When is a Graph Product of Groups Virtually-Free?

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Abstract

Those graph products of groups that are virtually-free are characterized.

1 Introduction

A finitely generated group G is called virtually-free if it has a free subgroup of finite index. Finitely generated virtually-free groups can be characterized in many different ways using notions from formal language theory [1, 13, 14], graph theory [3, 14], and mathematical logic [10, 14]. This makes virtually-free groups to an important research topic in combinatorial group theory. It is known that every virtually-free group is hyperbolic and that the class of virtually-free groups is closed under the operations of amalgamated free products and HNN extensions, when both operations are restricted to finite associated subgroups.

In this paper we study the operation of graph product, which is defined as follows: Let $X = (V, I)$ be a finite undirected graph, i.e., V is a finite set of nodes and I is an irreflexive and symmetric relation on I. For each node $v \in V$ let $G_v \neq 1$ be a nontrivial group. Then, the graph product $\mathbb{GP}(X, (G_v)_{v \in V})$ is the free product of the groups G_v subject to the relations $xyx^{-1}y^{-1} = 1$ for all $x \in G_u$, $y \in G_v$ such that $(u, v) \in I$. This operation generalizes both free and direct products. Graph products were first studied in [7]. Several nice closure properties are known for graph products. For instance, the following classes are all closed under graph products: residually finite groups [7], semihyperbolic groups [9], automatic groups [9]. In [12], a complete answer to the question under which conditions a graph product of hyperbolic groups is again hyperbolic was given (see Section 2 for definitions concerning graph theory): $\mathbb{GP}(X, (G_v)_{v \in V})$ is hyperbolic if and only if

(i) for every $v \in V$, G_v is hyperbolic,

(ii) if G_u and G_v are infinite groups and $u \neq v$, then $(u, v) \notin I$,

(iii) if G_u is an infinite group, G_v and G_w are finite groups such that $v \neq w$, and $(u, v), (u, w) \in I$, then also $(v, w) \in I$, and

(iv) the graph X does not contain an induced cycle of length 4.

The main result from this paper gives a similar characterization for virtually-free groups:

Theorem 1.1. *Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product with $X = (V, I)$. Then G is virtually-free if and only if:*

(1) for every $v \in V$, G_v is virtually-free,

(2) if G_u and G_v are infinite groups and $u \neq v$, then $(u, v) \notin I$,

(3) if G_u is an infinite group, G_v and G_w are finite groups such that $v \neq w$, and $(u, v), (u, w) \in I$, then also $(v, w) \in I$, and

(4) the graph X is chordal.

Note that condition (2) and (3) are identical to condition (ii) and (iii) in the corresponding characterization for hyperbolic groups and that condition (1) is just the obvious counterpart of condition (i). But condition (4) is strictly stronger than condition (iv) which only requires that X does not contain an induced cycle of length 4. In particular we see that a graph product $\mathbb{GP}(X, (G_v)_{v \in V})$, where every group G_v is finite and X is a cycle on 5 nodes, is hyperbolic but not virtually-free.

2 Preliminaries

For background in combinatorial group theory (resp. graph theory) see [11] (resp. [5]). In this paper we restrict to finitely generated groups. We will not mention this implicit assumption in the following. Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product as defined in Section 1, where $X = (V, I)$ is a finite undirected graph. A *clique* of $X = (V, I)$ is a subset $U \subseteq V$ such that $(u, v) \in I$ for all $u, v \in U$ with $u \neq v$. A cycle of length n is a graph of the form $(\{v_0, \ldots, v_{n-1}\}, J)$ such that: $(v_i, v_j) \in J$ if and only if $i = j + 1 \mod n$ or $j = i + 1 \mod n$. Let $U \subseteq V$. The subgraph of X *induced* by U is $X{\restriction_U} = (U, I \cap (U \times U))$. We say that $X = (V, I)$ contains an induced cycle of length n if there is $U \subseteq V$ such that $X{\restriction_U}$ is a cycle of length n. The graph $X = (V, I)$ is *chordal* if X does not contain an induced cycle of length at least 4. For $U \subseteq V$ we define the graph product $G{\restriction_U}$ as $\mathbb{GP}(X{\restriction_U}, (G_v)_{v \in U})$; it is a subgroup of G [7]. For groups G_1, G_2, and H, and injective homomorphisms $h_1 : H \to G_1$ and $h_2 : H \to G_2$, the *amalgamated free product* $G_1 *_H G_2$ is defined as the free product of G_1 and G_2 subject to the relations $h_1(x) = h_2(x)$ for every $x \in H$. W.l.o.g. we may assume that $G_1 \cap G_2 = H$ and that h_i is just the inclusion map. We need the following result on graph products.

Proposition 2.1 (p 48 of [7]). *Let $G = \mathbb{GP}(X, (G_v)_{v\in V})$ be a graph product, $X = (V, I)$, let $u \in V$, and let $N(u) = \{v \in V \mid (u, v) \in I\}$ be the neighbors of u. Then*

$$G \simeq (G{\restriction}_{V\setminus\{u\}}) *_{G{\restriction}_{N(u)}} (G{\restriction}_{N(u)} \times G_u).$$

We will also need some standard results from Basse-Serre theory of groups acting on trees, see [4] for more details. For this, we have to define another notion of graph, which differs from the one for undirected graphs that was used for graph products. For the rest of this section, a graph is a tuple $Y = (V, E, s, t, ^{-})$, where $V \neq \emptyset$ is the set of vertices, E is the set of edges ($V \cap E = \emptyset$), $s : E \to V$ maps every edge e to its source vertex $s(e)$, $t : E \to V$ maps every edge e to its target vertex $t(e)$, and $^{-}\colon E \to E$ is an involution without fixpoints such that: $s(e) = t(\bar{e})$ for all $e \in E$. A path in Y is a sequence of edges $p = (e_1, \cdots, e_n)$ $(n \geq 1)$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i < n$. We also say that the path p connects the vertices $s(e_1)$ and $t(e_n)$. We say that Y is *connected* if every two different nodes $u, v \in V$ are connected by a path. We say that Y is a *tree* if it is connected and for every path $p = (e_1, \cdots, e_n)$ with $s(e_1) = t(e_n)$ there exists $1 \leq i < n$ such that $e_i = \overline{e_{i+1}}$. A group G *acts* on the graph Y *without inversion of edges*, if there exists a morphism $f : G \to \text{Aut}(Y)$ such that $f(e) \neq \bar{e}$ for every edge $e \in E$. We write xg for $f(g)(x)$ ($x \in V \cup E$, $g \in G$). For a node $v \in V$, the *vertex stabilizer* of v is $\{g \in G \mid vg = v\}$.

Proposition 2.2 (p. 104, Corollary 1.9, in [4]). *A group G is virtually-free if and only if G acts on a tree without inversion of edges and such that all vertex stabilizers are finite.*

We will also use the following well-known facts about virtually-free groups:

- A direct product of two infinite groups is not virtually-free.
- Every finitely generated subgroup of a virtually-free group is virtually-free.
- If $G = G_1 *_A G_2$, where G_1 and G_2 are virtually free and A is a finite, then also G is virtually-free.
- If G is virtually-free and A is finite, then also $G \times A$ is virtually-free.

3 Proof of Theorem 1.1

Let us first prove the sufficiency of condition (1)–(4) in Theorem 1.1. We first consider the case of a graph product of finite groups.

Lemma 3.1. *If $G = \mathbb{GP}(X, (G_v)_{v\in V})$ is a graph product such that X is chordal and every group G_v is finite, then $\mathbb{GP}(X, (G_v)_{v\in V})$ is virtually-free.*

Proof. Let $X = (V, I)$ be chordal. We prove by induction on the number of vertices of X that G is virtually-free. By [6] there exists a simplical vertex in X, i.e., a vertex $u \in V$ such that the neighborhood $N_u = \{v \in V \mid (u, v) \in I\}$ of u is a clique of X. Thus, $F = G{\restriction}_{N(u)} = \prod_{v\in N(u)} G_v$ and $F \times G_u$ are both finite (and hence virtually-free). By induction, $H = G{\restriction}_{V\setminus\{u\}}$ is virtually-free. By Proposition 2.1 we have $G \simeq H *_F (F \times G_u)$, so G is virtually-free. $\square$

Now assume that condition (1)–(4) in Theorem 1.1 are true for the graph product $G = \mathbb{GP}(X, (G_v)_{v \in V})$. Let n be the number of vertices $u \in V$ such that G_u is infinite. We show by induction on n that G is virtually-free. The case $n = 0$ is dealt in Lemma 3.1. Now assume that $n > 0$ and let $u \in V$ such that G_u is infinite. Since for the graph product $H = G\restriction_{V \setminus \{u\}}$ again the conditions (1)–(4) from the theorem are satisfied, it follows by induction that H is virtually free. Let $N_u = \{v \in V \mid (u, v) \in I\}$ be the neighborhood of u. By (2) from Theorem 1.1, G_v has to be finite for every $v \in N_u$, and by (3), N_u is a clique of X. Thus, $F = G\restriction_{N(u)} = \prod_{v \in N(u)} G_v$ is finite and hence, $F \times G_u$ is still virtually-free. We can conclude as in the proof of Lemma 3.1.

Let us now prove the other direction in Theorem 1.1. The necessity of the conditions (1)–(3) is easy to see: For (1), just recall that every finitely generated subgroup of a virtually-free group is again virtually-free. Condition (2) has to be true, because the direct product of two infinite groups is not virtually-free. Similarly, if condition (3) would be false, then G would contain a subgroup of the form $G_u \times (G_v * G_w)$ with G_u infinite, and hence again a direct product of two infinite groups.

It remains to show that if $X = (V, I)$ is not chordal then $\mathbb{GP}(X, (G_v)_{v \in V})$ is not virtually-free. Thus, assume that (V, I) is not chordal, i.e., it contains an induced cycle of length at least 4. Since we may assume that condition (2) and (3) are satisfied, it follows, that already the subgraph of X that is induced by those $v \in V$ with G_v finite is non-chordal. Hence, since finitely generated subgroups of virtually-free groups are again virtually-free, it suffices to show the following lemma:

Lemma 3.2. *A graph product $G = \mathbb{GP}(X, (A_v)_{v \in V})$, where $X = (V, I)$ is a cycle of length $n \geq 4$ and every A_v is finite, is not virtually-free.*

We will present two proofs of Lemma 3.2: a very short proof using two difficult results concerning virtual cohomology dimension and a second proof based on Bass-Serre theory.

First proof of Lemma 3.2. It is known that a group is virtually-free if and only if it has virtual cohomology dimension 1, see e.g. [2]. On the other hand, Harlander and Meinert calculated the virtual cohomology dimension of graph products of finite groups [8]. In particular, if G is a graph product as described in Lemma 3.2, then its virtual cohomology dimension is 2 by [8, Thm. 6.6]. Hence, G cannot be virtually-free. □

Second proof of Lemma 3.2. Suppose that G is virtually-free. Let $V = \{0, \ldots, n-1\}$ and $I = \{(i, i+1) \mid 0 \leq i \leq n-1\}$, where here and in the following addition is modulo n. Let A_i be the finite group associated to the vertex i. By Proposition 2.2, G acts on a tree T without inversion of edges and such that all vertex stabilizers are finite. Let T_i be the maximal subtree of T fixed by A_i pointwise, $0 \leq i < n$. Since A_i commutes with A_{i+1}, $A_i \times A_{i+1} \leq G$ is a finite group acting on the tree T. By [4, p. 18, Corollary 4.9] $A_i \times A_{i+1}$ must stabilize a vertex v. Thus, v is stabilized by both A_i and A_{i+1}. Hence, $T_i \cap T_{i+1} \neq \emptyset$. Let us suppose that

$$\forall i \in \{0, \ldots, n-1\} : T_i \cap T_{i+1} \cap T_{i+2} = \emptyset. \tag{1}$$

By induction over $k \in \{0, \ldots, n-1\}$, (1) implies that there exists a length-minimal path of the form

$$v_0 \xrightarrow{*} v_1 \xrightarrow{*} \cdots \xrightarrow{*} v_i \xrightarrow{*} \cdots \xrightarrow{*} v_k$$

where $v_i \in T_i \cap T_{i+1}$, $v_{i-1} \neq v_i$, and the subpath $v_{i-1} \xrightarrow{*} v_i$ lies in T_i. Let us take $k = n - 1$. The length-minimal path from v_0 to v_{n-1} , also lies entirely in T_0 (because $v_0, v_{n-1} \in T_0$). This shows that $v_1 \in T_0 \cap T_1 \cap T_2$, violating hypothesis (1). Hence, there must exist $i \in \{0, \ldots, n-1\}$ such that $T_i \cap T_{i+1} \cap T_{i+2} \neq \emptyset$. Thus, $B = \langle A_i, A_{i+1}, A_{i+2} \rangle$ stabilizes a vertex of T. Hence, B is a finite group. But the groups A_i and A_{i+2} do not commute and therefore generate an infinite group. This is a contradiction. $\square$

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