Word problems on compressed words

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Motivation

In general, computational problems become harder, when inputs are represented in a compressed form.

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Two lines of research:

- Develop efficient algorithms on compressed data (strings, trees, pictures) that operate directly on compressed data without decompressing them first.
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Here we consider the compressed word problem for a fixed language $L \subseteq \Gamma^*$:

INPUT: A compressed representation of a word $w \in \Gamma^*$
QUESTION: $w \in L$?

Compressing strings

A straight-line program over the alphabet Γ is a context-free grammar $H = (V, \Gamma, P, S)$ in Chomsky normal form such that:

- For every $A \in V$ there exists exactly one production of the form $A \to \alpha$ in P.
- There exists a linear ordering $A_1, A_2, \ldots, A_n$ of V such that $S = A_1$ and for every production $A_i \to A_j A_k$ we have $i < j, k$.

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Let $|H|$ be the number of productions of H.

Compressing strings

Example: Let H_n be the straight-line program that consists of the following productions:

$$\begin{array}{rcl} S & \rightarrow & A_1 A_1 \\ A_1 & \rightarrow & A_2 A_2 \\ & \vdots & \\ A_{n-1} & \rightarrow & A_n A_n \\ A_n & \rightarrow & a \end{array}$$

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Several other compressed representations (e.g., Lempel-Ziv) can be efficiently transformed into straight-line programs and vice versa.

Known results

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Plandowski, Rytter: For every context-free language L, the compressed word problem for L is in PSPACE.

Plandowski, Rytter: There are context-free (even linear) languages (over a unary alphabet) with an NP-hard compressed word problem.

Main new result

There exists a fixed deterministic (and linear) context-free language with a PSPACE-complete compressed word problem.

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There exists a fixed deterministic (and linear) context-free language with a PSPACE-complete compressed word problem.

Upper bound: Holds even for the following uniform variant:

INPUT: A straight-line program H and a context-free grammar G
QUESTION: $\text{unfold}(H) \in L(G)$?

Upper bound

Goldschlager: For a given word w and a context-free grammar G in Chomsky normal form it can be checked in space $(\log(|w| + |G|))^2$ whether $w \in L(G)$.

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For a given straight-line program H and a context-free grammar G in Chomsky normal form we simulate the $(\log(|w| + |G|))^2$-space algorithm on $\mathrm{unfold}(H)$ and G without explicitly generating $\mathrm{unfold}(H)$.

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Critical fact: A position j in $\mathrm{unfold}(H)$ can be stored in polynomial space and the symbol at position j in $\mathrm{unfold}(H)$ can be calculated in polynomial time.

Lower bound

Let $\Gamma = \{b, c_0, c_1, c_2, \#, \$, \triangleright, 0\}$ and let R be the monadic string-rewriting system consisting of the following rules:

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0, 1, 2\}$
$\triangleright\,\$ \to \$$	$\#\,\$ \to \varepsilon$	$b\,c_2 \to 0$
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$\Rightarrow$ the language $\{w \in \Gamma^* \mid w \xrightarrow{*}_R 0\}$ is deterministic context-free.

We show that the compressed word problem for $\{w \in \Gamma^* \mid w \xrightarrow{*}_R 0\}$ is PSPACE-complete.

Lower bound

Let $G = (V, E)$ be a directed forest such that $V = \{v_1, v_2, \ldots, v_n\}$ and $(v_i, v_j) \in E \Rightarrow i < j$.

Fix a set $U \subseteq V$ of final nodes without outgoing edges.

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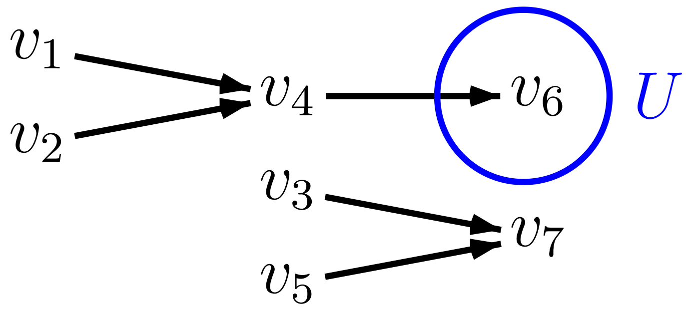

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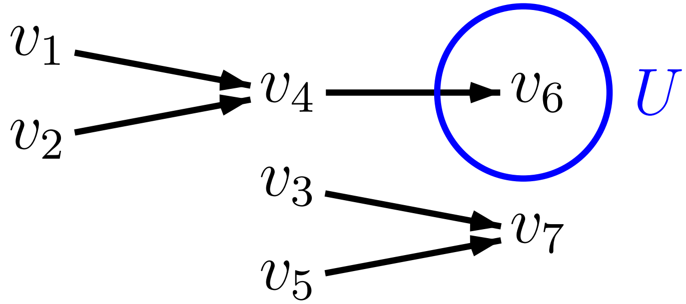

Define $w(G, U) = (\# \, b^n)^n \, \delta_1 \, \$ \, \delta_2 \, \$ \cdots \delta_n \$$ where

$$\delta_i = \begin{cases} c_0^{n+1-(j-i)} & \text{if } (v_i, v_j) \text{ is the unique outgoing edge at } v_i \\ c_1 & \text{if } v_i \in V \setminus U \text{ and } v_i \text{ has no outgoing edge} \\ c_2 & \text{if } v_i \in U \end{cases}$$

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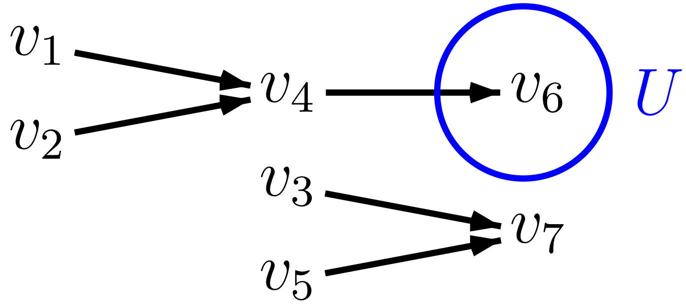

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For the above example: $w(G, U) = (\# b^7)^7 c_0^5 \, \$ \, c_0^6 \, \$ \, c_0^4 \, \$ \, c_0^6 \, \$ \, c_0^6 \, \$ \, c_2 \, \$ \, c_1 \, \$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\,c_i \to \triangleright$ for $i \in \{0,1,2\}$
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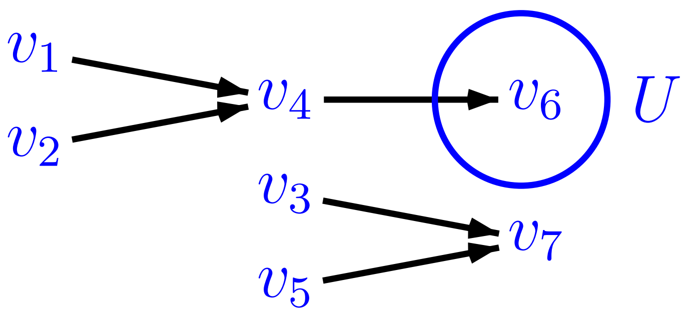

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7$

$$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\;\; c_0^5\;\$\;c_0^6\;\$\;c_0^4\;\$\;c_0^6\;\$\;c_0^6\;\$\;c_2\;\$\;c_1\;\$$$

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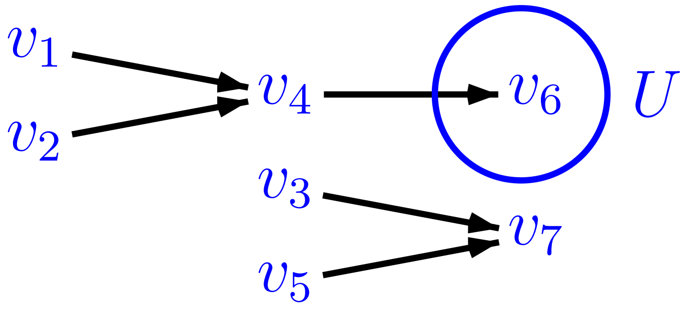

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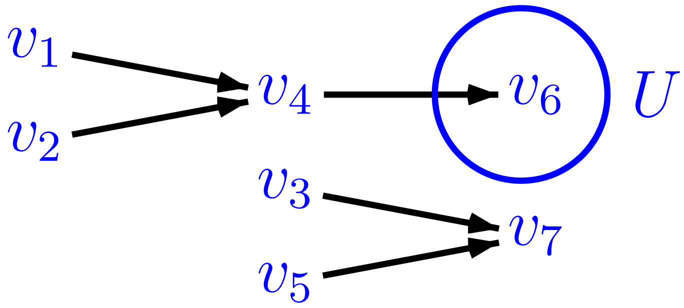

v_2 v_3 v_4 v_5 v_6 v_7

$$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b \quad b\,\$\;c_0^6\;\$\;c_0^4\;\$\;c_0^6\;\$\;c_0^6\;\$\;c_2\;\$\;c_1\;\$$$

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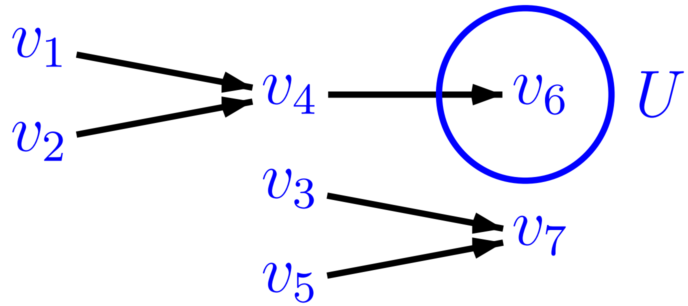

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$$\#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b \quad b\, \$\; c_0^6\, \$\; c_0^4\, \$\; c_0^6\, \$\; c_0^6\, \$\; c_2\, \$\; c_1\, \$$$

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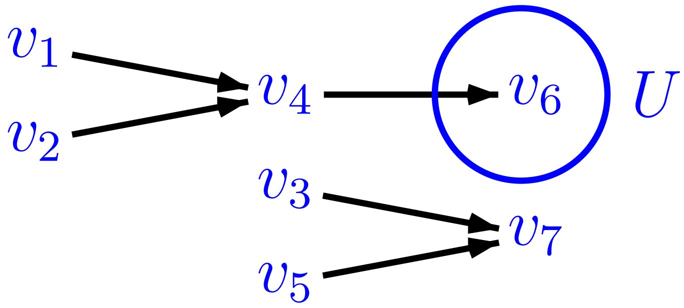

$$\begin{array}{ccccccc} & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b \quad \triangleright & c_0^6\,\$ & c_0^4\,\$ & c_0^6\,\$ & c_0^6\,\$ & c_2\,\$ & c_1\,\$ \end{array}$$

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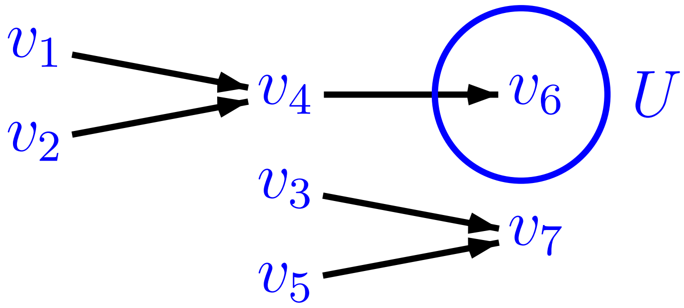

v_2 v_3 v_4 v_5 v_6 v_7

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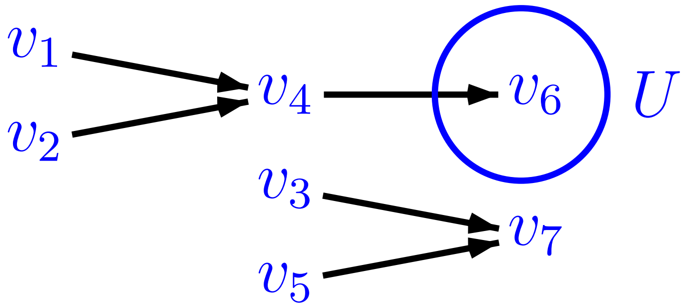

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$$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b \qquad \triangleright\,\$\;c_0^4\;\$\;c_0^6\;\$\;c_0^6\;\$\;c_2\;\$\;c_1\;\$$$

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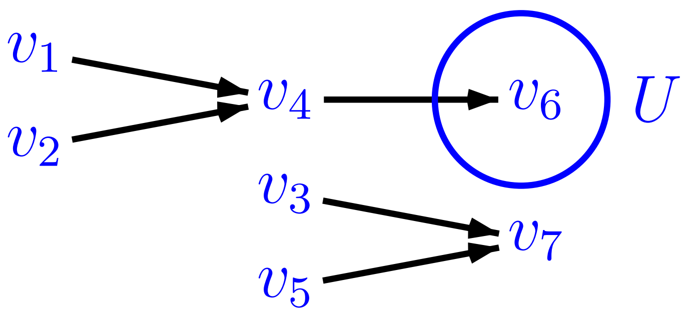

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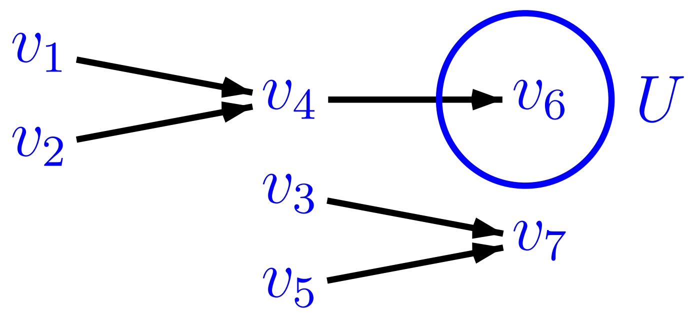

$$\underbrace{}_{}\qquad\qquad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7$$

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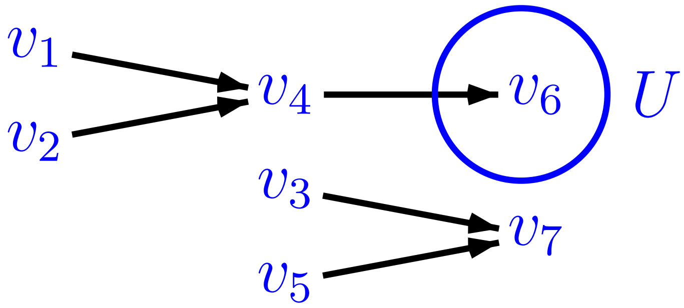

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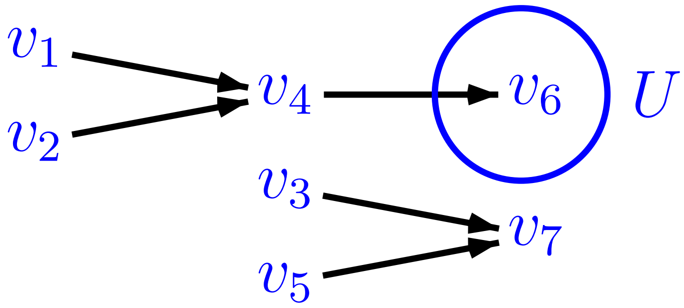

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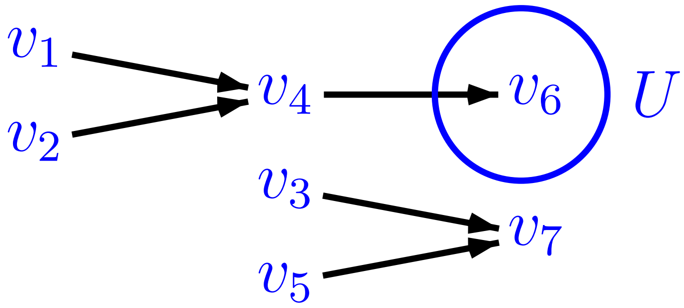

v_3 v_4 v_5 v_6 v_7

$$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \# \quad \triangleright\, c_0^4 \,\$\, c_0^6 \,\$\, c_0^6 \,\$\, c_2 \,\$\, c_1 \,\$$$

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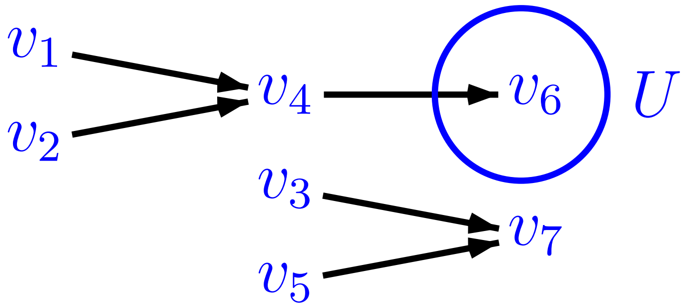

v_4 v_5 v_6 v_7

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$\triangleright\,\$ \to \$$	$\#\,\$ \to \varepsilon$	$b\,c_2 \to 0$
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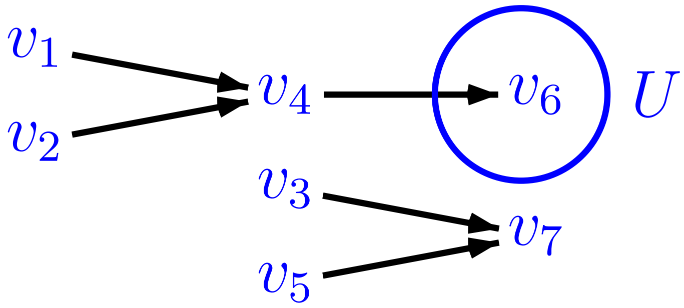

$v_4 \quad v_5 \quad v_6 \quad v_7$

$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\# \qquad \triangleright\,\$\; c_0^6\;\$\; c_0^6\;\$\; c_2\;\$\; c_1\;\$$

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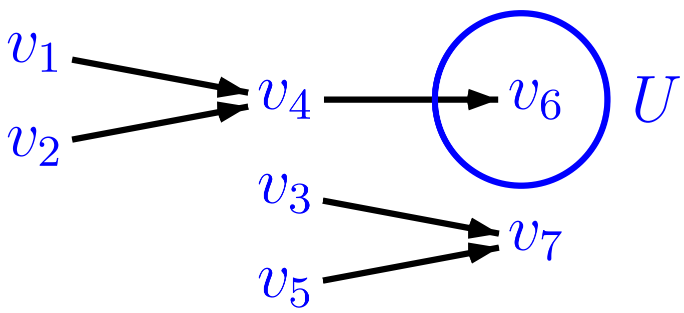

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$$\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\# \qquad \$\ c_0^6\ \$\ c_0^6\ \$\ c_2\ \$\ c_1\ \$$$

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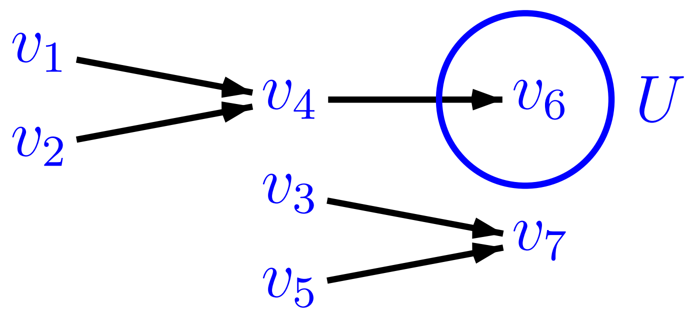

$$\begin{array}{ccccc} & v_4 & v_5 & v_6 & v_7 \\ \#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\# \quad \$ & c_0^6\ \$ & c_0^6\ \$ & c_2\ \$ & c_1\ \$ \end{array}$$

Lower bound

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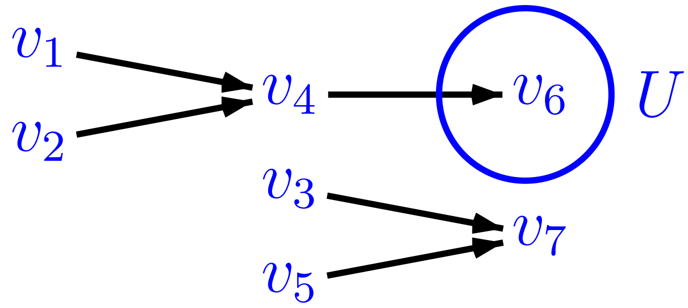

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v_4 \quad v_5 \quad v_6 \quad v_7 \\ \#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7 \qquad c_0^6\ \$\ c_0^6\ \$\ c_2\ \$\ c_1\ \$ \end{array}$$

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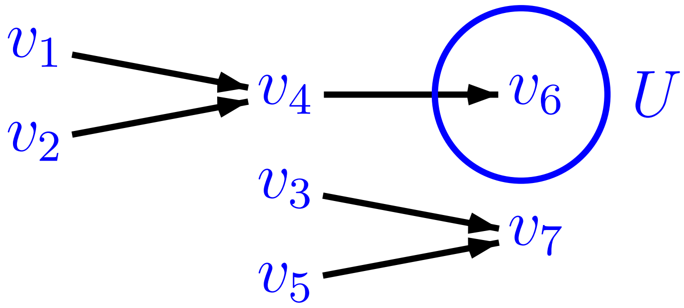

v_4 v_5 v_6 v_7

$\#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7 \quad c_0^6\, \$\, c_0^6\, \$\, c_2\, \$\, c_1\, \$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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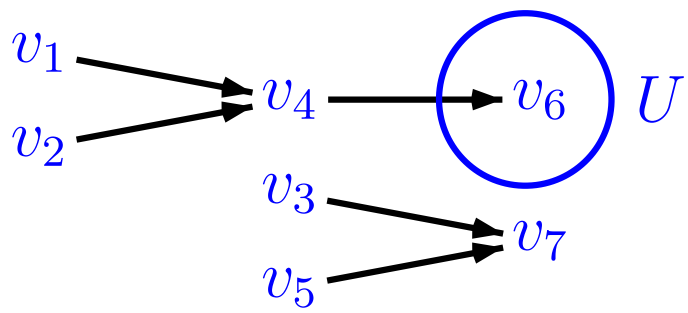

$v_5 \quad v_6 \quad v_7$

$$\# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \qquad b \, \$ \, c_0^6 \, \$ \, c_2 \, \$ \, c_1 \, \$$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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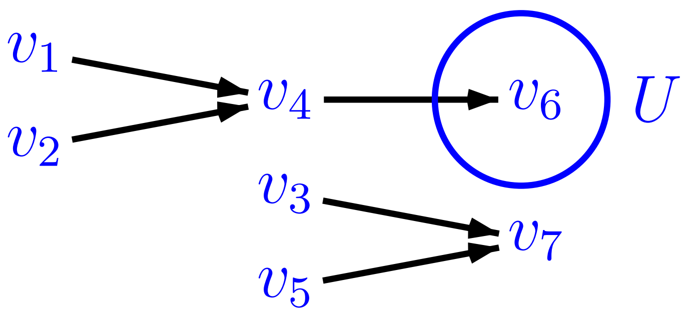

v_5 v_6 v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \#$ $\quad b\,\$\, c_0^6\, \$\, c_2\, \$\, c_1\, \$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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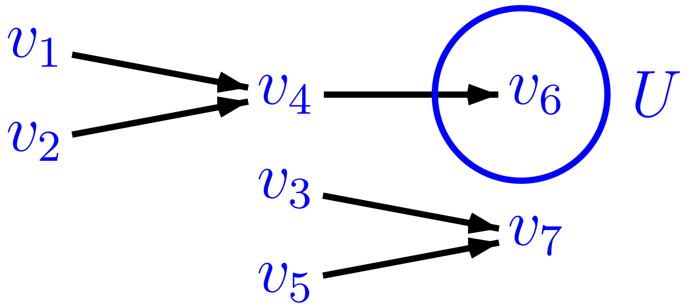

v_5 v_6 v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \#$ $\triangleright\, c_0^6\, \$\, c_2\, \$\, c_1\, \$$

Lower bound

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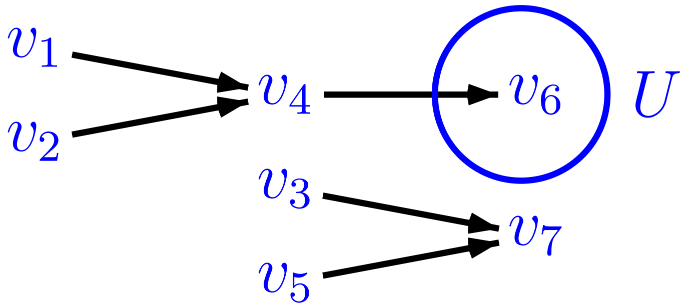

v_5 v_6 v_7

$$\#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \#\, b^7\, \# \quad \triangleright\, c_0^6\, \$\, c_2\, \$\, c_1\, \$$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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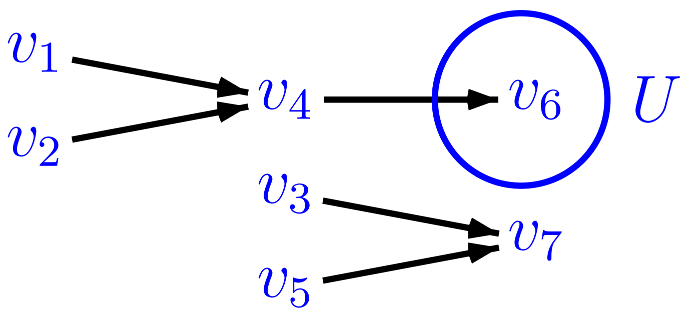

v_6 v_7

$\# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \, b^7 \, \# \qquad \triangleright \, \$ \, c_2 \, \$ \, c_1 \, \$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\,c_i \to \triangleright$ for $i \in \{0,1,2\}$
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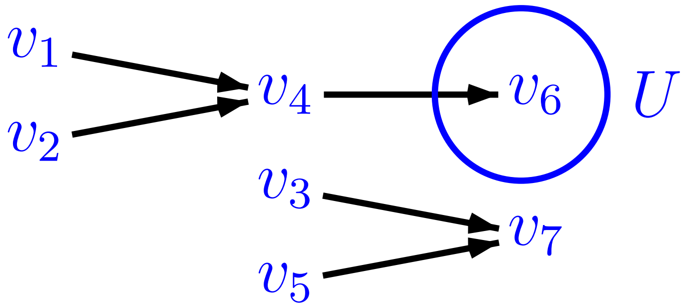

v_6 v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \#$ $\triangleright\,\$\ c_2\ \$\ c_1\ \$$

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$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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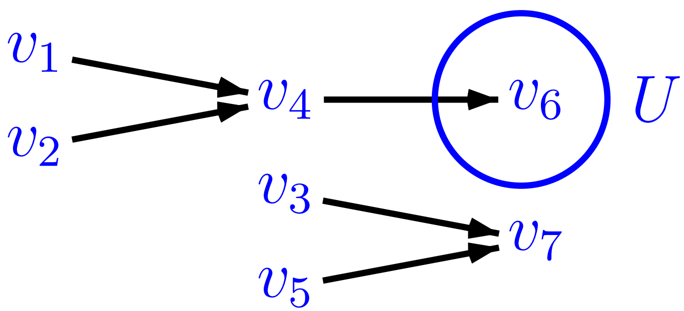

v_6 v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \# \quad \$\, c_2\, \$\, c_1\, \$$

Lower bound

$b\,c_0 \to \varepsilon$	$b\,\$ \to \triangleright$	$\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$
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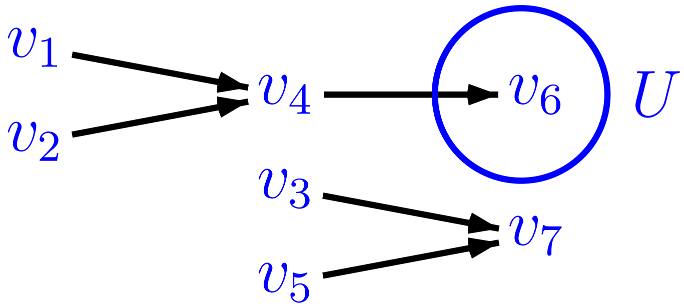

v_6 v_7

$\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\#\, b^7 \,\# \quad\quad \$\; c_2 \;\$\; c_1 \;\$$

Lower bound

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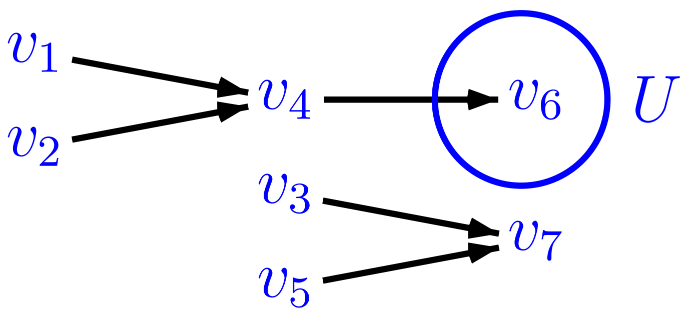

v_6 v_7

$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7 \quad \cdots \quad c_2\,\$\,c_1\,\$$

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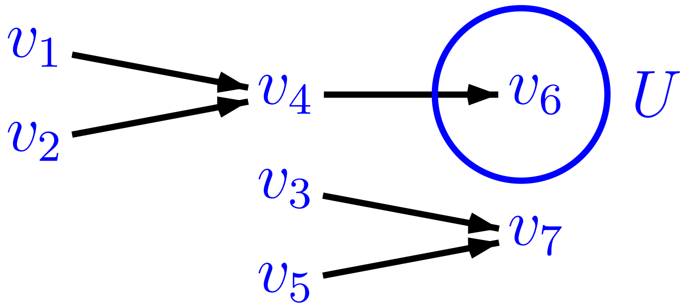

v_6 v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^7 \quad c_2\,\$\,c_1\,\$$

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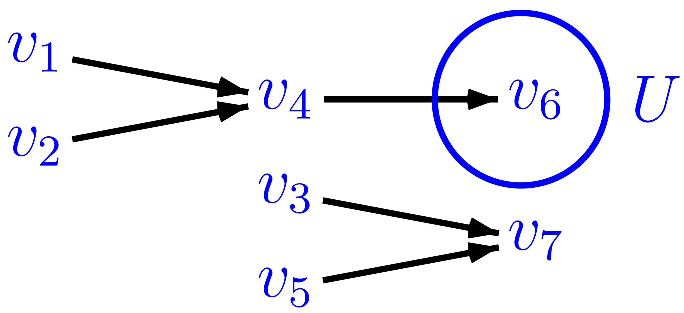

v_7

$\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^7\,\#\,b^6 \qquad 0\ \$\ c_1\ \$$

Lower bound

$b\,c_0 \to \varepsilon$ $\quad$ $b\,\$ \to \triangleright$ $\quad$ $\triangleright\, c_i \to \triangleright$ for $i \in \{0,1,2\}$

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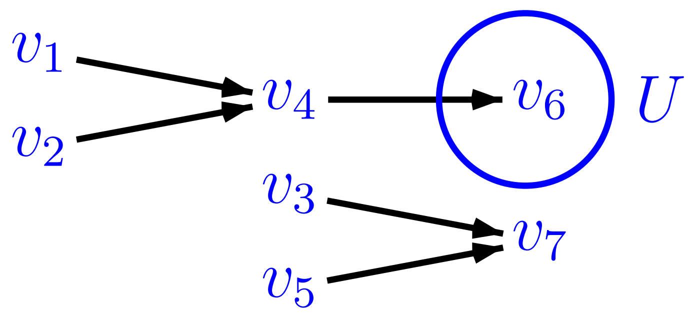

v_7

$\# b^7 \# b^7 \# b^7 \# b^7 \# b^6 \qquad 0 \; \$ \; c_1 \; \$$

Lower bound

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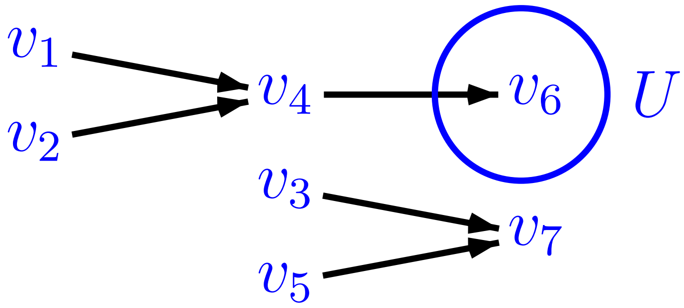

0

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Thus, $\mathrm{unfold}(H) \in L \Leftrightarrow \mathcal{A}$ accepts s.

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- The compressed word problem for the free group of rank 2 is P-complete.
- There exists a fixed context-sensitive language with an EXPSPACE-complete compressed word problem.