Theories of Automatic Structures and their Complexity

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Definition of Automatic Structures

Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure, $R_i \subseteq A^{n_i}$. We say that $\mathbb{A}$ is automatic, if the following data exist:

- a finite alphabet $\Sigma$
- a regular language $L \subseteq \Sigma^*$
- a bijection $h : L \to A$ such that for every $1 \leq i \leq n$ the relation

$$\{(u_1, u_2, \ldots, u_{n_i}) \in L^{n_i} \mid (h(u_1), h(u_2), \ldots, h(u_{n_i})) \in R_i\}$$

is synchronized rational.
Binary synchronized rational relations are recognized by synchronous 2-tape automata.

In order to accept a pair \((u, v) \in \Sigma^* \times \Sigma^*\) such an automaton operates as follows:

\[
\begin{array}{ccccccccccc}
\nu & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & \# & \cdots & \# \\
u & a_0 & a_1 & a_2 & \cdots & a_{m-1} & a_m & a_{m+1} & \cdots & a_n \\
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\begin{array}{cccccccc}
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\hline
v & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & \# & \cdots & \# \\
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The automaton transition is depicted as follows:

- \(q_m\) moves to a new state when the symbols \(u\) and \(v\) synchronize.
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The following structures are automatic:

- \((\mathbb{N}, +)\)
- \((\mathbb{Q}, \leq)\)
- Transition graphs of Turing-machines

The following structures are not automatic:

- \((\mathbb{N}, \cdot)\)
- the free monoid generated by two elements
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Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure.

Let $\Omega$ be an infinite set of variables ranging over $A$.

The set of all FO-formulas over $\mathbb{A}$ is defined as follows:

- $x = y$ and $R_i(x_1, \ldots, x_{n_i})$ are FO-formulas, where $x, y, x_1, \ldots, x_{n_i} \in \Omega$

- If $\phi$ and $\psi$ are FO-formulas then also
  
  $\neg \phi, \phi \land \psi, \phi \lor \psi, \exists x : \phi, \forall x : \phi$

are FO-formulas.

An FO-sentence is an FO-formula without free variables.

The FO-theory of $\mathbb{A}$ is the set of all FO-sentences that are true in the structure $\mathbb{A}$. 
Khoussainov, Nerode 1994: Every automatic structure has a decidable FO-theory.

A problem is called elementary decidable, if it can be decided in time \(2^{2^n}\), where the height of this tower of exponents is constant.

Blumensath, Grädel 2000: There are automatic structures which are not elementary decidable.

Example: \((\{0,1\}^*, s_0, s_1, \preceq)\), where \(s_i = \{(w, w^i | w \in \{0,1\}^*)\}\) and \(\preceq\) is the prefix relation.
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Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure. The Gaifman-graph of $\mathbb{A}$ is the undirected graph $(A, E)$, where

$$E = \{(a, b) \mid a \neq b, \ a \text{ and } b \text{ both belong to some tuple of some relation } R_i\}$$

The structure $\mathbb{A}$ has bounded degree if its Gaifman-graph has bounded degree, i.e., for some constant $\delta$, every element of $\mathbb{A}$ has at most $\delta$ many neighbors in the Gaifman-graph.
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**Main Results**

\( \text{ATIME}(a(n), t(n)) \) is the class of all problems that can be solved in

- alternating time \( t(n) \) with
- only \( a(n) \) many alternations.

Well-known: \( \text{ATIME}(a(n), t(n)) \subseteq \text{DSPACE}(t(n)) \)

**Theorem**

Let \( \mathbb{A} \) be an automatic structure of bounded degree. Then the FO-theory of \( \mathbb{A} \) belongs to \( \text{ATIME}(n, 2^{2^{c \cdot n}}) \) for some constant \( c \).

**Theorem**

There exists an automatic structure of bounded degree such that the FO-theory of \( \mathbb{A} \) is not in \( \text{ATIME}(c \cdot n, 2^{2^{c \cdot n}}) \) for some constant \( c \).
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Main ideas for the upper bound

Let $\mathbb{A} = (A, \ldots )$ be an automatic structure with degree bounded by $\delta \in \mathbb{N}$.

Let $\Gamma, L \subseteq \Gamma^*$, and $h : L \rightarrow A$ (bijective) witness the automaticity of $\mathbb{A}$.

For an element $a \in A$ of the structure $\mathbb{A}$ and $r \in \mathbb{N}$ let $S(a, r)$ be the substructure of $\mathbb{A}$ induced by the set

$$\{ b \in A \mid \text{the distance between } a \text{ and } b \text{ in the Gaifman-graph of } \mathbb{A} \text{ is at most } r \}$$

We prove: For every $a \in A$ and $r \in \mathbb{N}$ there exists $u \in L$ with:

- $S(a, r) \simeq S(h(u), r)$
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This allows to apply the machinery of Ferrante/Rackoff.
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We prove that there exists a finite automaton $B(a, r)$ such that

- the number of states of $B(a, r)$ is bounded by $2^{2^{O(r)}}$.
- $B(a, r)$ accepts the language $\{u \in L \mid S(a, r) \cong S(h(u), r)\}$.

Note that $m := |S(a, r)| \in 2^{O(r)}$, because the degree of the Gaifman-graph of $A$ is bounded by the constant $\delta$.

Let $S(a, r) = \{u_1, \ldots, u_m\}$ with $u = u_1$.

Take variables $x_1, \ldots, x_m$, where $x_i$ represents $u_i \in S(a, r)$. 
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- \( B(a, r) \) accepts the language \( \{ u \in L \mid S(a, r) \cong S(h(u), r) \} \).

Note that \( m := |S(a, r)| \in 2^{O(r)} \), because the degree of the Gaifman-graph of \( A \) is bounded by the constant \( \delta \).

Let \( S(a, r) = \{ u_1, \ldots, u_m \} \) with \( u = u_1 \).

Take variables \( x_1, \ldots, x_m \), where \( x_i \) represents \( u_i \in S(a, r) \).
Main ideas for the upper bound

For every $0 \leq n \leq \delta$ there exists an FO-formula (of constant size) $\deg_n(x)$, expressing that $x$ has degree $n$ in the Gaifman-graph of $\mathbb{A}$.

Let $\psi(x_1, \ldots, x_m)$ be the conjunction of the following formulas

- $x_i \neq x_j$ for $i \neq j$,
- $R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \in R$ ($R$ is an arbitrary relation of $\mathbb{A}$),
- $\neg R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \notin R$, and
- $\deg_n(x_i)$ if the degree of $u_i$ in the Gaifman-graph of $\mathbb{A}$ is precisely $n$.

Let $\theta(x_1) = \exists x_2 \cdots \exists x_m \psi(x_1, x_2 \ldots, x_m)$.

Then we have for every $b \in \mathbb{A}$:

$$\mathbb{A} \models \theta(b) \iff S(a, r) \simeq S(b, r)$$
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Then we have for every $b \in \mathbb{A}$:

$$\mathbb{A} \models \theta(b) \iff S(a, r) \preceq S(b, r)$$
Main ideas for the upper bound

We translate the formula \( \theta(x_1) = \exists x_2 \cdots \exists x_m \psi(x_1, x_2, \ldots, x_m) \) into an equivalent automaton \( B(a, r) \) of size \( 2^{2^{O(r)}} \):

Note that \( \psi(x_1, x_2, \ldots, x_m) \) is a conjunction of \( 2^{O(r)} \) formulas, each of which can be translated into an automaton of size \( O(1) \).

\[ \Rightarrow \psi(x_1, x_2, \ldots, x_m) \text{ can be translated into an automaton on } m \in 2^{O(r)} \text{ tracks with } 2^{2^{O(r)}} \text{ states (product construction).} \]

\[ \Rightarrow \text{Using projection, } \theta(x_1) = \exists x_2 \cdots \exists x_m \psi(x_1, x_2, \ldots, x_m) \text{ can be translated into an equivalent automaton of size } 2^{2^{O(r)}}. \]
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We translate the formula $\theta(x_1) = \exists x_2 \cdots \exists x_m \psi(x_1, x_2, \ldots, x_m)$ into an equivalent automaton $B(a, r)$ of size $2^{2^{O(r)}}$:

Note that $\psi(x_1, x_2, \ldots, x_m)$ is a conjunction of $2^{O(r)}$ formulas, each of which can be translated into an automaton of size $O(1)$.

$\Rightarrow \psi(x_1, x_2, \ldots, x_m)$ can be translated into an automaton on $m \in 2^{O(r)}$ tracks with $2^{2^{O(r)}}$ states (product construction).

$\Rightarrow$ Using projection, $\theta(x_1) = \exists x_2 \cdots \exists x_m \psi(x_1, x_2, \ldots, x_m)$ can be translated into an equivalent automaton of size $2^{2^{O(r)}}$. 
Main ideas for the lower bound

A binary tree with marked leaves is a structure \((A, s_0, s_1, P)\), where \((A, s_0, s_1)\) is a complete binary tree and \(P\) is a unary predicate on the leaves.

1. Construct a “hard” automatic structure \(\mathbb{A}\) of bounded degree: \(\mathbb{A}\) consists of countably many disjoint copies of every binary tree with marked leaves.

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A binary tree with marked leafs is a structure \((A, s_0, s_1, P)\), where \((A, s_0, s_1)\) is a complete binary tree and \(P\) is a unary predicate on the leafs.

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A “hard” automatic structure of bounded degree

Elements of the structure will be represented by words from the language \((\{a, a', b, b'\}*\{0, 1\}*\#)^*\).
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00#01#11#
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00#01#11#

**Automaton for \(s_0\):** In each \#\ldots\#-block, the first symbol \(x \in \{0, 1\}\) is replaced by \(a\) (if \(x = 0\)) resp. \(b'\) (if \(x = 1\)).
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\[
00\#01\#11#
\]

\[
a0\#a1\#b'1\#
\]

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\[ \begin{array}{c}
00\#01\#11#
\end{array} \]

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\[
\begin{align*}
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\[
\begin{array}{c}
00\#01\#11#
\
 a0\#a1\#b'1#
\
 aa\#ab'\#b' b' \\
 a'0\#a'1\#b1#
\
 a'a\#a'b'\# bb' \\
\end{array}
\]

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\[
\begin{array}{c}
00\#01\#11\\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\quad s_0 & s_1 & s_0 & s_1 \\
a0\#a1\#b'1\\
\downarrow \quad \downarrow \\
aa\#ab'\#b'\# & aa'\#ab\#b'b' \\
\downarrow \quad \downarrow \\
aa'\#ab\#b'b' & a'a\#a'b\#bb' \\
\downarrow \quad \downarrow \\
a'0\#a'1\#b1 & a'0\#a'1\#b1\\
\end{array}
\]

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![Diagram showing automata for different states and transitions.]

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Elements of the structure will be represented by words from the language \((\{a, a', b, b'\}^*\{0, 1\}^*\#)^*\).

![Diagram of automata]

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Monadic interpretation of addition

For an FO-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ over $A$ and $b_1, \ldots, b_m \in A$ let $\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^A$ be the $n$-ary relation

$$\{(a_1, \ldots, a_n) \mid \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m) \text{ is true in } A\}.$$  

For every $k \geq 0$ we can efficiently construct FO-formulas

$$\phi_k(x, y), \psi_k(x_1, x_2, x_3, y), \mu_k(x, y, z)$$

over $A$ such that there exists $a \in A$ with:

1. the structure $(\phi_k(x, a)^A, \psi_k(x_1, x_2, x_3, a)^A)$ is isomorphic to $(\{0, \ldots, 2^{2^k} - 1\}, \{(x, y, z) \mid x + y = z\})$, and
2. every subset of $\phi_k(x, a)^A$ is of the form $\mu_k(x, a, b)^A$ for some $b \in A$. 
Monadic interpretation of addition

For an FO-formula \( \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \) over \( A \) and \( b_1, \ldots, b_m \in A \) let \( \varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^A \) be the \( n \)-ary relation

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Choose for $a$ the root of a binary tree $T_k$ of height $2^k$ (labeling of the leaves irrelevant): Leafs of $T_k \leftrightarrow \{0, \ldots, 2^{2^k} - 1\}$. 
Monadic interpretation of addition

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Choose for $a$ the root of a binary tree $T_k$ of height $2^k$ (labeling of the leafs irrelevant): Leafs of $T_k \rightleftharpoons \{0, \ldots, 2^{2^k} - 1\}$.

We can express $x + y = z$ with an FO-formula of size $O(k)$: carry-look-ahead addition
For every $k \geq 0$ we can efficiently construct FO-formulas

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Let $B$ be an arbitrary subset of $\phi_k(x, a)^\mathbb{A}$, i.e., an arbitrary subset of the leafs of the tree $T_k$ rooted at $a$. 

![Diagram of a binary tree with labels $a$, $s_0$, $s_1$, $s_0$, $s_1$, $s_0$, $s_1$, and $B$ at the leaf nodes.](image)
For every $k \geq 0$ we can efficiently construct FO-formulas

$$
\phi_k(x, y), \psi_k(x_1, x_2, x_3, y), \mu_k(x, y, z)
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over $\mathbb{A}$ such that there exists $a \in A$ with:

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Tree automatic structures are defined similarly to automatic structures using tree automata.

$(\mathbb{N}, \cdot)$ is a tree automatic structure that is not automatic.

Let $A$ be a tree automatic structure of bounded degree. Then the FO-theory of $A$ belongs to $\text{ATIME}(n, 2^{2^{2^c \cdot n}})$ for some constant $c$. 
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Open Problems

- For automatic structures: Close that gap between ATIME$(c \cdot n, 2^{2^{c \cdot n}})$ (lower bound) and ATIME$(n, 2^{2^{2^{2^{c \cdot n}}}})$ (upper bound).
- For tree automatic structures: Close that gap between ATIME$(c \cdot n, 2^{2^{c \cdot n}})$ and ATIME$(n, 2^{2^{2^{2^{2^{c \cdot n}}}}})$.
- What about first-order logic with $\exists^\omega$-quantifiers?
- Is there a tree automatic structure of bounded degree that is not automatic?
- Other classes of (tree) automatic structures with elementary FO-theories.
- E.g. $(\mathbb{N}, +)$: Is there an automatic structure $\mathcal{A}$ of bounded degree such that $(\mathbb{N}, +)$ is first-order interpretable in $\mathcal{A}$. 

Markus Lohrey
Theories of Automatic Structures and their Complexity
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Open Problems

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- For tree automatic structures: Close that gap between $\text{ATIME}(c \cdot n, 2^{2c \cdot n})$ and $\text{ATIME}(n, 2^{2^{22c \cdot n}})$.

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Open Problems

For automatic structures: Close that gap between $\text{ATIME}(c \cdot n, 2^{2^c \cdot n})$ (lower bound) and $\text{ATIME}(n, 2^{2^{2^c} \cdot n})$ (upper bound).

For tree automatic structures: Close that gap between $\text{ATIME}(c \cdot n, 2^{2^c \cdot n})$ and $\text{ATIME}(n, 2^{2^{2^c} \cdot n})$.

What about first-order logic with $\exists^\omega$-quantifiers?

- Is there a tree automatic structure of bounded degree that is not automatic?

- Other classes of (tree) automatic structures with elementary FO-theories.

- E.g. $(\mathbb{N}, +)$: Is there an automatic structure $\mathfrak{A}$ of bounded degree such that $(\mathbb{N}, +)$ is first-order interpretable in $\mathfrak{A}$.
Open Problems

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Open Problems

- For automatic structures: Close that gap between $\text{ATIME}(c \cdot n, 2^{2c \cdot n})$ (lower bound) and $\text{ATIME}(n, 2^{22c \cdot n})$ (upper bound).
- For tree automatic structures: Close that gap between $\text{ATIME}(c \cdot n, 2^{2c \cdot n})$ and $\text{ATIME}(n, 2^{22c \cdot n})$.
- What about first-order logic with $\exists^\omega$-quantifiers?
- Is there a tree automatic structure of bounded degree that is not automatic?
- Other classes of (tree) automatic structures with elementary FO-theories.
  - E.g. $(\mathbb{N}, +)$: Is there an automatic structure $A$ of bounded degree such that $(\mathbb{N}, +)$ is first-order interpretable in $A$. 
Open Problems

- For automatic structures: Close that gap between
  $\text{ATIME}(c \cdot n, 2^{2c \cdot n})$ (lower bound) and $\text{ATIME}(n, 2^{2^{2c \cdot n}})$ (upper bound).

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  $\text{ATIME}(c \cdot n, 2^{2c \cdot n})$ and $\text{ATIME}(n, 2^{2^{22c \cdot n}})$.

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- E.g. $(\mathbb{N}, +)$: Is there an automatic structure $\mathfrak{A}$ of bounded degree such that $(\mathbb{N}, +)$ is first-order interpretable in $\mathfrak{A}$.