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# First-order and counting theories of $\omega$-automatic structures[1]


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**Abstract**

The logic $\mathcal{L}(\mathcal{Q}_u)$ extends first-order logic by a generalized form of counting quantifiers ("the number of elements satisfying ... belongs to the set $C$"). This logic is investigated for structures with an injective $\omega$-automatic presentation. If first-order logic is extended by an infinity-quantifier, the resulting theory of any such structure is known to be decidable [5]. It is shown that, as in the case of automatic structures [18], also modulo-counting quantifiers as well as infinite cardinality quantifiers ("there are $\varkappa$ many elements satisfying ...") lead to decidable theories. For a structure of bounded degree with injective $\omega$-automatic presentation, the fragment of $\mathcal{L}(\mathcal{Q}_u)$ that contains only effective quantifiers is shown to be decidable and an elementary algorithm for this decision is presented. Both assumptions ($\omega$-automaticity and bounded degree) are necessary for this result to hold.

# 1 Introduction

Automatic structures were introduced in [12, 15]. The idea goes back to the concept of automatic groups [8]. Roughly speaking, a structure is called automatic if the elements of the universe can be represented as words from a regular language and every relation of the structure can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic structures received increasing interest during the last years [1, 4, 13, 16, 17, 19, 21]. Recently, automatic structures were generalized to $\omega$-automatic structures by the use of Büchi-automata instead of automata on finite words [5]. One of the main motivations for investigating ($\omega$-)automatic structures is the fact that every ($\omega$-)automatic structure has a decidable first-order theory [5, 15]. For automatic structures, this result has been extended to first-order logic with modulo quantifiers [18] and the quantifier "there exist infinitely many" (infinity quantifier) [5]. The infinity quantifier was also shown to lead to decidable theories in the realm of $\omega$-automatic structures [3, 5] with injective presentations.[1] While there exist automatic structures with a non-elementary first-order theory [4], the first-order theory of any automatic structure of bounded degree is elementarily decidable; more precisely, an upper bound of triply exponential alternating time with a linear number of alternations was shown in [21].

The overall theme of this paper is to extend these results from automatic structures to $\omega$-automatic structures and to consider more involved logics. In a *first step*, we extend first-order logic by modulo-counting quantifiers as in [18] and exact counting quantifiers for infinite cardinals. We show that any injectively $\omega$-automatic structure has a decidable theory in this logic (Corollary 2.10). This extends [18, Theorem 3.2] from automatic to injectively $\omega$-automatic structures and [5, Theorem 2.1] from first-order logic with an infinity quantifier to a further extension of this logic. The proof is based on automata-theoretic constructions, in particular an analysis of successful runs in Muller automata.

In a *second step*, we consider an even more powerful logic that we call $\mathcal{L}(\mathcal{Q}_u)$, which is a finitary fragment of the logic $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^\omega$ from [14]. In this logic $\mathcal{L}(\mathcal{Q}_u)$ one may use generalized quantifiers of the form $\mathcal{Q}_\mathcal{C} y : (\psi_1(y), \ldots, \psi_n(y))$, where $y$ is a first-order variable and $\mathcal{C}$ is an $n$-ary relation on cardinals. To determine the truth of this formula in a model $\mathcal{A}$, one first determines the cardinalities of the sets defined by the formulas $\psi_i(y)$ $(1 \leq i \leq n)$. If the tuple of these cardinalities belongs to the relation $\mathcal{C}$, then the formula is true. All quantifiers mentioned so far are special instances of these generalized quantifiers. But, e.g., also the Härtig quantifier ("there are as many ... as ...") falls into this category.

Now let $\mathcal{L}$ be some fragment of $\mathcal{L}(\mathcal{Q}_u)$ that contains only countably many generalized quantifiers, and let $\mathcal{A}$ be some injectively $\omega$-automatic structure of bounded degree. We prove that the $\mathcal{L}$-theory of $\mathcal{A}$ can be decided by a Turing-machine with oracle access to the relations $\mathcal{C}$ that are allowed in the fragment $\mathcal{L}$. Moreover, this Turing-machine works in triply exponential space (Theorem 3.10). This extends [21, Theorem 3] since it applies to (1) injectively $\omega$-automatic structures as opposed to automatic structures and (2) to first-order logic extended by generalized quantifiers. This second main result rests on [14] where Hanf-locality is shown for the logic $\mathcal{L}(\mathcal{Q}_u)$. Our algorithm therefore has to determine how often a given neighborhood is realised (up to isomorphism) in the structure. Differently, the second author [21] used a similar locality principle to effectively bound the search space of quantifiers to short words.

Another corollary of the locality principle from [14] yields that any $\mathcal{L}$-definable relation in

[1] The decidability proof of [5, Theorem 2.1] assumes an injective $\omega$-automatic presentation. [5, Proposition 5.2] states that any $\omega$-automatic structure has such an injective presentation, but the proof is spurious (cf. Example 2.1). So we safely use the decidability for injective presentations, only.

an injectively $\omega$-automatic structure of bounded degree is necessarily first-order definable and therefore inherently regular (cf. [18]). But [14] does not provide a way to effectively translate $\mathcal{L}$ into first-order logic. Only our decidability result gives an effective (and even elementary) translation from $\mathcal{L}$ into first-order logic for any injectively $\omega$-automatic structure of bounded degree (Corollary 3.13).

Note that our results require a structure to be injectively $\omega$-automatic and of bounded degree. We finish the technical part of the paper showing that both these assumptions are necessary, namely that our results do not hold for recursive structures of bounded degree, nor for locally finite injectively $\omega$-automatic structures.

This paper can be understood as investigating the question which counting quantifiers $\mathcal{Q}_\mathcal{C}$ lead to theories that can be reduced to $\mathcal{C}$. Seen in this light, we show that this is the case (1) for semilinear sets $\mathcal{C}$ and arbitrary injectively $\omega$-automatic structures as well as (2) for arbitrary sets $\mathcal{C}$ and injectively $\omega$-automatic structures of bounded degree. It is therefore an open question whether there are non-semilinear sets $\mathcal{C}$ such that the first-order theory extended by the quantifier $\mathcal{Q}_\mathcal{C}$ of any injectively $\omega$-automatic structure can be reduced to $\mathcal{C}$. Towards the end of this paper, we exclude some non-semilinear sets from the list of possible candidates, but the general question remains open.

# 2 $\omega$-automatic structures, infinity and modulo quantifiers

## 2.1 Definitions and known results

This section introduces automata on finite and on infinite words, ($\omega$-)automatic structures, and logics, and recalls some basic results concerning these concepts. For more details, see [24, 26] for automata theoretic issues, [5, 15, 18] for $\omega$-automatic structures, and [11] as far as logics are concerned.

**Büchi-automata.** Let $\Gamma$ be a finite alphabet. With $\Gamma^*$ we denote the set of all finite words over the alphabet $\Gamma$. The set of all nonempty finite words is $\Gamma^+$. An *$\omega$-word* over $\Gamma$ is an infinite $\omega$-sequence $w = a_0a_1a_2\cdots$ with $a_i \in \Gamma$, we set $w(i) = a_i$ for $i \in \mathbb{N}$ and $w[i, j) = a_ia_{i+1}\dots a_{j-1}$ for natural numbers $i \leq j$. In the same spirit, $w[i, \infty)$ denotes the $\omega$-word $a_ia_{i+1}\dots$. The set of all $\omega$-words over $\Gamma$ is denoted by $\Gamma^\omega$. Similarly, for a set $V \subseteq \Gamma^*$ of finite words let $V^\omega \subseteq \Gamma^\omega$ be the set of all $\omega$-words of the form $v_1v_2v_3\cdots$ with $v_i \in V$.

A (nondeterministic) *Büchi-automaton* $M$ is a tuple $M = (Q, \Gamma, \delta, \iota, F)$, where $Q$ is a finite set of states, $\iota \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Gamma \times Q$ is the transition relation. If $\Gamma = \Sigma^n$ for some alphabet $\Sigma$, then we speak of an *$n$-dimensional Büchi-automaton over* $\Sigma$. A *run* of $M$ on an $\omega$-word $w = a_0a_1a_2\cdots$ is an $\omega$-word $r = p_0p_1p_2\cdots$ over the set of states $Q$ such that $(p_i, a_i, p_{i+1}) \in \delta$ for all $i \geq 0$. The run $r$ is *successful* if $p_0 = \iota$ and there exists a final state from $F$ that occurs infinitely often in $r$. The language $L_\omega(M) \subseteq \Gamma^\omega$ defined by $M$ is the set of all $\omega$-words for which there exists a successful run. An $\omega$-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi-automaton $M$ with $L_\omega(M) = L$.

For $\omega$-words $w_1, \dots, w_n \in \Gamma^\omega$, the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in (\Gamma^n)^\omega$ is given by

$$w_1 \otimes \cdots \otimes w_n = (w_1(1), \dots, w_n(1))\,(w_1(2), \dots, w_n(2))\,(w_1(3), \dots, w_n(3))\cdots$$

An $n$-ary relation $R \subseteq (\Gamma^\omega)^n$ is called *$\omega$-automatic* if the language $\{w_1 \otimes \cdots \otimes w_n \mid (w_1, \ldots, w_n) \in R\}$ is a regular $\omega$-language, i.e., accepted by some $n$-dimensional Büchi-automaton.

A Büchi-automaton $M = (Q, \Gamma, \delta, \iota, F)$ can also be considered as an ordinary finite automaton (on finite words). Then we denote with $L_*(M) \subseteq \Sigma^*$ the set of finite words accepted by $M$; these sets of finite words are called *regular*.

The definition of Büchi-automata implies that every regular $\omega$-language is a finite union of languages of the form $UV^\omega$, where $U$ and $V$ are regular languages of finite words. It is well-known that the class of all regular $\omega$-languages is closed under boolean operations and projections. For two Büchi-automata $M_1$ and $M_2$ with $n_1$ and $n_2$ many states, resp., there exists a Büchi-automaton with $3 \cdot n_1 \cdot n_2$ many states accepting the language $L_\omega(M_1) \cap L_\omega(M_2)$. The proof is based on a product construction for Büchi-automata, see e.g. [26].

**$\omega$-automatic structures.** A *signature* is a finite set $\tau$ of relational symbols, where each relational symbol $R \in \tau$ has an associated arity $n_R$. A (relational) *structure over the signature* $\tau$ or *$\tau$-structure* is a tuple $\mathcal{A} = (A, (R^\mathcal{A})_{R \in \tau})$, where $A$ is a set (the universe of $\mathcal{A}$) and $R^\mathcal{A}$ is a relation of arity $n_R$ over the set $A$, which interprets the relational symbol $R$. We will assume that every signature contains the equality symbol $=$ and that $=^\mathcal{A}$ is the identity relation on the set $A$. Usually, we denote the relation $R^\mathcal{A}$ also with $R$. We will also write $a \in \mathcal{A}$ for $a \in A$. For a subset $B \subseteq A$ we denote with $\mathcal{A}{\restriction}B$ the restriction $(B, (R^\mathcal{A} \cap B^{n_R})_{R \in \tau})$.

Let $\mathcal{A}$ be an arbitrary $\tau$-structure with universe $A$. An *$\omega$-automatic presentation* for $\mathcal{A}$ is a tuple $(\Gamma, L, h)$ such that

- $\Gamma$ is a finite alphabet,
- $L \subseteq \Gamma^\omega$ is a regular $\omega$-language,
- $h : L \to A$ is a surjection, and
- the relations

$$\begin{aligned} &\{(u, v) \in L^2 \mid h(u) = h(v)\} \\ \text{and } &\{(u_1, \ldots, u_{n_R}) \in L^{n_R} \mid (h(u_1), \ldots, h(u_{n_R})) \in R\} \end{aligned}$$

  are $\omega$-automatic for every $R \in \tau$.

An $\omega$-automatic presentation is *injective* if the function $h$ is injective (i.e., a bijection). We say that $\mathcal{A}$ is (injectively) *$\omega$-automatic* if there exists an (injective) $\omega$-automatic presentation for $\mathcal{A}$. *Automatic structures* are defined in the same way as $\omega$-automatic structures, except that finite automata over finite words instead of Büchi-automata are used (the convolution of finite words requires an additional letter $\perp$ that is appended to the arguments in order to make them the same length). By [3, Theorem 5.32], a countable structure is automatic if and only if it is injectively $\omega$-automatic. Furthermore, any automatic structure has an injective automatic presentation.

**Example 2.1.** *Let two sets $A$ and $B$ of natural numbers be equivalent ($A \approx B$) iff the symmteric difference $A \triangle B$ is finite. Then $\approx$ is a congruence wrt. union, intersection, and complementation of subsets of $\mathbb{N}$. Hence the quotient $\mathcal{B}$ of the powerset of $\mathbb{N}$ wrt. $\approx$ is a Boolean algebra. It has an $\omega$-automatic presentation: Let $\Gamma = \{0, 1\}$, $L = \Gamma^\omega$, and $h(w) = [\{i \in \mathbb{N} \mid w(i) = 1\}]_\approx$. Then $h(u) = h(w)$ iff $u$ and $w$ are eventually equal which can be tested by a Büchi-automaton* *with only two states. Similarly, $h(u) \leq h(v)$ in the Boolean algebra $\mathcal{B}$ iff $u(i) \leq v(i)$ for almost all $i$.*

*Any infinite $\omega$-regular set $K$ contains two $\omega$-words that are eventually equal. Hence there is no $\omega$-regular subset $K \subseteq L$ such that, for any $u \in L$, there is precisely one $v \in K$ with $h(u) = h(v)$. The $\omega$-automatic presentation $(\Gamma, L, h)$ of the Boolean algebra $\mathcal{B}$ can therefore not be restricted to an injective one $(\Gamma, K, h)$. This shows that the proof of [5, Proposition 5.2] does not work. It is therefore open as to whether every $\omega$-automatic structure has an injective $\omega$-automatic presentation.*

**Logic.** In addition to the usual first-order quantifier $\exists$, this section is concerned with quantifiers $\exists^\infty$, $\exists^\varkappa$ for a cardinal $\varkappa$, and $\exists^{(t,k)}$ for $0 \leq t < k > 1$ two natural numbers. The semantics of these quantifiers are defined as follows:

- $\mathcal{A} \models \exists^\infty x\, \psi$ if and only if there are infinitely many $a \in \mathcal{A}$ with $\mathcal{A} \models \psi(a)$.
- $\mathcal{A} \models \exists^\varkappa x\, \psi$ if and only if the set $\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}$ has cardinality $\varkappa$.
- $\mathcal{A} \models \exists^{(t,k)} x\, \psi$ if and only if the set $\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}$ is finite and $t = |\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}| \bmod k$.

We will denote by FO the set of first-order formulas. For a class of cardinals $C$, we denote by $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa \in C}, (\exists^{(t,k)})_{0 \leq t < k > 1})$ the set of formulas using $\exists$ and the quantifiers listed. For any of these sets $\mathcal{L}$ of formulas, the *$\mathcal{L}$-theory* of a structure $\mathcal{A}$ is the set of sentences (i.e., formulas without free variables) that hold in $\mathcal{A}$.

The following result can be shown by induction on the structure of the formula $\varphi$.

**Proposition 2.2 (cf. [5, 15, 18]).** *Let $(\Gamma, L, h)$ be an automatic presentation for the structure $\mathcal{A}$ and let $\varphi(x_1, \ldots, x_n)$ be a formula of $\mathrm{FO}(\exists^\infty, (\exists^{(t,k)})_{0 \leq t < k \geq 2})$ over the signature of $\mathcal{A}$. Then the relation*

$$\{(u_1, \ldots, u_n) \in L^n \mid \mathcal{A} \models \varphi(h(u_1), \ldots, h(u_n))\}$$

*is effectively automatic. It is effectively $\omega$-automatic if $(\Gamma, L, h)$ is an injective $\omega$-automatic presentation for the structure $\mathcal{A}$ and $\varphi$ belongs to $\mathrm{FO}(\exists^\infty)$.*

This proposition implies the following result, which is one of the main motivations for investigating automatic structures.

**Theorem 2.3 ([5, 18]).** *If $\mathcal{A}$ is an injectively $\omega$-automatic structure, then the $\mathrm{FO}(\exists^\infty)$-theory of $\mathcal{A}$ is decidable. If $\mathcal{A}$ is an automatic structure, then even the $\mathrm{FO}(\exists^\infty, (\exists^{(t,q)})_{0 \leq t < q \geq 2})$-theory of $\mathcal{A}$ is decidable.*

Note that any automatic structure $\mathcal{A}$ is at most countably infinite. Hence the quantifiers $\exists^\infty$ and $\exists^{\aleph_0}$ are equivalent in this setting. Furthermore, no formula $\exists^\varkappa x\, \psi$ with $\varkappa > \aleph_0$ holds in $\mathcal{A}$. Hence, for any countable set of cardinals $C$, the $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa \in C}, (\exists^{(t,k)})_{0 \leq t < k > 1})$-theory of an automatic structure is decidable.[2] In the rest of Section 2 we extend this result to injectively $\omega$-automatic structures.

To the knowledge of the authors, the modulo quantifiers $\exists^{(t,k)}$ have not yet been considered for $\omega$-automatic structures. Concerning the counting quantifiers $\exists^\varkappa$, the situation is more

[2] $C$ has to be countable for otherwise the set of formulas would become uncountable rendering the decidability question nonsense.

involved than in the setting of automatic structures since an $\omega$-automatic structure can have up to $2^{\aleph_0}$ many elements. Thus, it makes sense to consider quantifiers of the form $\exists^{\varkappa}$ with $\aleph_0 \leq \varkappa \leq 2^{\aleph_0}$.

## 2.2 Cardinality quantifier $\exists^{\varkappa}$ for $\omega$-automatic structures

Two infinite words $v$ and $w$ are *ultimately equal*, briefly $v \sim w$, if there exists $i \in \mathbb{N}$ with $v[i,\infty) = w[i,\infty)$. The following lemma is our main combinatorial tool for analyzing $\omega$-automatic structures.

**Lemma 2.4.** *Let $M$ be a Büchi-automaton with $n$ states over $\Sigma \times \Gamma$, $u \in \Sigma^\omega$, and $V = \{v \in \Gamma^\omega \mid u \otimes v \in L_\omega(M)\}$. Then:*

- $|V| = 2^{\aleph_0}$ *if and only if* $|V/{\sim}| > n$ *and*
- $|V| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

*Proof.* Since equivalence classes wrt. $\sim$ are at most countably infinite, the implication "$\Rightarrow$" of the first statement is obvious. So assume $|V/{\sim}| > n$. Then there are mutually non-equivalent words $v_1, v_2, \ldots, v_{n+1} \in V$. For $1 \leq i \leq n+1$, let $r_i$ be a successful run of $M$ on the word $u \otimes v_i$. For $1 \leq i < j \leq n+1$, set $x_{ij} = \sup\{k \in \mathbb{N} \mid r_i(k) = r_j(k)\}$ and let $x = \max\{x_{ij} \mid 1 \leq i < j \leq n+1\}$. If $x \in \mathbb{N}$, then the states $r_i(x+1)$ for $1 \leq i \leq n+1$ are mutually distinct which is impossible since there are only $n$ states. Hence, there are $1 \leq i < j \leq n+1$ with $x_{ij} = \omega$, w.l.o.g. we assume $i = 1$ and $j = 2$. Since $v_1 \not\sim v_2$, since $x_{12} = \omega$, and since $r_1$ and $r_2$ are successful, there exist $0 = i_0 < i_1 < i_2 \ldots$ such that for any $j \in \mathbb{N}$

- $v_1[i_j, i_{j+1}) \neq v_2[i_j, i_{j+1})$ and $r_1(i_j) = r_2(i_j)$
- there exist $k, \ell$ with $i_j \leq k, \ell < i_{j+1}$ and $r_1(k), r_2(\ell) \in F$.

Hence, the Büchi-automaton $M$ accepts any $\omega$-word of the form $u \otimes (y_0 y_1 y_2 \cdots)$, where $y_j \in \{v_1[i_j, i_{j+1}), v_2[i_j, i_{j+1})\}$ for all $j \in \mathbb{N}$. This gives $2^{\aleph_0}$ many distinct elements of $V$, i.e., we showed $|V| = 2^{\aleph_0}$.

If $|V| > \aleph_0$, then $V/{\sim}$ contains infinitely many equivalence classes since any of them is at most countable. Thus, $|V| = 2^{\aleph_0}$ follows, which gives us the second statement from the lemma. ☐

Setting $u = a^\omega \in \Sigma^\omega$, an $\omega$-regular language $L \subseteq \Gamma^\omega$ can be considered as the set $V$ in the lemma above. Thus, any uncountable $\omega$-regular language $L$ contains $2^{\aleph_0}$ many words, a result that can also be found in [20, Lemma 5.41].

**Proposition 2.5.** *Let the relation $R \subseteq (\Gamma^\omega)^{n+1}$ be $\omega$-automatic (thus, $(\Gamma^\omega, R)$ is an injectively $\omega$-automatic structure) and let $\varkappa$ be some cardinal. Then*

$$R_\varkappa = \{(u_1, \ldots, u_n) \mid (\Gamma^\omega, R) \models \exists^{\varkappa} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})\}$$

*is effectively $\omega$-automatic.*

*Proof.* By Lemma 2.4, $R_\varkappa = \emptyset$ for $\varkappa \notin \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. If $\varkappa$ is finite, we can define the relation $R_\varkappa$ in first-order logic from $R$; hence the result follows from Proposition 2.2. To deal with the two remaining cases, recall that the relation $\sim$ (ultimate equality), defined at the beginning of this section, is $\omega$-automatic. Hence $\mathcal{A} = (\Gamma^\omega, \sim, R)$ is an injectively $\omega$-automatic structure. Let $m$ be the number of states of some $(n+1)$-dimensional Büchi-automaton $M$ accepting $R$. Thus we have to construct a Büchi-automaton accepting

$$\{u_1 \otimes \ldots \otimes u_n \mid \mathcal{A} \models Qx_{n+1} : R(u_1, \ldots, u_n, x_{n+1})\}$$

where $Q \in \{\exists^{\aleph_0}, \exists^{2^{\aleph_0}}\}$.

Let $u_1, \ldots, u_n \in \Gamma^\omega$ and consider $u = u_1 \otimes u_2 \cdots \otimes u_n \in \Sigma^\omega$ with $\Sigma = \Gamma^n$. Let $V = \{v \in \Gamma^\omega \mid \mathcal{A} \models R(u_1, \ldots, u_n, v)\} = \{v \in \Gamma^\omega \mid u \otimes v \in L_\omega(M)\}$. Hence, by Lemma 2.4, we have $\mathcal{A} \models \exists^{2^{\aleph_0}} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})$ (i.e., $|V| = 2^{\aleph_0}$) if and only if

$$\mathcal{A} \models \exists x_0 \cdots \exists x_m \left( \bigwedge_{0 \le i < j \le m} x_i \not\sim x_j \wedge \bigwedge_{0 \le i \le m} R(u_1, \ldots, u_n, x_i) \right).$$

Lemma 2.4 also ensures that $\mathcal{A} \models \exists^{\aleph_0} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})$ if and only if

$$\mathcal{A} \models \exists^\infty x_{n+1} : R(u_1, \ldots, u_n, x_{n+1}) \wedge \neg \exists^{2^{\aleph_0}} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1}).$$

Now Proposition 2.2 allows to construct the Büchi-automata accepting the convolution of $R_\varkappa$. $\square$

## 2.3 Modulo quantifier $\exists^{(t,k)}$ for $\omega$-automatic structures

We now want to prove a result similar to Proposition 2.5 for modulo quantifiers. Therefore, let $R \subseteq (\Gamma^\omega)^{n+1}$ be $\omega$-automatic and let $0 \le t < k \ge 2$. It is our aim to show the $\omega$-automaticity of the relation $S$ of all those tuples $(u_1, \ldots, u_n) \in (\Gamma^\omega)^n$ such that $\{v \in \Gamma^\omega \mid (u_1, \ldots, u_n, v) \in R\}$ is finite and contains, modulo $k$, precisely $t$ elements. For the following, it is convenient to write $\Sigma = \Gamma^n$ and consider $R$ as an $\omega$-automatic subset of $\Sigma^\omega \times \Gamma^\omega$.

For the further considerations, we will need the concept of a deterministic *Muller-automaton*: it is a tuple $M = (Q, \Gamma, \delta, \iota, \mathcal{F})$, where $Q$ is a finite set of states, $\iota \in Q$ is the initial state, $\mathcal{F} \subseteq 2^Q$ is a table of accepting states, and $\delta : Q \times \Sigma \to Q$ is the transition function. A *run* of $M$ on an $\omega$-word $w = a_0 a_1 a_2 \cdots$ is an $\omega$-word $r = p_0 p_1 p_2 \cdots$ over the set of states $Q$ such that $p_{i+1} = \delta(p_i, a_i)$ for all $i \ge 0$. Let $\inf(r)$ denote the set of states appearing infinitely often in the run $r$. Then $r$ is *successful* if $p_0 = \iota$ and $\inf(r) \in \mathcal{F}$. The language $L_\omega(M) \subseteq \Gamma^\omega$ defined by $M$ is the set of all $\omega$-words for which there exists a successful run. By McNaughton's theorem, any Büchi-automaton $M$ can effectively be transformed into a Muller automaton $M'$ with $L_\omega(M) = L_\omega(M')$ and vice versa, see e.g. [24].

Since the convolution of $R$ is $\omega$-regular, it can be accepted by some deterministic Muller-automaton $M = (Q, \Sigma \times \Gamma, \delta, \iota, \mathcal{F})$. Now consider the alphabet $\Delta = \Sigma \times \Gamma \times \{0, \ldots, k-1\}^Q \times \{0,1\}^Q$ and let $\pi : \Delta \to \Sigma \times \Gamma$ be the canonical projection morphism.

**Lemma 2.6.** *One can construct a Büchi-automaton $M'$ over the alphabet $\Delta$ that accepts an $\omega$-word $(a_i, b_i, f_i, g_i)_{i \ge 0}$ if and only if we have for all $i \ge 0$ and all $p \in Q$*

*(1)* $f_i(p) = |\{w \in \Gamma^* \mid |w| = i, \delta(\iota, a_0 a_1 \ldots a_{i-1} \otimes w) = p\}| \bmod k$ *and*

*(2) $g_i(p) = 1$ if and only if the $\omega$-word $a_i a_{i+1} \cdots \otimes b_i b_{i+1} \cdots$ has an accepting run in $M$ from the state $p$.*

Note that $f_i(p)$ is the number of possible partners (modulo $k$) that allow $a_0 \dots a_{i-1}$ to move from the initial state of $M$ into $p$. Furthermore, $g_i(p)$ tells whether the remaining word (obtained by discarding the information $f_i$ and $g_i$) is accepted by $M$.

*Proof.* Since $M$ is deterministic, (1) above holds if and only if the following two local conditions hold:

- $f_0(\iota) = 1$ and $f_0(p) = 0$ for $p \neq \iota$
- for any $i \in \mathbb{N}$ and $q \in Q$, we have

$$f_{i+1}(q) = \left( \sum_{\substack{p \in Q, b \in \Gamma, \\ \delta(p,(a_i,b))=q}} f_i(p) \right) \bmod k.$$

Thus, one can construct a Büchi-automaton over $\Delta$ that precisely accepts all sequences satisfying (1).

Let $M_p$ ($p \in Q$) be the Muller automaton that results from $M$ by making $p$ the unique initial state. Then $w \in \Delta^\omega$ violates (2) if and only if there is a state $p \in Q$ and a suffix of $w$ that belongs to

$$\begin{aligned} & \left[\pi^{-1}(L_\omega(M_p)) \cap \{(a,b,f,g) \in \Delta \mid g(p) = 0\}\Delta^\omega\right] \\ \cup \; & \left[\Delta^\omega \setminus \pi^{-1}(L_\omega(M_p)) \cap \{(a,b,f,g) \in \Delta \mid g(p) = 1\}\Delta^\omega\right]. \end{aligned}$$

Thus, we can also construct a Büchi-automaton over $\Delta$ that accepts a sequence if and only if it satisfies (2). □

Note that for any $u \in \Sigma^\omega$ and $v \in \Gamma^\omega$, there is precisely one $\omega$-word $x \in L(M')$ with $\pi(x) = u \otimes v$.

**Lemma 2.7.** *Let $u \in \Sigma^\omega$ and $v \in \Gamma^\omega$ and let $x = (a_1, b_1, f_1, g_1)(a_2, b_2, f_2, g_2) \cdots \in L(M')$ be the unique $\omega$-word with $\pi(x) = u \otimes v$. If $\{w \in \Gamma^\omega \mid w \sim v, (u,w) \in R\}$ is finite, then there exists $i \in \mathbb{N}$ such that for all $j \geq i$, we have*

$$\sum_{\substack{p \in Q, \\ g_j(p)=1}} f_j(p) \equiv |\{w \in \Gamma^\omega \mid w \sim v, (u,w) \in R\}| \bmod k. \tag{1}$$

*Proof.* Let $H = \{w \in \Gamma^\omega \mid w \sim v, (u,w) \in R\}$. Since $H$ is a finite set of words that are ultimately equal to $v$, there exists $i \in \mathbb{N}$ such that, for all $w \in H$, we have $w[i,\infty) = v[i,\infty)$. Now let $j \geq i$. We show that

$$H = \{wv[j,\infty) \mid |w| = j, \delta(\iota, u[0,j) \otimes w) = p \text{ with } g_j(p) = 1\},$$

which immediately implies (1).

First, let $x \in H$. Since $j \geq i$, we get $x[j,\infty) = v[j,\infty)$, i.e., there exists a word $w \in \Sigma^*$ with $|w| = j$ and $x = wv[j,\infty)$. Let $p = \delta(\iota, u[0,j) \otimes w)$. Since $x \in H$, we get $u \otimes x \in L_\omega(M)$ and therefore $u[j,\infty) \otimes x[j,\infty) \in L_\omega(M_p)$. Since $x[j,\infty) = v[j,\infty)$, this implies $g_j(p) = 1$. Thus, $x$ is in the set above.

Conversely, let $w \in \Sigma^*$ with $|w| = j$, $p = \delta(\iota, u[0,j) \otimes w)$, and $g_j(p) = 1$. Set $x = wv[j,\infty)$ ensuring $x \sim v$. Furthermore, $g_j(p) = 1$ implies $u[j,\infty) \otimes v[j,\infty) \in L_\omega(M_p)$ and therefore $u \otimes x \in L_\omega(M)$. Hence $x \in H$. □

**Proposition 2.8.** *Let the relation $R \subseteq (\Gamma^\omega)^{n+1}$ be $\omega$-automatic (thus, $(\Gamma^\omega, R)$ is an injectively $\omega$-automatic structure) and let $0 \leq t < k \geq 2$. Then*

$$S = \{(u_1, \ldots, u_n) \mid (\Gamma^\omega, R) \models \exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})\}$$

*is effectively $\omega$-automatic.*

*Proof.* Let $R' \subseteq R$ comprise all those tuples $(u_1, \ldots, u_n, v) \in R$ such that there are only finitely many $w \in \Gamma^\omega$ with $(u_1, \ldots, u_n, w) \in R$. Then, by Proposition 2.2, the relation $R'$ is effectively $\omega$-automatic, i.e., $(\Gamma^\omega, R')$ is an injectively $\omega$-automatic structure. Moreover, $(\Gamma^\omega, R) \models \exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})$ if and only if $(\Gamma^\omega, R') \models \exists^{(t,k)} x_{n+1} : R'(u_1, \ldots, u_n, x_{n+1})$, i.e., replacing $R$ by $R'$ in the definition of $S$ does not change the set $S$. Thus, we can assume $R = R'$. This has the advantage that the finiteness assumption in Lemma 2.7 is trivially satisfied in the further discussion.

For $0 \leq s < k$ let $R_s \subseteq R$ comprise all $n+1$-tuple $(u_1, u_2, \ldots, u_n, v) \in R$ such that, modulo $k$, there are $s$ words $w$ ultimately equal to $v$ with $(u_1, u_2, \ldots, u_n, w) \in R$. To show that $R_s$ is $\omega$-automatic, let $\Delta_s \subseteq \Delta$ comprise all tuples $(a, b, f, g)$ that satisfy $s = \sum\{f(p) \mid p \in Q, g(p) = 1\} \bmod k$. Then the set $L_\omega(M') \cap \Delta^* \Delta_s^\omega$ is $\omega$-regular (where $M'$ is the Büchi-automaton from Lemma 2.6). Hence the same holds for the projection $P$ of this language to $(\Sigma \times \Gamma)^\omega$. Then, by Lemma 2.7, $u_1 \otimes u_2 \ldots u_n \otimes v \in P$ if and only if, modulo $k$, there are $s$ words $w$ ultimately equal to $v$ such that $(u_1, \ldots, u_n, w) \in R$, i.e., $(u_1, \ldots, u_n, v) \in R_s$. Hence $R_s$ is $\omega$-automatic.

Since $R$ is $\omega$-automatic, there is a Büchi-automaton with, say, $m$ states accepting the convolution of $R$. Let $\bar{u} \in (\Gamma^\omega)^n$. Since, by our assumption on $R$, the set $\{v \in \Gamma^\omega \mid (u, v) \in R\}$ is finite, there are $r$ (for some $r \leq m$) many $\omega$-words $v_1, \ldots, v_r$ in this set that are mutually not ultimately equal (Lemma 2.4). Thus, we have $(\Gamma^\omega, R) \models \exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})$ if and only if there exist $r \leq m$, mutually not ultimately equivalent words $v_1, \ldots, v_r \in \Gamma^\omega$ and integers $0 \leq t_i < k$ for $1 \leq i \leq r$ such that

1. $R(u_1, \ldots, u_n, v_i)$ for $1 \leq i \leq r$,
2. for any $v \in \Gamma^\omega$ with $R(u_1, \ldots, u_n, v)$, there exists $1 \leq i \leq r$ with $v \sim v_i$,
3. $t = \sum_{i=1}^r t_i \bmod k$ and $(u_1, u_2 \ldots u_n, v_i) \in R_{t_i}$ for $1 \leq i \leq r$.

Since $m$ is a constant depending on $R$, only, these conditions can be expressed in FO over the injectively $\omega$-automatic structure $(\Gamma^\omega, \sim, R, R_0, R_1, \ldots, R_{k-1})$. Hence Proposition 2.2 implies that $S$ is effectively $\omega$-regular. □

Together with Propositions 2.2 and 2.5, we obtain:

**Theorem 2.9.** *Let $(\Gamma, L, h)$ be an injective $\omega$-automatic presentation for the structure $\mathcal{A}$, let $C$ be an at most countably infinite set of cardinals, and let $\varphi(x_1, \ldots, x_n)$ be a formula of $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa \in C}, (\exists^{(t,k)})_{0 \leq t < k \geq 2})$ over the signature of $\mathcal{A}$. Then the relation*

$$R = \{(u_1, \ldots, u_n) \in L^n \mid \mathcal{A} \models \varphi(h(u_1), \ldots, h(u_n))\}$$

*is effectively $\omega$-automatic.*

*Proof.* The proof is by induction on the construction of a formula. If $\varphi$ is an atomic formula, the relation $R$ is $\omega$-automatic since $\mathcal{A}$ is $\omega$-automatic. If $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$, or $\varphi = \exists x\, \psi$, the result holds since $\omega$-regular languages are effectively closed under Boolean operations and projections. Finally, if $\varphi = Qx\,\psi$ for some quantifier $Q$, then we invoke [3] for $Q = \exists^\infty$, Proposition 2.5 for $Q = \exists^\varkappa$, and Proposition 2.8 for $Q = \exists^{(t,k)}$. ☐

**Corollary 2.10.** *Let $\mathcal{A}$ be an injectively $\omega$-automatic structure and let $C$ be an at most countably infinite set of cardinals. Then the* $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa\in C}, (\exists^{(t,k)})_{0\le t<k>1})$*-theory of $\mathcal{A}$ is decidable.*

*Proof.* This follows immediately from Theorem 2.9 since the emptiness of $L_\omega(M)$ is decidable for a Büchi-automaton $M$. ☐

# 3 $\omega$-automatic structures of bounded degree and complexity of theories

Consider the structure $(\{0,1\}^*, s_0, s_1, \preceq)$, where $s_i(w) = wi$ for $w \in \{0,1\}^*$ and $i \in \{0,1\}$, and $\preceq$ is the prefix order on finite words. It is easily seen to be automatic, hence its first-order theory is decidable. But the time complexity of this theory is non-elementary, i.e., cannot be bounded by a fixed tower of exponents, see e.g. [7, Example 8.3]. Thus, as first observed in [4], there are automatic structures with a non-elementary first-order theory.

Our aim in this section is to single out a class of $\omega$-automatic structures such that the $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}}, (\exists^{(t,k)})_{0\le t<k>1})$-theory is elementarily decidable. In doing so, we will find that even more general quantifiers give rise to elementarily decidable theories provided we constrain ourselves to structures of bounded degree.

## 3.1 Definitions and known results

**Structures of bounded degree.** Let $\tau$ be a relational signature and let $\mathcal{A}$ be a $\tau$-structure with universe $A$. The *Gaifman-graph* $G_\mathcal{A}$ of the structure $\mathcal{A}$ is the following undirected graph:

$$G_\mathcal{A} = (A, \{(a,b) \in A \times A \mid \exists R \in \tau\, \exists (c_1, \ldots, c_{n_R}) \in R\, \exists j,k : c_j = a \neq b = c_k\}).$$

Thus, the set of nodes is the universe of $\mathcal{A}$ and there is an edge between two elements, if and only if they are contained in some tuple belonging to one of the relations of $\mathcal{A}$. The structure $\mathcal{A}$ is *locally finite*, if every node of the Gaifman-graph $G_\mathcal{A}$ has only finitely many neighbors. It has *bounded degree*, if its Gaifman-graph $G_\mathcal{A}$ has bounded degree, i.e., there exists a constant $d$ such that every $a \in A$ is adjacent to at most $d$ other nodes in $G_\mathcal{A}$.

In contrast to the structure $(\{0,1\}^*, s_0, s_1, \preceq)$, if the degree of an automatic structure $\mathcal{A}$ is bounded, an elementary upper bound for the first-order theory of $\mathcal{A}$ is due to the second author ($\mathrm{ATIME}(f(n), g(n))$ is the class of problems that can be solved by an alternating Turing machine in time $g(n)$ with at most $f(n)$ many alternations on an input of size $n$):

**Theorem 3.1 ([21]).** *The following holds:*

1. *If $\mathcal{A}$ is an automatic structure of bounded degree, then the* FO*-theory of $\mathcal{A}$ can be decided in* $\mathrm{ATIME}(O(n), \exp(3, O(n)))$.

2. *There exists an automatic structure $\mathcal{A}$ of bounded degree such that for some constant c,* $\mathrm{ATIME}(c \cdot n, \exp(2, c \cdot n))$ *is a hereditary lower bound (see [7] for the definition) for the* FO*-theory of $\mathcal{A}$.*

This result was not known to apply to more general quantifiers nor to $\omega$-automatic structures. An important tool in the proof of Theorem 3.1 as well as in our extension, is the concept of a sphere that we introduce next.

With $d_{\mathcal{A}}(a,b)$, where $a, b \in A$, we denote the distance between $a$ and $b$ in $G_{\mathcal{A}}$, i.e., it is the length of a shortest path connecting $a$ and $b$ in $G_{\mathcal{A}}$. For $a \in A$ and $r \geq 0$ we denote with $S_{\mathcal{A}}(r,a) = \{b \in A \mid d_{\mathcal{A}}(a,b) \leq r\}$ the *$r$-sphere around $a$*. If $\bar{a} = (a_1, \ldots, a_n) \in A^n$ is a tuple, then $S_{\mathcal{A}}(r,\bar{a}) = \bigcup_{i=1}^{n} S_{\mathcal{A}}(r,a_i)$. The *neighborhood* $N_{\mathcal{A}}(r,\bar{a}) = \mathcal{A}{\restriction}S_{\mathcal{A}}(r,\bar{a})$ of radius $r$ around $\bar{a}$ is the substructure of $\mathcal{A}$ induced by $S_{\mathcal{A}}(r,\bar{a})$.

**Generalized quantifiers and locality.** Let us fix a relational signature $\tau$. In this section, we will consider the logic $\mathcal{L}(\mathcal{Q}_u)$. Formulas of the logic $\mathcal{L}(\mathcal{Q}_u)$ are built from atomic formulas of the form $R(x_1, \ldots, x_{n_R})$, where $R \in \tau$ is a relational symbol and $x_1, \ldots, x_{n_R}$ are first-order variables ranging over the universe of the underlying structure, using boolean connectives and quantifications of the form $\mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{x},y), \ldots, \psi_n(\bar{x},y))$. Here, $\psi_i(\bar{x},y)$ is already a formula of $\mathcal{L}(\mathcal{Q}_u)$, $\bar{x}$ is a sequence of variables, and $\mathcal{C}$ is an $n$-ary relation over cardinals, i.e., $\mathcal{C} = \{(\varkappa_{i,1}, \ldots, \varkappa_{i,n}) \mid i \in J, \varkappa_{i,j} \text{ is a cardinal}\}$ for some index set $J$. The semantics of the $\mathcal{Q}_{\mathcal{C}}$-quantifier is defined as follows: Let $\mathcal{A}$ be a $\tau$-structure with universe $A$ and let $\bar{u}$ be a tuple of values from $A$ of the same length as $\bar{x}$. Then $\mathcal{A} \models \mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{u},y), \ldots, \psi_n(\bar{u},y))$ if and only if $(c_1, \ldots, c_n) \in \mathcal{C}$, where $c_i$ is the cardinality of the set $\{a \in A \mid \mathcal{A} \models \psi_i(\bar{u},a)\}$. In the above situation, we call the quantifier $\mathcal{Q}_{\mathcal{C}}$ also an *$n$-dimensional counting quantifier*. The *quantifier rank* $\mathrm{qfr}(\varphi)$ of a formula $\varphi$ is inductively defined as follows: Every atomic formula has quantifier rank 0, and $\mathrm{qfr}(\neg\varphi) = \mathrm{qfr}(\varphi)$, $\mathrm{qfr}(\varphi \wedge \psi) = \mathrm{qfr}(\varphi \vee \psi) = \max\{\mathrm{qfr}(\varphi), \mathrm{qfr}(\psi)\}$, and $\mathrm{qfr}(\mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{x},y), \ldots, \psi_n(\bar{x},y))) = 1 + \max\{\mathrm{qfr}(\psi_i(\bar{x},y) \mid 1 \leq i \leq n\}$. The logic $\mathcal{L}(\mathcal{Q}_u)$ is a finitary fragment of the logic $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^{\omega}$ from [14], which allows infinite conjunctions and disjunctions but restricts to finite quantifier rank.

In [14], counting quantifiers where introduced slightly differently using classes of structures with only unary predicates: Let $K$ be a class of structures, where every structure in $K$ consists of exactly $n$ unary predicates. Then, in [14] the formula $\mathcal{Q}_K y : (\psi_1(\bar{u},y), \ldots, \psi_n(\bar{u},y))$ expresses that the structure $(A, (\{a \in A \mid \mathcal{A} \models \psi_i(\bar{u},a)\})_{1 \leq i \leq n})$ is isomorphic to a structure in $K$. It is easy to see that the resulting logic is equivalent with respect to expressive power to our variant.

Let us consider some examples for generalized quantifiers. The ordinary existential quantifier $\exists y : \varphi(\bar{x},y)$ is equivalent to $\mathcal{Q}_{\mathcal{C}}\, y : \varphi(\bar{x},y)$, where $\mathcal{C}$ is the class of all non-zero cardinals. Similarly, we can obtain the counting quantifier $C_K\, y : \varphi(\bar{x},y)$ for $K$ some class of cardinals ("the number of $y$ satisfying $\varphi(\bar{x},y)$ belongs to $K$"). Well-known special cases of the latter quantifier are the quantifiers $\exists^{\infty}$, $\exists^{\varkappa}$ and $\exists^{(t,q)}$ from the previous section. All these counting quantifiers are one-dimensional. A well-known two-dimensional counting quantifier is the Härtig quantifier $I\, y : (\psi_1(\bar{x},y), \psi_2(\bar{x},y))$ [10] ("the number of $y$ satisfying $\psi_1(\bar{x},y)$ equals the number of $y$ satisfying $\psi_2(\bar{x},y)$"). For this we have to choose for $\mathcal{C}$ the identity relation on cardinals.

For a class $\mathbb{C}$, where every $\mathcal{C} \in \mathbb{C}$ is a relation on cardinals, $\mathcal{L}(\mathbb{C})$ denotes those formulas of $\mathcal{L}(\mathcal{Q}_u)$ that only use quantifiers of the form $\mathcal{Q}_{\mathcal{C}}$ with $\mathcal{C} \in \mathbb{C}$. Furthermore, $\mathrm{FO}(\mathbb{C})$ denotes the logic $\mathcal{L}(\{\exists\} \cup \mathbb{C})$, i.e., the extension of first-order logic by quantifiers $\mathcal{Q}_{\mathcal{C}}$ for $\mathcal{C} \in \mathbb{C}$. For a singleton class $\mathbb{C} = \{\mathcal{C}\}$ we also write $\mathrm{FO}(\mathcal{C})$ instead of $\mathrm{FO}(\mathbb{C})$. For a logic $\mathcal{L}$ and a structure $\mathcal{A}$, the $\mathcal{L}$-theory of $\mathcal{A}$ denotes the set of all sentences from $\mathcal{L}$ that hold in $\mathcal{A}$.

We will make use of the following locality principle for the logic $\mathcal{L}(\mathcal{Q}_u)$:

**Theorem 3.2 ([14]).** *Let $\mathcal{A}$ be a locally finite structure, let $\varphi(x_1, \dots, x_k)$ be an $\mathcal{L}(\mathcal{Q}_u)$-formula of quantifier rank at most $d$, and let $\bar{a}, \bar{b} \in \mathcal{A}^k$ be $k$-tuples with $(N_{\mathcal{A}}(2^d, \bar{a}), \bar{a}) \cong (N_{\mathcal{A}}(2^d, \bar{b}), \bar{b})$.*[3] *Then $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{A} \models \varphi(\bar{b})$.*

*Proof.* Keisler and Lotfallah proved in [14] the statement of the theorem for locally finite *countable* structures and the infinitary extension $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^\omega$ of $\mathcal{L}(\mathcal{Q}_u)$, see Option 2 in [14]. As an intermediate step, they considered the fragment of $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^\omega$ where only counting quantifiers of the form $C_A$ with $A = \{0, 1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ are allowed. Considering, instead, the fragment where counting quantifiers $C_A$ with $A = \{\lambda \mid \lambda \le \varkappa\}$ for $\varkappa$ a cardinal are allowed, one obtains the above general theorem (which does not restrict to countable structures) without any further modifications of [14]. ☐

**An auxiliary result on regular $\omega$-languages.** Recall that, by Lemma 2.4, the cardinality of a regular $\omega$-language $L$ belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Thus, it makes sense to ask whether this cardinality can be computed effectively from a given Büchi-automaton $M$ for $L$. The decidability of this question was shown in [20, Satz 5.48]. Here, we give a bound on the complexity, which will be needed in Section 3.2 in order to derive our upper complexity bounds for $\mathcal{L}(\mathbb{C})$-theories.

**Lemma 3.3.** *From a given Büchi-automaton $M$ we can compute in polynomial space the cardinality $|L_\omega(M)|$ of the $\omega$-language accepted by $M$.*

*Proof.* Let $F$ be the set of final states of $M$, $\Sigma$ its alphabet, and $n$ the number of its states. We may assume that all states of $M$ are reachable from the initial state and that for every final state $p \in F$ there exists a nonempty path from $p$ to $p$. Suppose there is a final state $p \in F$ and two distinct words $v$ and $w$ of equal length that label paths from $p$ back to $p$. Let $u$ be the label of some path from the initial state to $p$. Then $u\{v, w\}^\omega$ contains $2^{\aleph_0}$ many elements. Hence, this is the size of $L_\omega(M)$. Using a simple pigeonhole argument, the length of $v$ and $w$ can be bounded by $n^2$. Thus, a nondeterministic machine can check in polynomial space the existence of $p$, $v$, and $w$ as required.

Now suppose that for any final state $p$ and any $m \in \mathbb{N}$, there is at most one word $w$ of length $m$ that labels a path from $p$ to $p$. For $p \in P$, let $u_p$ be the unique shortest word labeling a loop at $p$. Then $|u_p| \le n$. Furthermore, let $v_p$ be the unique primitive word[4] such that $u_p \in v_p^*$. Clearly, also the length of $v_p$ is bounded by $n$. Then $L_\omega(M) = \bigcup_{p \in F} L_*(M_p) v_p^\omega$ where the only difference of $M$ and $M_p$ is that $p$ is the only accepting state of $M_p$. Choose the unique factorization $v_p = x_p y_p$ such that $y_p x_p$ is the lexicographically minimal word (w.r.t. some fixed order on the alphabet $\Sigma$) in the language $\{yx \mid v_p = xy\}$. Then $L_*(M_p) v_p^\omega = L_*(M_p) x_p (y_p x_p)^\omega$. Let $w_p = y_p x_p$ and $L_p = L_*(M_p) x_p$. Thus, $L_*(M_p) v_p^\omega = L_p w_p^\omega$ and $w_p$ is the lexicographically minimal cyclic rotation of $v_p$ (and hence of itself). Since any cyclic rotation of a primitive word remains primitive, $w_p$ is still primitive [22].

We claim that $L_p w_p^\omega \cap L_q w_q^\omega = \emptyset$ whenever $w_p \neq w_q$. To show this by contraposition, let $s w_p^\omega = t w_q^\omega$ for some $s \in L_p$ and $t \in L_q$. W.l.o.g. $|s| \ge |t|$. Thus, there exists $m \ge 0$ and a factorization $w_q = xy$ such that $s = t w_q^m x$. Hence, $w_p^\omega = y w_q^\omega = (yx)^\omega$. Since $w_p$ and $yx$ are

[3] Thus, there exists an isomorphism $f : N_{\mathcal{A}}(2^d, \bar{a}) \to N_{\mathcal{A}}(2^d, \bar{b})$ mapping for every $1 \le i \le k$ the $i$-th entry of $\bar{a}$ to the $i$-th entry of $\bar{b}$.

[4] i.e., $v_p \neq \varepsilon$ is no power of any word different from itself.

both primitive, we obtain $w_p = yx$. Now since both $w_p$ and $w_q = xy$ are their lexicographically minimal cyclic rotations, we get $w_p = w_q$, a contradiction.

For two accepting states $p$ and $q$ with $w_p = w_q$, we write $p \approx q$. Thus, by the previous paragraph, $L_\omega(M)$ is the disjoint union of the sets $\bigcup_{p\in X} L_p w_p^\omega$, where $X$ is an equivalence class of $\approx$. Therefore it suffices to calculate the cardinality of every set $\bigcup_{p\in X} L_p w_p^\omega$, where $X$ is an equivalence class of $\approx$. Let us fix such an equivalence class $X$. Let $w = w_p = y_p x_p$ (for an arbitrary $p \in X$). By adding a thread labeled with the word $x_p$ to every state $p \in X$ of the automaton $M$, we can easily construct a finite automaton for the language $K = \bigcup_{p\in X} L_p \subseteq \Sigma^*$ with $O(n^2)$ states (recall that $L_p = L_*(M_p)x_p$). We have to calculate the cardinality of the set $Kw^\omega$. Define

$$H = \{x \in \Sigma^* \mid xw^* \cap K \neq \emptyset, x \notin \Sigma^* w\} \ .$$

Then $Hw^\omega = Kw^\omega$: if $u \in H$, then there exists $m \geq 0$ with $uw^m \in K$ implying $uw^\omega \in Kw^\omega$. Conversely, let $u \in K$. Then let $m \in \mathbb{N}$ be maximal such that $w^m$ is a suffix of $u$ and write $u = tw^m$. Then $t \in H$ implying $uw^\omega \in Hw^\omega$.

In addition, $|H| = |Kw^\omega|$: Consider the function $f : H \to \Sigma^\omega : u \mapsto uw^\omega$. It maps $H$ surjectively onto $Kw^\omega$ by the above. To show injectivity, let $s, t \in H$ with $sw^\omega = tw^\omega$. Then there is $m \in \mathbb{N}$ and a proper prefix $x$ of $w$ such that w.l.o.g. $s = tw^m x$. Let $w = xy$. Since $sw$ and $tw^{m+1}x$ are both prefixes of the $\omega$-word $sw^\omega = tw^\omega$ and have the same length, we obtain $sxy = sw = tw^{m+1}x = tw^m xyx$. Hence $xy = w = yx$. If both, $x$ and $y$ are nonempty, then they have a common root [22, Proposition 1.3.2]. But this contradicts the primitivity of $w$. Since $x$ is a proper prefix of $w$, it must therefore be empty, i.e., $s = tw^m \in H$. Since $w$ is no suffix of any word in $H$, we obtain $m = 0$ and therefore $s = t$. Thus, indeed, $f$ is bijective.

Thus, we have to calculate the cardinality of the set $H$. By calculating all states in a finite state automaton for $K$ from which a final state can be reached by a $w^*$-labeled path, we can easily construct a finite state automaton $A$ for the language $\{x \in \Sigma^* \mid xw^* \cap K \neq \emptyset\}$. Then $H$ is the set $L_*(A) \setminus \Sigma^* w$. We will calculate the cardinality of the set $H^{\mathrm{rev}} = L_*(A)^{\mathrm{rev}} \setminus w^{\mathrm{rev}}\Sigma^*$. Note that a deterministic and complete automaton for $w^{\mathrm{rev}}\Sigma^*$ has $|w^{\mathrm{rev}}| + 2 \leq n + 2$-many states. Thus, by the product construction we can compute a nondeterministic finite automaton $A'$ for $H^{\mathrm{rev}}$ with $O(n^3)$ states. The infinity of the language $L_*(A')$ can be checked in nondeterministic logarithmic space by searching for a reachable loop in $A'$. Thus, within the given space bound, we can check infinity of $H$. On the other hand, if $L_*(A')$ is finite, then every word in $L_*(A')$ is of length $O(n^3)$. In order to calculate the size of $L_*(A')$ we test all words of length $O(n^3)$ in lexicographic order. This can be done in polynomial space. ☐

**Remark 3.4.** *In the above proof, we reduce the calculation of the cardinality of $L_\omega(M)$ in polynomial time to the calculation of the size of the language accepted by an acyclic finite automaton. The latter problem is easily seen to be #P-complete, where #P is the class of all counting functions $f : \{0,1\}^* \to \mathbb{N}$ for which there exists a nondeterministic polynomial time Turing machine $M$ such that $f(x)$ is the number of accepting paths of $M$ on input $x$, see e.g. [23]. For a given nondeterministic polynomial time Turing machine $M$ running in time $p(n)$ and an input $x$ of length $n$, one can construct an acyclic automaton $A$ such that $L_*(A)$ is precisely the set of all words $w$ with $|w| = (p(n)+1)p(n)$ that do not encode an accepting computation of $M$ on input $x$. This construction is similar to the proof that universality for nondeterministic finite automata is PSPACE-complete [25]. This also shows that the PSPACE-bound in Lemma 3.3 cannot be improved to deterministic polynomial time unless P = NP.*

## 3.2 Complexity of the $\mathcal{L}(Q_u)$-theory

In Section 3.4 we will show that there exists a locally finite automatic structure $\mathcal{A}$ and a recursive set $U \subseteq \mathbb{N}$ such that the $\mathrm{FO}(C_U)$-theory of $\mathcal{A}$ is undecidable. To obtain a decidability result, we therefore consider an injectively $\omega$-automatic structure $\mathcal{A}$ of bounded degree. We will consider the $\mathcal{L}(\mathbb{C})$-theory of $\mathcal{A}$, where every $\mathcal{C} \in \mathbb{C}$ is a relation over cardinals. Furthermore, we make the following assumptions:

**Assumption 3.5.** *(1)* $(\Gamma, L, \mathrm{id})$ *is an $\omega$-automatic presentation for $\mathcal{A}$, i.e., in particular $L$ is the universe of $\mathcal{A}$.*

*(2)* $\delta \in \mathbb{N}$ *is a bound for the degrees of the nodes in the Gaifman graph $G_{\mathcal{A}}$.*

*(3) For every $0 \leq n \leq \delta$ the signature $\tau$ contains a unary predicate $\deg_n$ with $\mathcal{A} \models \deg_n(u)$ if and only if the degree of $u$ in the Gaifman-graph $G_{\mathcal{A}}$ is exactly $n$.*

*(4)* $\mathbb{C}$ *is a countable set of relations on* $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Clearly, neither (1) nor (2) imposes restrictions on (the isomorphism type of) $\mathcal{A}$. On the other hand, (3) and (4) seem to be severe restrictions on the class of structures and logics, we are considering. Concerning (3), note that the set $D_n$ of nodes $w$ of degree $n$ in $G_{\mathcal{A}}$ is FO-definable in $\mathcal{A}$ also without assuming (3). Hence, by Proposition 2.2, this language $D_n \subseteq L$ is $\omega$-regular, i.e., extending $\mathcal{A}$ by the unary relations $D_n$ for $0 \leq n \leq \delta$ results again in an $\omega$-automatic structure of bounded degree. Thus, assumption (3) above is no restriction. Finally, consider (4). If $\mathbb{C}$ allows more than countably many quantifiers, then the $\mathcal{L}(\mathbb{C})$-theory of $\mathcal{A}$ becomes uncountable, so it does not make sense to ask for the decidability. Since the $\omega$-automatic structure $\mathcal{A}$ contains at most $2^{\aleph_0}$ many elements, we can assume that every $\mathcal{C} \in \mathbb{C}$ is in fact a relation over $\{\varkappa \mid \varkappa \leq 2^{\aleph_0}\}$. If we assume the continuum hypothesis (i.e., there is no cardinal $\varkappa$ with $\aleph_0 < \varkappa < 2^{\aleph_0}$), (4) is therefore no restriction at all. This is the case even without this controversial assumption from set theory, as we will see following Corollary 3.11 (we omit this discussion here to not distract the reader from the main line of argument).

We will prove that under the above four restrictions, the $\mathcal{L}(\mathbb{C})$-theory of $\mathcal{A}$ can be reduced in triply exponential space to the relations in $\mathbb{C}$. For this, we need the following concept: A pair $(\mathcal{B}, \bar{b})$ is a *potential* $(D,k)$*-sphere* ($D, k \in \mathbb{N}$) if the following holds:

- $\mathcal{B}$ is a finite $\tau$-structure whose Gaifman-graph has degree at most $\delta$,
- $\bar{b}$ is a $k$-tuple of elements from $\mathcal{B}$,
- $N_{\mathcal{B}}(2^D, \bar{b}) = \mathcal{B}$, i.e., every element of $\mathcal{B}$ has distance at most $2^D$ from some entry of the tuple $\bar{b}$,
- for any $y \in S_{\mathcal{B}}(2^D - 1, \bar{b})$, we have $\mathcal{B} \models \deg_n(y)$ if and only if $n$ is the degree of $y$ in the Gaifman-graph of $\mathcal{B}$, and
- for any $y \in \mathcal{B} \setminus S_{\mathcal{B}}(2^D - 1, \bar{b})$ there is a unique $0 \leq n \leq \delta$ such that $\mathcal{B} \models \deg_n(y)$ and the degree of $y$ in the Gaifman-graph of $\mathcal{B}$ is at most $n$.

Thus, a potential $(D,k)$-sphere is a candidate for a $2^D$-sphere around some $k$-tuple in the structure $\mathcal{A}$.

Let $\{b_1, b_2, \ldots, b_n\}$ be the universe of $\mathcal{B}$ with $\bar{b} = (b_1, \ldots, b_k)$ ($k \leq n$). Since $\bar{b}$ is not necessarily repetition-free, we may have $b_i = b_j$ for some $i < j \leq k$, but we may assume that $b_{k+1}, \ldots, b_n$ are pairwise different and different from $b_1, \ldots, b_k$. Then let $\psi(x_1, \ldots, x_n)$ denote the conjunction of the following formulas:

- $x_i = x_j$ if $b_i = b_j$ and $x_i \neq x_j$ if $b_i \neq b_j$ for $i, j \in \{1, 2, \ldots, n\}$
- $R(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ if $(b_{i_1}, b_{i_2}, \ldots, b_{i_m}) \in R$ for $R \in \tau$ with $m = n_R$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$
- $\neg R(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ if $(b_{i_1}, b_{i_2}, \ldots, b_{i_m}) \notin R$ for $R \in \tau$ with $m = n_R$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$.

Then set $\varphi_{(\mathcal{B},\bar{b})} = \exists x_{k+1} \cdots \exists x_n : \psi$.

**Lemma 3.6.** *There exists a constant $c \in \mathbb{N}$ such that for any potential $(D, k)$-sphere $(\mathcal{B}, \bar{b})$, the existential FO-formula $\varphi_{(\mathcal{B},\bar{b})}$ has size at most $\exp(2, c(D + k))$. For any $k$-tuple $\bar{u} \in L^k$, we have $(\mathcal{A}, \bar{u}) \models \varphi_{(\mathcal{B},\bar{b})}$ if and only if $(N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong (\mathcal{B}, \bar{b})$.*

*Proof.* Recall that the Gaifman graph of $\mathcal{B}$ has degree at most $\delta$ and that the tuple $\bar{b}$ has length $k$. Hence the $2^D$-sphere in $\mathcal{B}$ around $\bar{b}$ has at most $h := k \cdot \delta^{2^D}$ many elements. Note that the number $n$ of variables that are used in the formula $\varphi_{(\mathcal{B},\bar{b})}$ is at most $k + h$. Thus, the size of the formula $\varphi_{(\mathcal{B},\bar{b})}$ is bounded by $c'(n^2 + \sum_{R \in \tau} n^{n_R}) \leq \exp(2, c(D + k))$ for appropriate constants $c'$ and $c$.

Now let $\bar{u} = (u_1, \ldots, u_k) \in L^k$ be a $k$-tuple of words in $L$. If $(N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong (\mathcal{B}, \bar{b})$, it is obvious that $\mathcal{A} \models \varphi_{(\mathcal{B},\bar{b})}(\bar{u})$. Conversely, suppose $\mathcal{A} \models \varphi_{(\mathcal{B},\bar{b})}(\bar{u})$. Since the formula $\varphi_{(\mathcal{B},\bar{b})}$ describes completely the relations between the nodes $x_i$ for $1 \leq i \leq n$, the structure $(\mathcal{B}, \bar{b})$ can be embedded into $(N_{\mathcal{A}}(2^D, \bar{u}), \bar{u})$ by the mapping $f : \mathcal{B} \to N_{\mathcal{A}}(2^D, \bar{u})$ with $f(b_i) = u_i$ ($1 \leq i \leq k$). We show surjectivity of $f$ by induction on the distance to $\bar{u}$. To start, let $v \in L$ with $d_{\mathcal{A}}(v, \bar{u}) = 0$, i.e., $v$ is an entry in the tuple $\bar{u}$. Since $\bar{b}$ is mapped onto $\bar{u}$, this solves the base case. Now let $d_{\mathcal{A}}(v, \bar{u}) = r + 1 \leq 2^D$ and suppose that any $w \in L$ with $d_{\mathcal{A}}(w, \bar{u}) \leq r$ is in the image under $f$. There exists a path $v_0, v_1, \ldots, v_r, v$ in the Gaifman-graph of $\mathcal{A}$, where $v_0$ belongs to the tuple $\bar{u}$. By induction, we find a path $c_0, c_1, \ldots, c_r$ in the Gaifman-graph of $\mathcal{B}$ such that $f(c_i) = v_i$. Thus, $c_0$ belongs to $\bar{b}$. Let $m$ be the degree of $c_r$ in the Gaifman graph $G_{\mathcal{B}}$ of $\mathcal{B}$. Since $d_{\mathcal{B}}(c_r, \bar{b}) \leq r \leq 2^D - 1$, we get $\mathcal{B} \models \deg_m(c_r)$. Hence $\mathcal{A} \models \deg_m(v_r)$, i.e., $v_r$ has precisely $m$ neighbors in the Gaifman graph of $\mathcal{A}$. Since the $m$ neighbors of $c_r$ are mapped by $f$ to distinct neighbors of $v_r$, there is $b \in \mathcal{B}$ with $f(b) = v$. $\square$

**Lemma 3.7.** *There are functions $\# : \mathbb{N}^2 \to \mathbb{N}$ and $\Phi : \mathbb{N}^3 \to \mathrm{FO}$ such that*

*(0) $\#(D, k)$ is computable in space $\exp(2, O(D+k))$ and $\Phi(D, k, i)$ in space $\exp(2, O(D+k)) + \log i$*

*(1) for any $D, k \in \mathbb{N}$, $\#(D, k)$ is the number of potential $(D, k)$-spheres,*

*(2) for any $D, k, i \in \mathbb{N}$, there is a potential $(D, k)$-sphere $\mathcal{B}(D, k, i)$ with $\varphi_{\mathcal{B}(D,k,i)} = \Phi(D, k, i)$, and*

*(3) for any $D, k \in \mathbb{N}$ and any potential $(D, k)$-sphere $(\mathcal{B}, \bar{b})$, there exists $i \leq \#(D, k)$ with $\varphi_{(\mathcal{B},\bar{b})} = \Phi(D, k, i)$.*

*Proof.* Let $c$ be the constant from Lemma 3.6. Given a formula $\varphi$ of size at most $\exp(2, c(D+k))$, one can decide in linear space whether there exists a potential $(D,k)$-sphere $(\mathcal{B}, \bar{b})$ with $\varphi = \varphi_{(\mathcal{B},\bar{b})}$: First, the formula has to be existential and list all possible relations between the variables. Secondly, the unique structure obtained this way has to be a potential $(D,k)$-sphere.

Now, to compute $\Phi(D,k,i)$, enumerate all existential formulas of size at most $\exp(2, c(D+k))$ and search for the $i$-th such formula that arises from some potential $(D,k)$-sphere. Since the number of existential formulas of the given size is triply exponential in $D+k$, in order to compute $\#(D,k)$, we have to count up to $\exp(3, D+k)$ which is possible in doubly exponential space. □

Note that $\mathcal{B}(D,k,1), \ldots, \mathcal{B}(D,k,\#(D,k))$ enumerates the isomorphism types of potential $(D,k)$-spheres for any $D,k \in \mathbb{N}$.

In the following we identify a tuple $\bar{u} = (u_1, \ldots, u_k)$ with its convolution $u_1 \otimes u_2 \otimes \cdots \otimes u_k$. We write $k = |\bar{u}|$ for the length of the tuple $\bar{u}$.

**Lemma 3.8.** *The following can be computed in space* $\exp(3, O(D+k)) + \log i$*:*
*INPUT:* $D, k, i \in \mathbb{N}$
*OUTPUT: a $k$-dimensional Büchi-automaton $M$ of size* $\exp(3, O(D+k))$ *with*
$L_\omega(M) = \{\bar{u} \mid (N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong \mathcal{B}(D,k,i)\}$.

*Proof.* Let $\varphi = \Phi(D,k,i) = \varphi_{\mathcal{B}(D,k,i)}$. By Lemma 3.7, it can be computed in space $\exp(2, O(D+k)) + \log i$. From this formula, using the product construction for Büchi-automata [26], we can build the Büchi-automaton $M$ of size $O((3q)^{|\varphi|})$ where $q$ is the maximal size of an automaton used in the presentation of $\mathcal{A}$. Hence the size of $M$ (as well as the space needed for its construction) is in $\exp(3, O(D+k))$. Furthermore, $L_\omega(M) = \{\bar{u} \in L^k \mid (N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \models \varphi\} = \{\bar{u} \in L^k \mid (N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong \mathcal{B}(D,k,i)\}$ by Lemma 3.6. □

Let us fix a function $s(D+k) \in \exp(3, O(D+k))$ bounding the space in Lemma 3.8. For a word $u \in \Sigma^\omega$, let the *norm* $\lambda(u)$ be given by

$$\lambda(u) = \inf\{|vw| \mid u = vw^\omega\}$$

with $\lambda(u) = \infty$ if $u$ is not ultimately periodic, i.e., not of the form $vw^\omega$ for some $v, w \in \Sigma^*$. Let UP denote the class of all ultimately periodic $\omega$-words over some alphabet. In the algorithms below, we will often handle $\omega$-words $u \in \mathrm{UP}$ that can be given as a pair $(v, w)$ with $u = vw^\omega$ and $|vw| = \lambda(w)$. Note that if $M$ is a Büchi-automaton with $n$ states and $L_\omega(M) \neq \emptyset$, then we find an $\omega$-word $u \in L_\omega(M)$ such that $\lambda(u) \leq 2n$. Note that for $\bar{u} = (u_1, \ldots, u_k)$ we have $\lambda(\bar{u}) = \lambda(u_1 \otimes u_2 \cdots \otimes u_k) \leq \prod_{1 \leq i \leq k} \lambda(u_i)$. Since we can build a $(k+1)$-dimensional Büchi-automaton with $\lambda(\bar{u})$ many states that accepts the language $\bar{u} \otimes \Sigma^\omega$, the product construction for Büchi-automata and Lemma 3.8 gives:

**Lemma 3.9.** *The following can be computed in space* $3 \cdot s(D+k+1) \cdot \lambda(\bar{u}) + \log i$ *if* $k = |\bar{u}| > 0$ *and in space* $s(D+1) + \log i$ *if* $k = |\bar{u}| = 0$*:*
*INPUT:* $D, k, i \in \mathbb{N}$ *and* $\bar{u} \in L^k \cap \mathrm{UP}$
*OUTPUT: a $(k+1)$-dimensional Büchi-automaton $M$ with* $L_\omega(M) = \{\bar{u}w \in L^{k+1} \mid (N_{\mathcal{A}}(2^D, \bar{u}w), \bar{u}w) \cong \mathcal{B}(D, k+1, i)$.
*Moreover, if $L_\omega(M) \neq \emptyset$, then we can compute within the same space bound a word $w \in L \cap \mathrm{UP}$ with $\bar{u}w \in L_\omega(M)$ and*

$$\lambda(w) \leq \begin{cases} 6 \cdot s(D+k+1) \cdot \lambda(\bar{u}) & \text{if } k > 0 \\ 2 \cdot s(D+1) & \text{if } k = 0. \; (*) \end{cases}$$

Now consider the two algorithms **size** and **check** in Figure 1 and Figure 2, respectively. The algorithm **size** shall return the number of words $v \in \Sigma^\omega$ with $\mathcal{A} \models \varphi(\bar{u}v)$. The algorithm **check** shall check whether $\mathcal{A} \models \varphi(\bar{u})$.

**check**$(\varphi(\bar{x}), \bar{u}) : \{0, 1\}$
    $(\varphi(\bar{x})$ formula with $|\bar{u}| = |\bar{x}|$ many free variables,
    $\bar{u}$ tuple of ultimately periodic words from $L)$
  **case** $\varphi = R(\bar{x})$
    **if** $\bar{u} \in R$ **then return**$(1)$ **else return**$(0)$ **endif**
  **case** $\varphi = \varphi_1 \wedge \varphi_2$
    **return**(**check**$(\varphi_1, \bar{u}) \wedge$ **check**$(\varphi_2, \bar{u})$)
  **case** $\varphi = \neg\varphi_1$
    **return**($\neg$**check**$(\varphi_1, \bar{u})$)
  **case** $\varphi = \mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{x}, y), \ldots, \psi_n(\bar{x}, y))$
    **for** $i = 1$ **to** $n$ **do**
      $\varkappa_i :=$ **size**$(\psi_i, \bar{u})$
    **endfor**
    **if** $(\varkappa_1, \ldots, \varkappa_n) \in \mathcal{C}$ **then return**$(1)$ **else return**$(0)$ **endif**

Figure 1: The algorithm **check**

1 **size**$(\varphi, \bar{u}) : \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$
2     $(\varphi$ formula with $|\bar{u}| + 1$ many free variables,
3     $\bar{u}$ tuple of ultimately periodic words from $L)$
4   $D := \mathrm{qfr}(\varphi)$; $\varkappa := 0$;
5   **for** $i := 1$ **to** $\#(D, |\bar{u}| + 1)$ **do**
6     **calculate** an $|\bar{u}| + 1$-dimensional Büchi-automaton $M$ with
      $L_\omega(M) = \{\bar{u}w \in L^{|\bar{u}|+1} \mid (N_{\mathcal{A}}(2^D, \bar{u}w), \bar{u}w) \cong \mathcal{B}(D, |\bar{u}| + 1, i)\}$
7     **if** $L_\omega(M) \neq \emptyset$ **then**
8       **choose** $w \in \Sigma^\omega$ with $\bar{u}w \in L_\omega(M)$ and $\lambda(w) \leq 6 \cdot s(D + |\bar{u}| + 1) \cdot \lambda(\bar{u})$
9       **if check**$(\varphi, \bar{u}w)$ **then**
10        $\varkappa := \varkappa + |L_\omega(M)|$
11       **endif**
12     **endif**
13   **endfor**
14   **return**$(\varkappa)$

Figure 2: The algorithm **size**

Let us first verify the correctness of the algorithms **check** and **size**. If **size** behaves as intended, the correctness of **check** is rather obvious. We now discuss **size**. By Lemma 3.7, line 5 iterates over all potential $(D, |\bar{u}| + 1)$-spheres. Since $D = \mathrm{qfr}(\varphi)$, Theorem 3.2 implies that if $\mathcal{A} \models \varphi(\bar{u}v)$ and $(N_{\mathcal{A}}(2^D, \bar{u}v), \bar{u}v) \cong (N_{\mathcal{A}}(2^D, \bar{u}w), \bar{u}w)$, then also $\mathcal{A} \models \varphi(\bar{u}w)$. Thus, there exists a tuple $\bar{u}w \in L_\omega(M)$ with $\mathcal{A} \models \varphi(\bar{u}w)$ if and only if $\mathcal{A} \models \varphi(\bar{u}v)$ for all $\bar{u}v \in L_\omega(M)$, where $M$ is the Büchi-automaton calculated in line 6. To check this, we select in line 8 a "short" tuple $\bar{u}w \in L_\omega(M)$ and check in line 9 whether $\mathcal{A} \models \varphi(\bar{u}w)$ using algorithm **check**. If this is true, then we add to the current $\varkappa$ the size of the language $L_\omega(M)$, which can be calculated by Lemma 3.3 in polynomial space wrt. the size of $M$.

Next we discuss the space complexity of a call **check**$(\psi, \varepsilon)$ (where $\varepsilon$ is the empty tuple) for a sentence $\psi$ of quantifier rank $D_0$. There are at most $D_0$ nested calls to **size** since each time, the quantifier rank decreases. Moreover, note that when we call **size** with parameters $\varphi$ and $\bar{u}$, then we have $\mathrm{qfr}(\varphi)+|\bar{u}|+1 \leq D_0$. Thus, the Büchi-automaton $M$ in line 6 can be calculated in space $3 \cdot s(D+|\bar{u}|+1) \cdot \lambda(\bar{u}) \leq 3 \cdot s(D_0) \cdot \lambda(\bar{u})$ by Lemma 3.9 (since $i \leq \#(D, |\bar{u}|+1) \in \exp(3, O(D_0))$, we can forget the summand $\log i$) and also the bound $6 \cdot s(D+|\bar{u}|+1) \cdot \lambda(\bar{u}) \leq 6 \cdot s(D_0) \cdot \lambda(\bar{u})$ in line 8 for the $\omega$-word $w$ follows from Lemma 3.9. Assume that $(u_1, u_2, \ldots, u_{D_0})$ is the tuple of ultimately periodic $\omega$-words calculated by the algorithm. If we set $\bar{u}_k = (u_1, u_2, \ldots, u_k)$, then we obtain:

$$\begin{aligned}
\lambda(\bar{u}_1) &\leq 2 \cdot s(D_0) \qquad \text{(by } (*) \text{ in Lemma 3.9)} \\
\lambda(\bar{u}_{k+1}) &\leq \lambda(\bar{u}_k) \cdot \lambda(u_{k+1}) \leq 6 \cdot s(D_0) \cdot \lambda(\bar{u}_k)^2
\end{aligned}$$

From this, we obtain by induction $\lambda(\bar{u}_k) \leq 2^{2^k} \cdot 6^{2^k-1} \cdot s(D_0)^{2^k-1}$. Since $s(D_0) \in \exp(3, O(D_0))$ and $k \leq D_0$, it follows $\lambda(\bar{u}_k) \in \exp(3, O(D_0))$. Hence, each of the Büchi-automata $M$ in line 6 can be constructed in triply-exponential space. Since the recursion depth of the overall algorithm is bounded by the size of the input formula and for each recursive call only a triply exponential amount of information has to be stored, the whole algorithm can be executed in space triply exponential in the size of the input formula. Thus, we proved:

**Theorem 3.10.** *Let $\mathbb{C} = \{\mathcal{C}_i \mid i \in \mathbb{N}\}$, where $\mathcal{C}_i$ is a relation on $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Let $\mathcal{A}$ be an injectively $\omega$-automatic structure of bounded degree. Then the $\mathcal{L}(\mathbb{C})$-theory of $\mathcal{A}$ can be decided in triply exponential space by a Turing machine with oracle $\{(i, \bar{c}) \mid i \in \mathbb{N}, \bar{c} \in \mathcal{C}_i\}$.*

*Proof.* Extending the signature of $\mathcal{A}$ by unary relations $\deg_n$, we can ensure that Assumption 3.5 holds. Then the statement follows easily from the above algorithms. Oracle access to $\{(i, \bar{c}) \mid i \in \mathbb{N}, \bar{c} \in \mathcal{C}_i\}$ is needed in line 14 of **check**. $\square$

## 3.3 Expressiveness of the logic $\mathcal{L}(Q_u)$

From Theorem 3.2 we can easily deduce that the logic $\mathcal{L}(Q_u)$ has a quite restricted expressive power over structures of bounded degree:

**Corollary 3.11.** *Let $\mathcal{A}$ be a $\tau$-structure of bounded degree, and let $\varphi(\bar{x}) \in \mathcal{L}(\mathcal{Q}_u)$. There exists a formula $\psi(\bar{x}) \in \mathrm{FO}$ such that $\mathcal{A} \models \forall \bar{x}(\varphi \leftrightarrow \psi)$.*

*Proof.* Extending the signature of $\mathcal{A}$ by the first-order definable unary relations $\deg_n$, we can ensure that Assumption 3.5(2,3) holds. In the further consideration, we will use Lemmas 3.6 and 3.7 that have only been shown for injectively $\omega$-automatic structures satisfying Assumption 3.5. But it is straightforward to verify that all we actually used was Assumption 3.5(2,3). Let $d$ be the quantifier-rank of the $\mathcal{L}(\mathcal{Q}_u)$-formula $\varphi$. Furthermore, let $\#$ and $\Phi$ be the functions from Lemma 3.7 that compute the number of potential $(d, k)$-spheres and a formula describing the $i$th such $(d, k)$-sphere, respectively. Then set

$$I = \{i \mid 1 \leq i \leq \#(d, k), \mathcal{A} \models \forall \bar{x} : (\Phi(d, k, i) \rightarrow \varphi)\}$$

and $\psi = \bigvee_{i \in I} \Phi(d,k,i)$.

We show $\mathcal{A} \models \forall \bar{x}(\psi \leftrightarrow \varphi)$: The implication "$\rightarrow$" is obvious by the definition of the set $I$. So assume $\mathcal{A} \models \varphi(\bar{u})$. Then, by Lemma 3.7(3), there exists $1 \leq i \leq \#(d,k)$ with $\Phi(d,k,i) = \varphi_{(N_\mathcal{A}(2^d,\bar{u}),\bar{u})}$. Let $\bar{v} \in A^k$ with $(\mathcal{A},\bar{v}) \models \Phi(d,k,i)$. Then, by Lemma 3.6, $(N_\mathcal{A}(2^d,\bar{v}),\bar{v}) \cong (N_\mathcal{A}(2^d,\bar{u}),\bar{u})$. Hence, by Theorem 3.2, we get $(\mathcal{A},\bar{v}) \models \varphi$, i.e., we showed $\mathcal{A} \models \forall \bar{x}(\Phi(d,k,i) \rightarrow \varphi)$ and therefore $i \in I$. Hence $\mathcal{A} \models \psi(\bar{u})$. Thus, $R$ is indeed first-order definable by $\psi$. $\square$

**Discussion 3.12.** *Recall that we postponed the discussion concerning point (4) in Assumption 3.5 and the influence of the continuum hypothesis to our results. More formally, we restricted attention to counting quantifiers $\mathcal{Q}_\mathcal{C}$ where $\mathcal{C}$ is a relation on cardinals in $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. We now show that allowing cardinals $\varkappa$ with $\aleph_0 < \varkappa < 2^{\aleph_0}$ does not change the results on $\omega$-automatic structures. So let $\mathcal{A}$ be some injectively $\omega$-automatic structure of bounded degree with presentation $(\Gamma, L, \mathrm{id})$ and let $\mathcal{C} \in \mathbb{C}$ be an arbitrary $n$-ary relation on cardinals. Furthermore, let $\mathcal{D} = \{(\varkappa_1,\ldots,\varkappa_n) \in \mathcal{C} \mid \varkappa_1,\ldots,\varkappa_n \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}\}$ be the restriction of $\mathcal{C}$ to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Now let $\psi_i(\bar{x},y)$ be some $\mathcal{L}(\mathcal{Q}_u)$-formula for $1 \leq i \leq n$. Then, by Corollary 3.11, there are first-order formulas $\psi'_i(\bar{x},y)$ such that $\mathcal{A} \models \forall \bar{x} \forall y(\psi_i \leftrightarrow \psi'_i)$ for $1 \leq i \leq n$. Hence, by Proposition 2.2, the relations*

$$R_i = \{(\bar{u},u) \in L^k \times L \mid \mathcal{A} \models \psi_i(\bar{u},u)\}$$

*are $\omega$-automatic. Let $K_i = \{u_1 \otimes \cdots \otimes u_k \otimes u \mid (\bar{u},u) \in R_i\}$, which is $\omega$-regular. Since for every fixed $\bar{u} = (u_1,\ldots,u_k) \in L^k$ we have*

$$|\{u \in L \mid \mathcal{A} \models \psi_i(\bar{u},u)\}| = |\{u \mid (u_1 \otimes \cdots \otimes u_k) \otimes u \in K_i\}|,$$

*Lemma 2.4 implies that the former cardinality belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Hence*

$$\mathcal{A} \models Q_\mathcal{C} y(\psi_1(\bar{u},y),\ldots,\psi_n(\bar{u},y)) \quad \Leftrightarrow \quad \mathcal{A} \models Q_\mathcal{D} y(\psi_1(\bar{u},y),\ldots,\psi_n(\bar{u},y)).$$

*Thus, the quantifiers $Q_\mathcal{C}$ and $Q_\mathcal{D}$ are equivalent and Assumption 3.5(4) does not impose a restriction as far as expressiveness is concerned.*

The above proof of Corollary 3.11 is not effective since it does not give a way to compute the set $I$. For injectively $\omega$-automatic structures of bounded degree, the situation changes:

**Corollary 3.13.** *Let $\mathbb{C} = \{\mathcal{C}_i \mid i \in \mathbb{N}\}$, where $\mathcal{C}_i$ is a relation on $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Let $(\Gamma, L, \mathrm{id})$ be an injective $\omega$-automatic presentation of the structure $\mathcal{A}$ of bounded degree. For a given formula $\varphi(\bar{x}) \in \mathcal{L}(\mathbb{C})$, one can construct in elementary space (modulo $\mathbb{C}$) a first-order formula $\psi(\bar{x})$ and a $k$-dimensional Büchi-automaton $M$ (where $k = |\bar{x}|$) such that for any $\bar{u} \in L^k$*

$$\mathcal{A} \models \varphi(\bar{u}) \iff \mathcal{A} \models \psi(\bar{u}) \iff \bar{u} \in L_\omega(M) .$$

*Proof.* In view of the proof of Corollary 3.11, it remains to be shown that the set $I$ can be computed in elementary space: By Lemma 3.6, the formula $\Phi(d,k,i)$ has size at most $\exp(2, c(d+k))$. Hence, in order to calculate $I$, one has to decide validity of formulas of size $\exp(2, O(d+k))$ which can be done in space $\exp(5, O(d+k))$ by Theorem 3.10. Hence, indeed, $\psi$ can be computed in elementary space. By Lemma 3.8, we can translate each of the formulas $\Phi(d,k,i)$ for $i \in I$ into a Büchi-automaton in elementary space. The disjoint union of these automata is $M$ that can, again, be computed in elementary space. $\square$

Note the similarity of Theorem 2.9 and Corollary 3.13 (both state that definable relations are $\omega$-automatic) as well as that of Corollary 2.10 and Theorem 3.10 (both state that some theories are decidable). But the proof strategies are different: while Corollary 2.10 was derived from Theorem 2.9, the corresponding statement Theorem 3.10 was used to prove Corollary 3.13.

## 3.4 Optimality

Our main results deal with structures satisfying two assumptions: they are $\omega$-automatic and of bounded degree. For these structures, we showed how to decide the $\mathcal{L}(\mathbb{C})$-theory (modulo $\mathbb{C}$) and how to translate formulas from $\mathcal{L}(\mathbb{C})$ effectively (modulo $\mathbb{C}$) into equivalent ones from FO. In this section, we show that the two assumptions we made cannot be relaxed. First, it is shown that relaxing "automatic" to "recursive" makes the results fail:

**Theorem 3.14.** *There exists a recursive structure $\mathcal{A}$ of bounded degree such that the* FO*-theory of $\mathcal{A}$ is decidable and the* FO$(\exists^\infty)$*-theory of $\mathcal{A}$ is undecidable.*

*Proof.* Let $L \subseteq \{0,1\}^*$ be a recursively enumerable, but not recursive set. Then there exists a deterministic Turing machine $M$ such that, on input of $w \in \{0,1\}^*$, the machine $M$ eventually stops if and only if $w \in L$. Let $f(w) \in \mathbb{N} \cup \{\omega\}$ denote the number of steps $M$ performs on input $w$. Intuitively, we consider a structure that consists of $f(w)$ many copies of the word $\triangleright w \triangleleft$ for any $w \in \{0,1\}^*$. More formally, we set

$$\begin{aligned}
A &= \bigcup_{w\in\{0,1\}^*} \{w\} \times \{0,1,\ldots,|w|+1\} \times \{i \mid 0 \leq i < f(w)\}\\
s &= \{((w,i,j),(w,i+1,j)) \mid (w,i,j),(w,i+1,j) \in A\}\\
P_0 &= \{(w,i,j) \in A \mid w = a_1\cdots a_n, a_i = 0\}\\
P_1 &= \{(w,i,j) \in A \mid w = a_1\cdots a_n, a_i = 1\}\\
P_\triangleright &= \{(w,i,j) \in A \mid i = 0\}\\
P_\triangleleft &= \{(w,i,j) \in A \mid i = |w|+1\}\\
\mathcal{A} &= (A, s, P_0, P_1)
\end{aligned}$$

Then $\mathcal{A}$ is a labeled directed graph whose degree is bounded by 2. In what follows, we write $s(x) = y$ for $(x,y) \in s$. For $w = a_1a_2\cdots a_n \in \{0,1\}^*$, let $\varphi_w(x)$ denote the following formula

$$x \in P_\triangleright \wedge \bigwedge_{1\leq i\leq n} s^i(x) \in P_{a_i} \wedge s^{n+1}(x) \in P_\triangleleft$$

Note that $\mathcal{A} \models \varphi_w(x)$ if and only if $x$ is the $\triangleright$-node of some copy of $\triangleright w \triangleleft$ in $\mathcal{A}$. Hence $\varphi_w(x)$ is satisfied by precisely $f(w)$ many nodes in $\mathcal{A}$. Therefore, $w \in L$ if and only if $\mathcal{A} \models \neg\exists^\infty x : \varphi_w(x)$. This shows that the FO$(\exists^\infty)$-theory of $\mathcal{A}$ is undecidable.

On the other hand, the first-order theory of $\mathcal{A}$ is decidable: By Gaifman's theorem [9], it suffices to decide sentences of the form

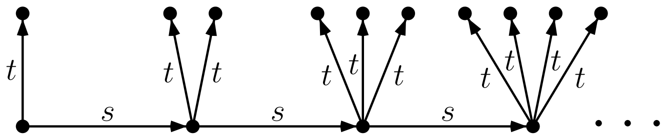
"there are at least $n$ nodes $x$ with $(N_\mathcal{A}(r,x),x) \cong \mathcal{B}$"

where $n, r \in \mathbb{N}$ and $\mathcal{B}$ is some finite structure. This is only interesting if $\mathcal{B}$ is a line labeled in $\{0,1,\triangleright,\triangleleft\}$ (which we will identify with the sequence of labels, i.e., a word $v \in \{0,1,\triangleright,\triangleleft\}^+$), and the position of $x$). If $\mathcal{B}$ is of the form $\triangleright w \triangleleft$ with $w \in \{0,1\}^*$, then the above statement holds if and only if $f(w) \geq n$ which can be decided. Any other structure $\mathcal{B}$ that can be found in $\mathcal{A}$ at all appears infinitely often in $\mathcal{A}$, i.e., the statement is true for them. ☐

By choosing a more complicated but still recursive counting quantifier, we can show that Theorem 3.10 even fails for locally finite automatic structures.

**Theorem 3.15.** *There is a recursive set $U \subseteq \mathbb{N}$ and a locally finite automatic structure $\mathcal{A}$ such that the* $\mathrm{FO}(C_U)$*-theory of $\mathcal{A}$ is undecidable.*

*Proof.* We first claim that the structure $\mathcal{A} = (A, s, t)$, where $s$ and $t$ are binary relations, that looks as follows is automatic:




To see this, let $L = a^+ \cup b^+ a^*$. The automaton for $s$ simply reads a word of the form $a^n$ ($n \geq 1$) on the first tape and $a^{n+1}$ on the second tape. The automaton for $t$ reads a word of the form $a^n$ ($n \geq 1$) on the first tape and a word of the form $b^i a^{n-i}$ (for some $1 \leq i \leq n$) on the second tape.

Now let $A \subseteq \mathbb{N}$ be a nonrecursive but recursively enumerable set of natural numbers. Let $a_1, a_2, a_3, \ldots$ be a recursive enumeration of $A$. Let $U = \{a_1 + \cdots + a_i \mid i \geq 1\}$. Then $U$ is recursive. We claim that the $\mathrm{FO}(C_U)$-theory of $\mathcal{A}$ is undecidable. Let $\varphi_U(x)$ be the formula $C_U y : t(x, y)$. Then $m \in A$ if and only if

$$\mathcal{A} \models \exists y, z : \varphi_U(y) \wedge \varphi_U(z) \wedge s^m(y) = z \wedge \bigwedge_{1 \leq k < m} \neg \varphi_U(s^k(y)) \ .$$

This proves the theorem. □

# 4 An open problem

In view of Corollary 2.10 and Theorem 3.15, it might be an interesting problem to characterize those subsets $U \subseteq \mathbb{N}$ such that for every ($\omega$-) automatic structure (not necessarily of bounded degree), the $\mathrm{FO}(C_U)$-theory of $\mathcal{A}$ is decidable. Note that by Corollary 2.10, this is true for every semilinear set $U$. Since $(\mathbb{N}, \leq)$ is automatic and since $x \in U$ can be expressed as $C_U y : y < x$, the set $U$ has to be decidable. It turns out that this class is rather restricted as the following two results from the literature indicate:

First, let $U = \{p(n) \mid n \in \mathbb{N}\}$ be the range of a polynomial $p$ over $\mathbb{N}$ of degree at least two. Then the FO-theory of $(\mathbb{N}, +, U)$ is undecidable [6]. This implies that the $\mathrm{FO}(C_U)$-theory of the automatic structure $(\mathbb{N}, +)$ is undecidable (express $x \in U$ as $C_U y : y < x$).

Secondly, let $U \subseteq \mathbb{N}$ be not semilinear but $k$-recognizable for some $k \geq 2$, i.e., the set of all base-$k$ expansions of the elements of $U$ is a regular language over the alphabet $\{0, \ldots, k-1\}$. Choose a number $\ell \geq 2$ multiplicatively independent from $k$ (i.e., $k^a \neq \ell^b$ for any $a, b \geq 1$). Let $|_\ell$ (for $\ell \geq 2$) be the set of all pairs $(n, m)$ such that $n$ is a power of $\ell$ dividing $m$. By [2, Theorem 4.9] the FO-theory of $(\mathbb{N}, +, |_\ell, U)$ is undecidable. As above, the $\mathrm{FO}(C_U)$-theory of the automatic structure $(\mathbb{N}, +, |_\ell)$ is therefore undecidable.

Thus, in order to make the $\mathrm{FO}(C_U)$-theory of any automatic structure decidable, the set $U$ cannot be the range of a non-linear polynomial nor can it be $k$-recognizable but not semilinear.

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