Theories of Automatic Structures and their Complexity

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Definition of Automatic Structures

Let $\mathbb{A} = (A, R_1, \dots, R_n)$ be a relational structure, $R_i \subseteq A^{n_i}$.
We say that $\mathbb{A}$ is automatic, if the following data exist:

- a finite alphabet Σ
- a regular language $L \subseteq \Sigma^*$
- a bijection $h : L \to A$ such that for every $1 \leq i \leq n$ the relation

  $$\{(u_1, u_2, \dots, u_{n_i}) \in L^{n_i} \mid (h(u_1), h(u_2), \dots, h(u_{n_i})) \in R_i\}$$

  is synchronized rational.

Synchronized Rational Relations

Binary synchronized rational relations are recognized by synchronous 2-tape automata.

In order to accept a pair $(u, v) \in \Sigma^* \times \Sigma^*$ such an automaton operates as follows:

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

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q_0

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	#	$\cdots$	#
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	q_1								
v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
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			q_2						
v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

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q_{m+1}

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

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Examples of Automatic Structures

The following structures are automatic:

- $(\mathbb{N}, +)$
- $(\mathbb{Q}, \leq)$
- Transition graphs of Turing-machines

The following structures are not automatic:

- $(\mathbb{N}, \cdot)$
- the free monoid generated by two elements

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First-Order Logic (FO)

Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure.

Let Ω be an infinite set of variables ranging over A.

The set of all FO-formulas over $\mathbb{A}$ is defined as follows:

- $x = y$ and $R_i(x_1, \ldots, x_{n_i})$ are FO-formulas, where $x, y, x_1, \ldots, x_{n_i} \in \Omega$
- If ϕ and ψ are FO-formulas then also

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \exists x : \phi, \quad \forall x : \phi$$

  are FO-formulas.

An FO-sentence is an FO-formula without free variables.
The FO-theory of $\mathbb{A}$ is the set of all FO-sentences that are true in the structure $\mathbb{A}$.

FO-theories of automatic structures

Khoussainov, Nerode 1994: Every automatic structure has a decidable FO-theory.

A problem is called elementary decidable, if it can be decided in time $2^{2^{\cdot^{\cdot^{2^n}}}}$, where the height of this tower of exponents is constant.

Blumensath, Grädel 2000: There are automatic structures which are not elementary decidable.

Example: $(\{0,1\}^*, s_0, s_1, \preceq)$, where $s_i = \{(w, w\,i \mid w \in \{0,1\}^*\}$ and $\preceq$ is the prefix relation.

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Structures of Bounded Degree

Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure.
The Gaifman-graph of $\mathbb{A}$ is the undirected graph (A, E), where

$$E = \{(a, b) \mid a \neq b, \; a \text{ and } b \text{ both belong to some tuple of some relation } R_i\}$$

The structure $\mathbb{A}$ has bounded degree if its Gaifman-graph has bounded degree, i.e., for some constant δ, every element of $\mathbb{A}$ has at most δ many neighbors in the Gaifman-graph.

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- Cayley-graphs of automatic groups

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Main Results

ATIME$(a(n), t(n))$ is the class of all problems that can be solved in

- alternating time $t(n)$ with
- only $a(n)$ many alternations.

Well-known: $\mathrm{ATIME}(a(n), t(n)) \subseteq \mathrm{DSPACE}(t(n))$

Theorem

Let $\mathbb{A}$ be an automatic structure of bounded degree. Then the FO-theory of $\mathbb{A}$ belongs to $\mathrm{ATIME}(n, 2^{2^{2^{c \cdot n}}})$ for some constant c.

Theorem

There exists an automatic structure of bounded degree such that the FO-theory of $\mathbb{A}$ is not in $\mathrm{ATIME}(c \cdot n, 2^{2^{c \cdot n}})$ for some constant c.

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Main ideas for the upper bound

Let $\mathbb{A} = (A, \ldots)$ be an automatic structure with degree bounded by $\delta \in \mathbb{N}$.

Let Γ, $L \subseteq \Gamma^*$, and $h : L \to A$ (bijective) witness the automaticity of $\mathbb{A}$.

For an element $a \in A$ of the structure $\mathbb{A}$ and $r \in \mathbb{N}$ let $S(a, r)$ be the substructure of $\mathbb{A}$ induced by the set

$$\{b \in A \mid \text{the distance between } a \text{ and } b \text{ in the Gaifman-graph of } \mathbb{A} \text{ is at most } r\}$$

We prove: For every $a \in A$ and $r \in \mathbb{N}$ there exists $u \in L$ with:

- $S(a, r) \simeq S(h(u), r)$
- $|u| \leq 2^{2^{c \cdot r}}$ for a constant c

This allows to apply the machinery of Ferrante/Rackoff.

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- $S(a, r) \simeq S(h(u), r)$
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We prove that there exists a finite automaton $B(a, r)$ such that

- the number of states of $B(a, r)$ is bounded by $2^{2^{O(r)}}$.
- $B(a, r)$ accepts the language $\{u \in L \mid S(a, r) \simeq S(h(u), r)\}$.

Note that $m := |S(a, r)| \in 2^{O(r)}$, because the degree of the Gaifman-graph of $\mathbb{A}$ is bounded by the constant δ.

Let $S(a, r) = \{u_1, \ldots, u_m\}$ with $u = u_1$.

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For every $0 \leq n \leq \delta$ there exists an FO-formula (of constant size) $\deg_n(x)$, expressing that x has degree n in the Gaifman-graph of $\mathbb{A}$.

Let $\psi(x_1, \ldots, x_m)$ be the conjunction of the following formulas

- $x_i \neq x_j$ for $i \neq j$,
- $R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \in R$ (R is an arbitrary relation of $\mathbb{A}$),
- $\neg R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \notin R$, and
- $\deg_n(x_i)$ if the degree of u_i in the Gaifman-graph of $\mathbb{A}$ is precisely n.

Let $\theta(x_1) = \exists x_2 \cdots \exists x_m \; \psi(x_1, x_2 \ldots, x_m)$.

Then we have for every $b \in \mathbb{A}$:

$$\mathbb{A} \models \theta(b) \quad \Leftrightarrow \quad S(a, r) \simeq S(b, r)$$

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We translate the formula $\theta(x_1) = \exists x_2 \cdots \exists x_m \, \psi(x_1, x_2, \ldots, x_m)$ into an equivalent automaton $B(a,r)$ of size $2^{2^{O(r)}}$:

Note that $\psi(x_1, x_2, \ldots, x_m)$ is a conjunction of $2^{O(r)}$ formulas, each of which can be translated into an automaton of size $O(1)$.

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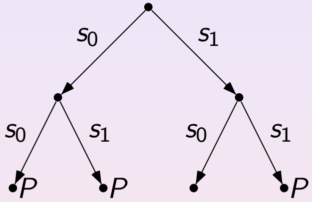

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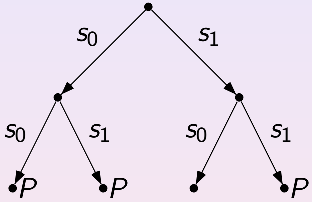

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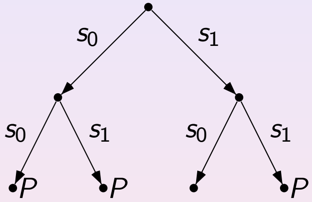

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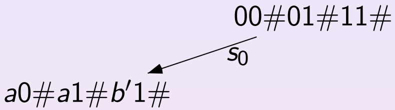

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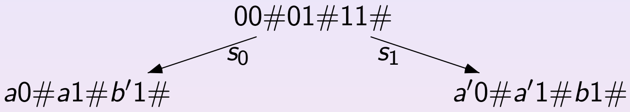

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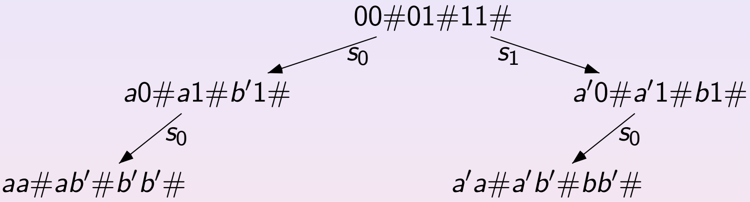

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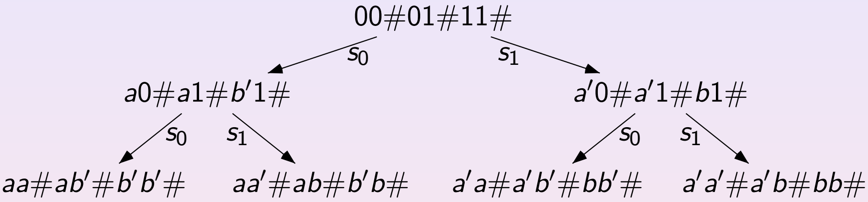

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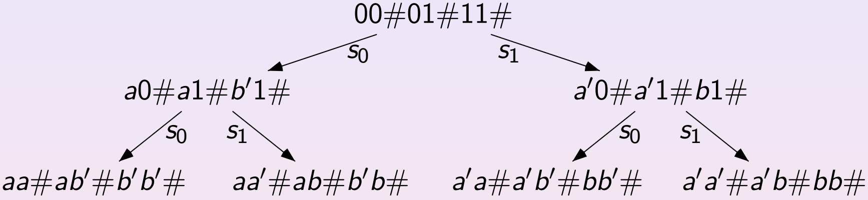

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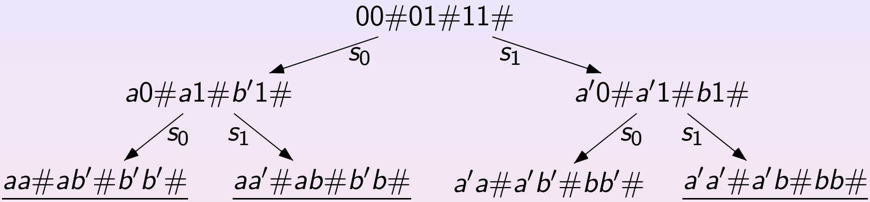

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For an FO-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ over $\mathbb{A}$ and $b_1, \ldots, b_m \in A$ let $\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^{\mathbb{A}}$ be the n-ary relation

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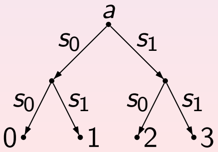

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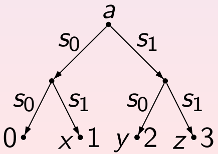

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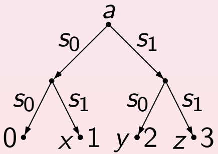

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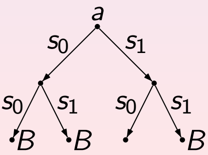

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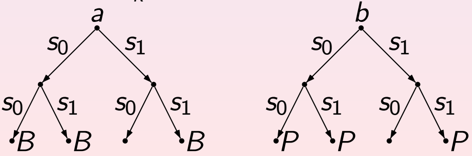

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- For tree automatic structures: Close that gap between $\mathrm{ATIME}(c \cdot n, 2^{2^{c \cdot n}})$ and $\mathrm{ATIME}(n, 2^{2^{2^{2^{c \cdot n}}}})$.
- What about first-order logic with $\exists^{\omega}$-quantifiers?
- Is there a tree automatic structure of bounded degree that is not automatic?
- Other classes of (tree) automatic structures with elementary FO-theories.
- E.g. $(\mathbb{N}, +)$: Is there an automatic structure $\mathbb{A}$ of bounded degree such that $(\mathbb{N}, +)$ is first-order interpretable in $\mathbb{A}$.

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