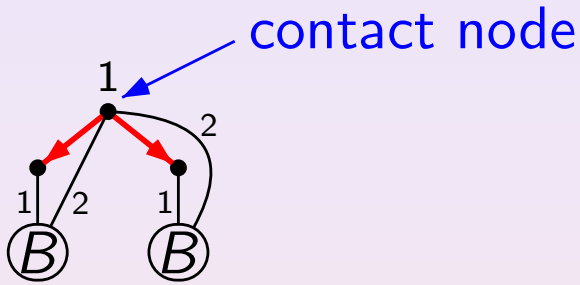
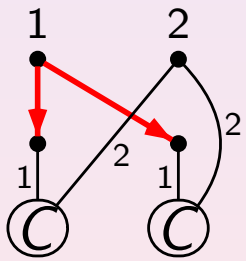
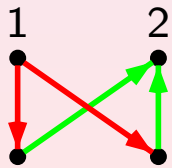
$A :=$ 

$B :=$ 

$C :=$ 

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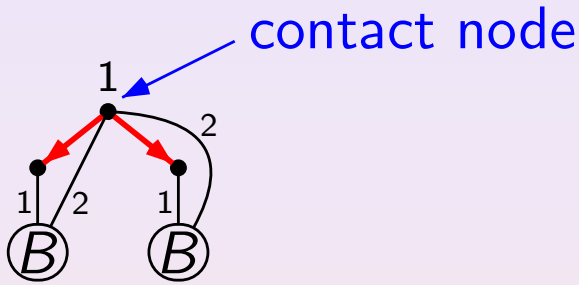
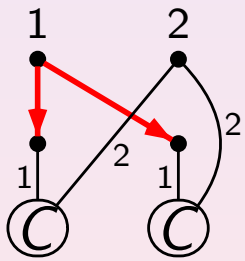
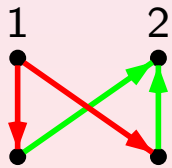
$A :=$ 

$B :=$ 

$C :=$ 

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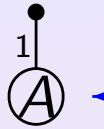
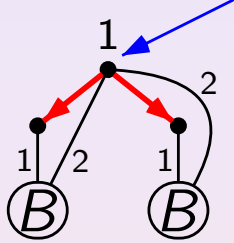
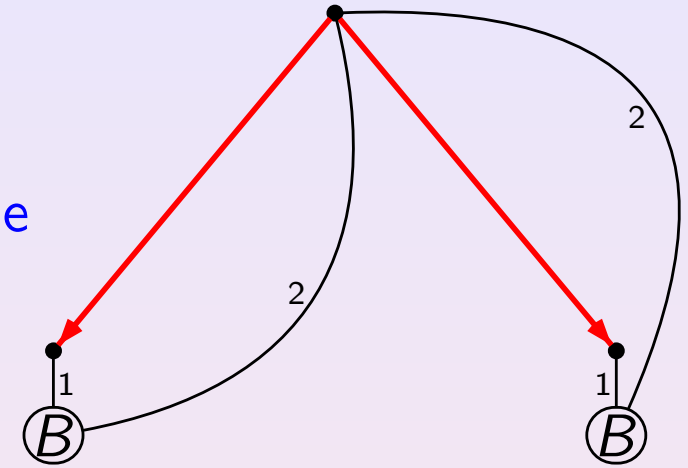
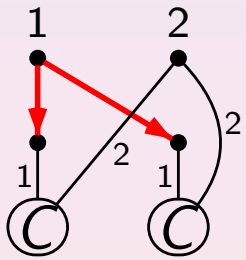
$S :=$ 

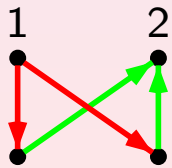
macro

contact node

$A :=$ 



$B :=$ 

$C :=$ 

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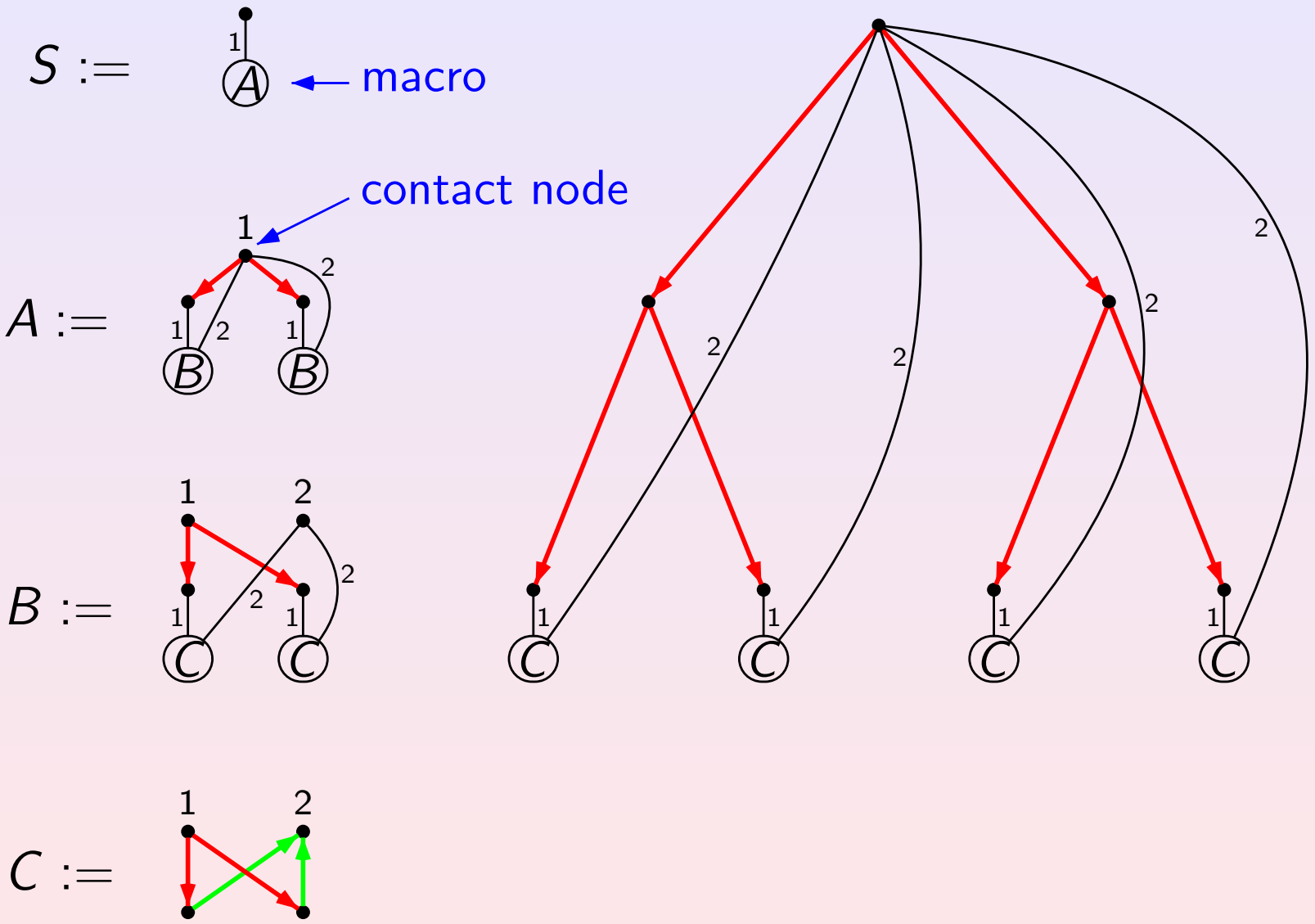

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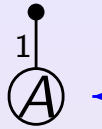
$S :=$ 

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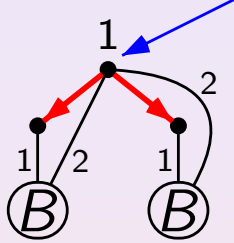
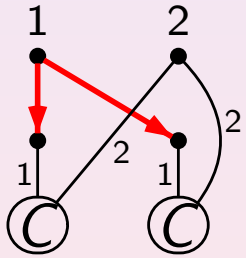
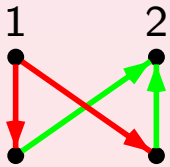
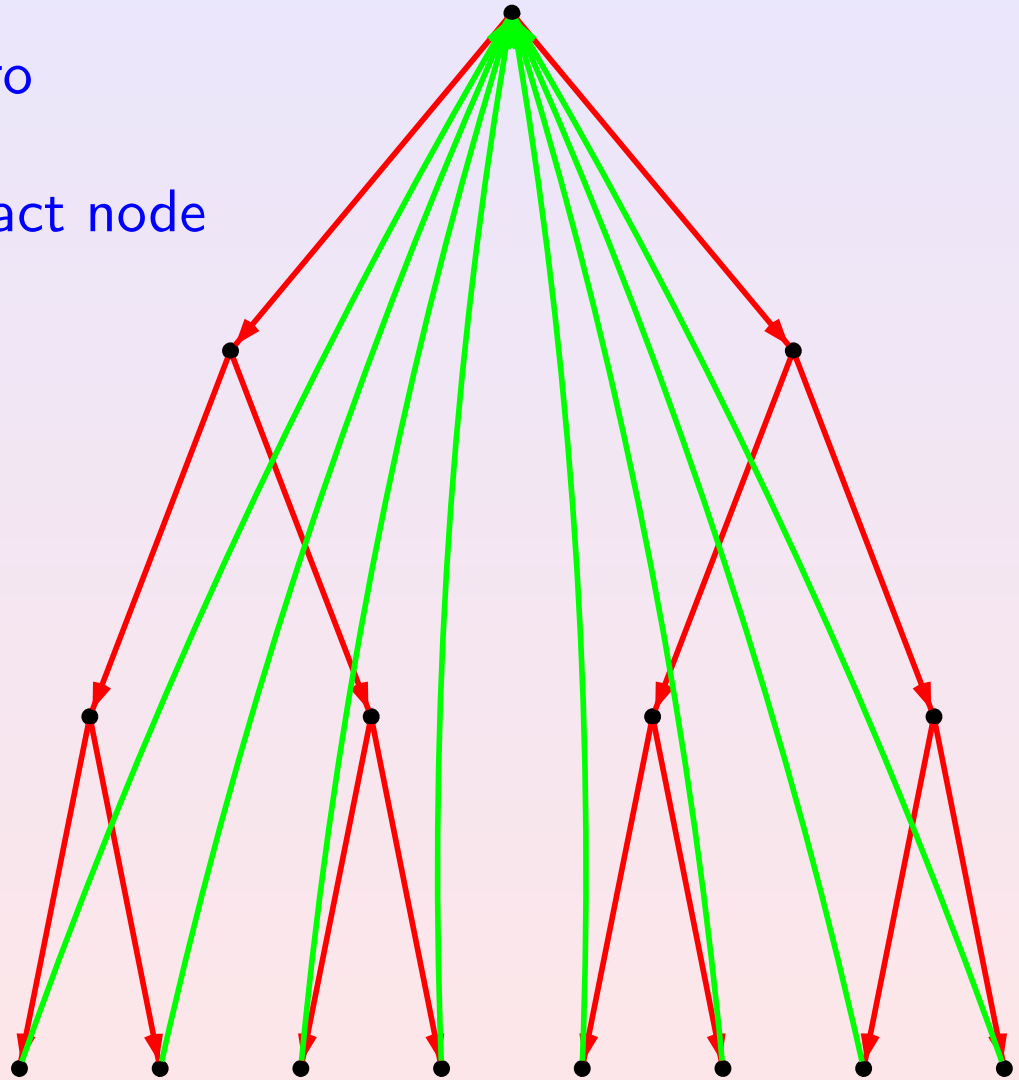
$A :=$ 

$B :=$ 

$C :=$ 

unfold(D)



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Consequence: Computational problems tend to be more difficult, when input structures are given hierarchical!

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W.l.o.g. at most two references in each right-hand side (Chomsky normal form)

Hierarchical graph definitions (HGDs) III

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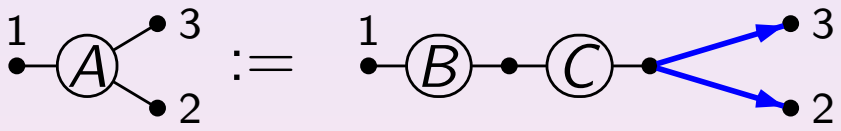
A HGD D is non-passing if macros are not connected to contact nodes in right-hand sides.

E.g. not allowed: 

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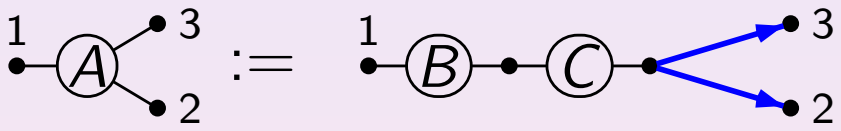
E.g. not allowed: 

A HGD D is c-bounded for $c \in \mathbb{N}$ if the rank of every macro is at most c.

History and related formalisms

Previous work:

- Lengauer, Wanke (around 1990): Specific algorithmic problems on hierarchically defined graphs.
- Marathe, Hunt, Stearns, Radhakrishnan (around 1995): Approximation algorithms for hierarchically defined graphs.
- Alur, Yannakakis 2001: Model-checking temporal logics for hierarchical state machines.
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Here: Model-Checking of first-order logic (FO) and monadic second-order logic (MSO) on hierarchically defined input graphs.

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$$\forall x \in V \, \exists y_1 \in V \, \exists y_2 \in V : y_1 \neq y_2 \ \wedge \ E(x, y_1) \ \wedge \ E(x, y_2) \wedge$$
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Monadic second-order logic (MSO): quantification over elements and subsets of a structure
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$$\exists x \in V \, \exists y \in V \, \exists X \subseteq V : x \in X \wedge y \notin Y \wedge \\ \forall u \in V \, \forall v \in V : u \in X \wedge E(u, v) \Rightarrow v \in X$$

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Complexity of FO

data complexity	non-hierarchical	apex HGD	c-bounded HGD	general HGD
Σ_k-FO	AC^0_k	NL	NL $\cdots$ P	NL $(k = 1)$ $\Sigma^{\mathrm{poly}}_{k-1}$ $(k > 1)$

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Σ_k-FO	AC^0_k	NL	NL $\cdots$ P	NL $(k = 1)$ $\Sigma^{\mathrm{poly}}_{k-1}$ $(k > 1)$

combined complexity	non-hierarchical	apex HGD	c-bounded HGD	general HGD
Σ_k-FO	Σ^{poly}_k			

Data complexity of Σ_2-FO is NP-hard

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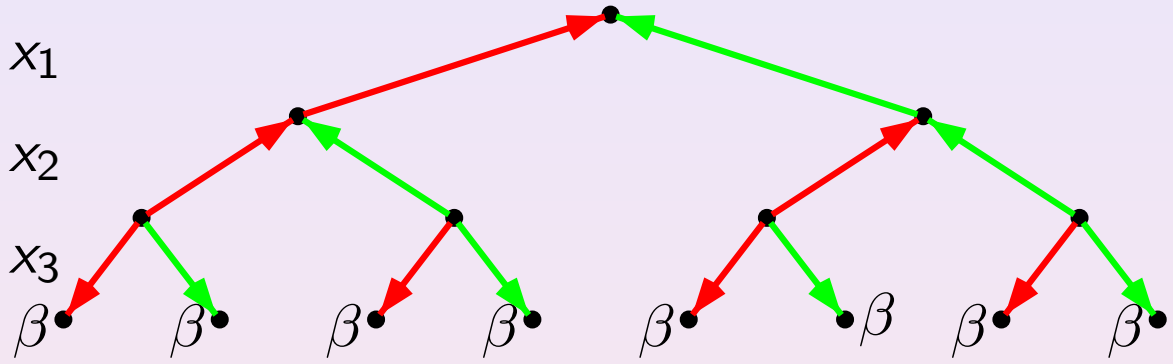
Reduction from the NP-hard problem 3-SAT

Example: $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$



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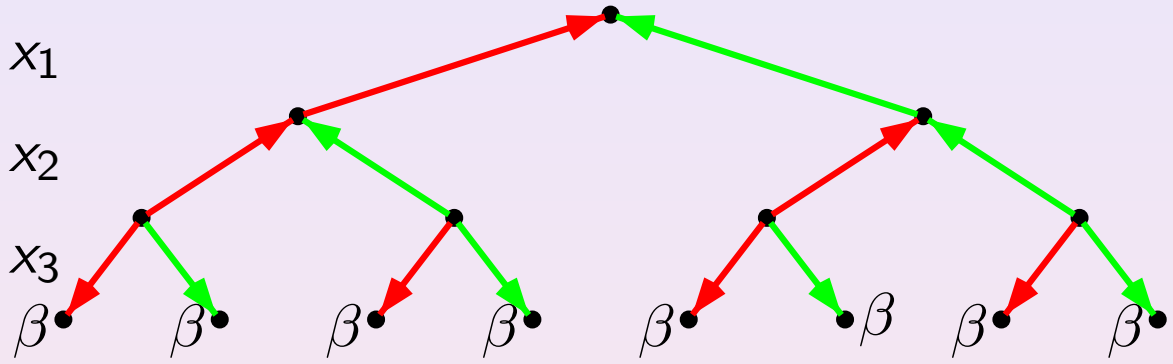

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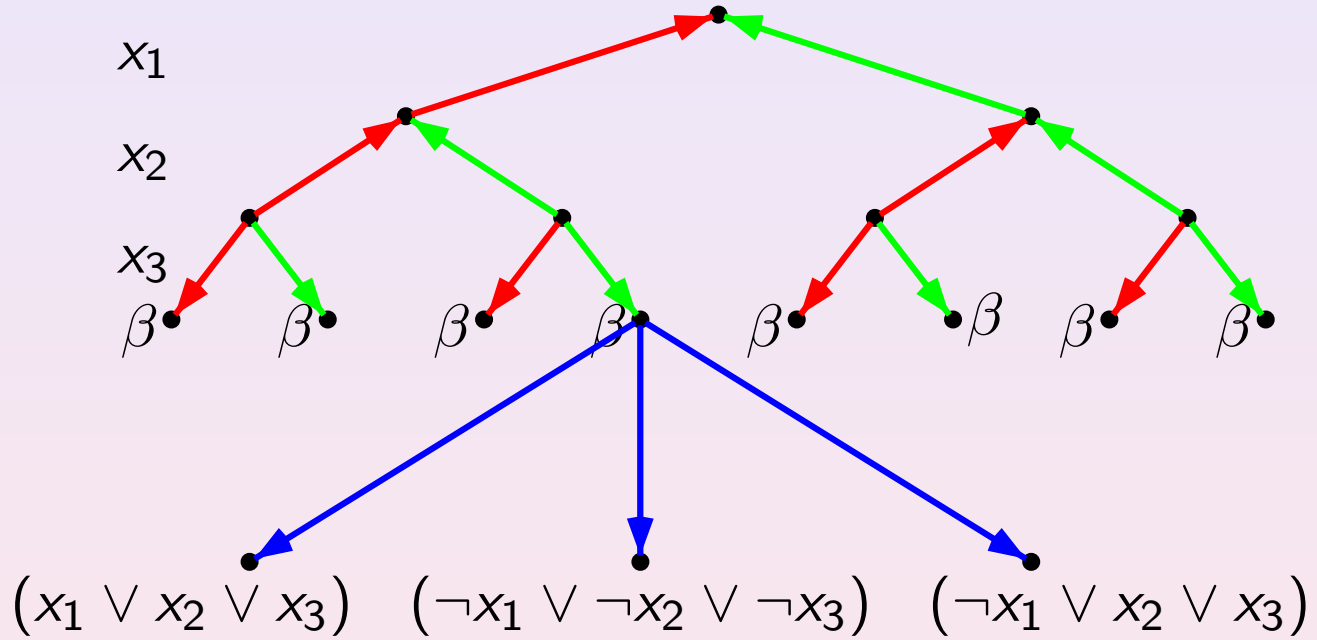

$\exists x : \beta(x) \wedge \forall y : x \longrightarrow y \Rightarrow \cdots$

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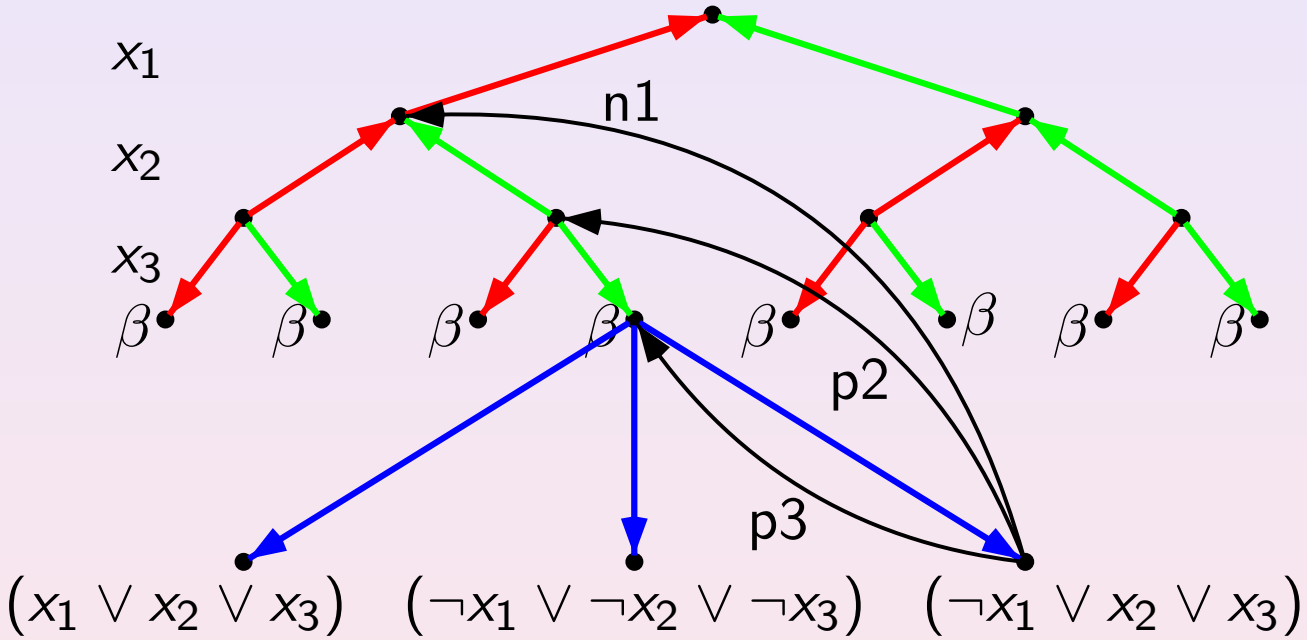

$$\exists x : \beta(x) \wedge \forall y : x \longrightarrow y \Rightarrow \bigvee_{i=1}^{3} \exists z, z' : y \xrightarrow{\text{pi}} z \longrightarrow z' \vee y \xrightarrow{\text{ni}} z \longrightarrow z'$$

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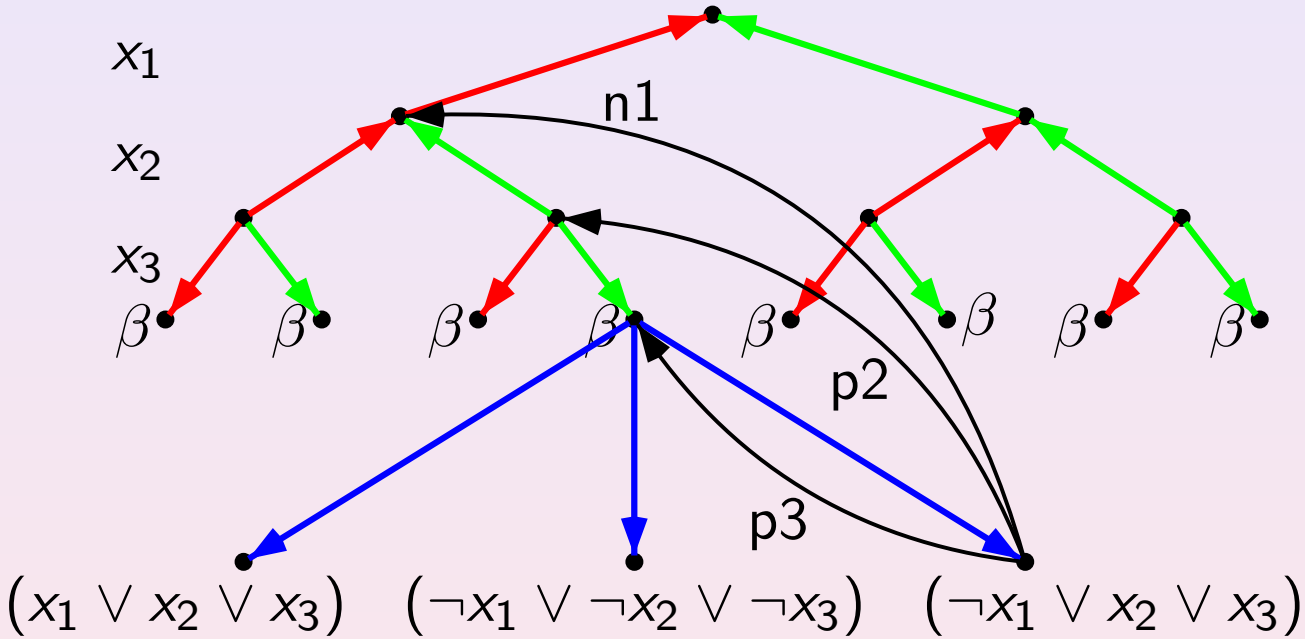

$$\exists x : \beta(x) \wedge \forall y : x \longrightarrow y \Rightarrow \bigvee_{i=1}^{3} \forall z, z' : \neg(y \xrightarrow{\text{pi}} z \longrightarrow z') \wedge \neg(y \xrightarrow{\text{ni}} z \longrightarrow z')$$

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data complexity	non-hierarchical	c-bounded HGD	general HGD
Σ_k-MSO	Σ_k^{poly}		Σ_k^{exp}

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Remark: The apex restriction does not help in the context of MSO.

Future Work/Open Problems

- Close the gap between NL and P for the data complexity of FO for c-bounded HGDs.

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- Close the gap between NL and P for the data complexity of FO for c-bounded HGDs.
- Under which conditions is hierarchical model-checking for a logic harder than "flat" model-checking?
- When can hierarchy be exploited?