Axiomatising Divergence

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Abstract

When a process is capable of executing an unbounded number of non-observable actions it is said to be *divergent*. Different capabilities of an observer to identify this phenomen along the execution leads to different divergent sensitive semantics. This paper develops sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalence. The axiomatisations separates the axioms concerning recursion and those that capture the essence of diverging behaviour.

Key words: process algebra, axiomatisation, divergence

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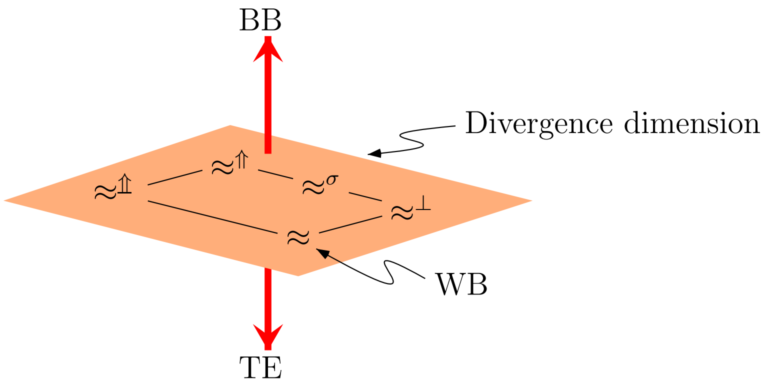


Fig. 1. A zoom into the van Glabbeek spectrum with silent moves

1 Motivation

The study of comparative concurrency semantics is concerned with a uniform classification of process behaviour, and has cumulated in Rob van Glabbeek's seminal papers on the *linear time-branching time spectrum* [6,7]. The main ('vertical') dimension of the spectrum with silent moves [7] spans between trace equivalence (TE) and branching bisimulation (BB), and identifies different ways to discriminate processes according to their branching structure, where BB induces the finest, and TE the coarsest reasonable semantics. Due to the presence of silent moves, this spectrum is spread in another ('horizontal') dimension, determined by the semantics of *divergence*. In the fragment spanning from weak bisimulation (WB) to BB, seven different criteria to distinguish divergence induce a 'horizontal' lattice, and this lattice appears for all the bisimulation relations. Five of the induced relations are equivalences, the other two are preorders. The weak bisimulation equivalence spectrum fragment is depicted in Figure 1.

To illustrate the spectrum, van Glabbeek lists a number of examples and counterexamples showing the differences among the various semantics [7]. *Process algebra* provides a different – and to our opinion more elegant – way to compare semantic issues, by providing distinguishing axioms that capture the essence of an equivalence (or preorder). For the 'vertical' dimension of the spectrum, these distinguishing axioms are well-known (see e.g. [6,9,2]). Typical axioms of this kind are Milner's three law for WB [14], van Glabbeek and Weijland axiom for BB [9], and the axioms for failure semantics [5,4]. In comparison, little effort has been made on finding axioms that capture the essence of divergence, that is, that characterise the 'horizontal' dimension. We believe that this is mainly due to the fact that divergence only makes sense in the presence of recursion, and that recursion is hard to tackle axiomatically. Isolated points in the 'horizontal' dimension have however been axiomatised, most notably Milner's weak bisimulation (WB) congruence [13], and also convergent WB preorder [15], as well as divergence insensitive BB congruence [8] and stable WB congruence [10]. It is worth to mention the works of [4] and [1], which axiomatised divergence sensitive WB congruence and convergent WB preorder, respectively, but without showing completeness in the presence of recursion.

This paper develops complete axiomatisations for the 'horizontal' dimension of weak bisimulation equivalence. A lattice of distinguishing axioms is shown to characterise the distinct semantics of divergence, and to precisely reflect the 'horizontal' lattice structure of the spectrum. This lattice forms the basis of a complete axiomatisation for the bisimulation spectrum spanning from WB to BB.

The paper is organised as follows. Section 2 introduces the necessary notation and definitions, while Section 3 recalls the weak bisimulation equivalences and Section 4 introduces the axiom systems. Section 5 is devoted to soundness of the axioms and sets the ground for the completeness proof. Section 6 presents the main step of the proof, only focusing on closed expressions, while Section 7 covers open expressions. Section 8 concludes the paper.

A short version of this paper has appeared in [11].

2 Preliminaries

We assume a set of variables $\mathbb{V}$ and a set of actions $\mathbb{A}$, containing the silent action τ. We consider the set of open finite state agents with silent moves and explicit divergence, given as the set $\mathbb{E}$ of expressions generated by the grammar

$$\mathcal{E} \quad ::= \quad 0 \quad | \quad a.\mathcal{E} \quad | \quad \mathcal{E}+\mathcal{E} \quad | \quad recX.\mathcal{E} \quad | \quad X \quad | \quad \Delta(\mathcal{E})$$

where $X \in \mathbb{V}$ and $a \in \mathbb{A}$. The expression $\Delta(E)$ has the same behaviour as E, except that divergence is explicitly added to the root of E.

A variable X is said to occur free in E if it occurs in E outside the scope of any binding $recX$-operator. $\mathbb{V}(E)$ denote the set of all free variables in E. We define $\mathbb{P} = \{E \in \mathbb{E} \mid \mathbb{V}(E) = \emptyset\}$. We use $E, F, G, H, \ldots$ (resp. $P, Q, R, \ldots$) to range over expressions from $\mathbb{E}$ (resp. $\mathbb{P}$). The syntactic equality on $\mathbb{E}$ up to renaming of bounded variables is denoted by $\equiv$. If $\vec{F} = (F_1, \ldots, F_n)$ is a sequence of expressions, $\vec{X} = (X_1, \ldots, X_n)$ is a sequence of variables, and $E \in \mathbb{E}$, then $E\{\vec{F}/\vec{X}\}$ denotes the expression that results from E by simultaneously replacing all free occurrences of X_i in E by F_i ($1 \leq i \leq n$).[5] The variable X is *guarded* (resp. *weakly guarded*) in E, if every free occurrence of X in E lies within a subexpression of the form $a.F$ with $a \in \mathbb{A} \setminus \{\tau\}$ (resp. $a \in \mathbb{A}$),

[5] If necessary we have to rename bounded variables in E in order to avoid that free variables of the F_i get bounded.

otherwise X is called unguarded (resp. *totally unguarded*) in E. Thus for instance X is unguarded but weakly guarded in $\tau.X$. Furthermore X is totally unguarded in $\Delta(X)$. The expression E is guarded if for every subexpression $recY.F$ of E the variable Y is guarded in F.

The semantics of $\mathbb{E}$ is given as the least transition relations on $\mathbb{E}$ satisfying the following rules, where $a \in \mathbb{A}$:

$$\frac{}{a.E \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{F+E \xrightarrow{a} E'}$$

$$\frac{E\{recX.E/X\} \xrightarrow{a} E'}{recX.E \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{\Delta(E) \xrightarrow{a} E'} \qquad \frac{}{\Delta(E) \xrightarrow{\tau} \Delta(E)}$$

The rules are standard, except that, as noted before, $\Delta(E)$ can diverge, in addition to exhibiting all the behaviour of E. It should be noted that $\Delta(E)$ and $recX.(\tau.X + E)$ generate the same transition system when $X \notin \mathbb{V}(E)$, indicating that the Δ-operator is not essential. We however prefer to keep the Δ-operator explicitly, since it will simplify notations considerably. Operators similar to Δ have already appeared in the literature (see e.g. [4,15,10]), all of them as constants (instead of unary operations) and usually bearing also the meaning of "undefinedness", not present in our work. From all of them, the semantics of Δ is closest to that of [4].

A useful observation is that $E \xrightarrow{a} F$ and $Y \in \mathbb{V}(F)$ implies $Y \in \mathbb{V}(E)$. This can be shown by induction on the height of the derivation tree for the transition $E \xrightarrow{a} F$.

Since we are working in the context of silent steps, we define a few standard abbreviations. The relation $\Longrightarrow$ denotes the transitive reflexive closure of the relation $\xrightarrow{\tau}$. We write $E \stackrel{a}{\Longrightarrow} F$ if $E \Longrightarrow E' \xrightarrow{a} F' \Longrightarrow F$ for some $E', F' \in \mathbb{E}$. We write $E \stackrel{\widehat{a}}{\Longrightarrow} F$ if ($a \neq \tau$ and $E \stackrel{a}{\Longrightarrow} F$) or ($a = \tau$ and $E \Longrightarrow F$). We write $E \xrightarrow{\tau}$ if $E \xrightarrow{\tau} F$ for some $F \in \mathbb{E}$. Finally, we write $E \longrightarrow$ if $E \xrightarrow{a} F$ for some $a \in \mathbb{A}$ and $F \in \mathbb{E}$.

Next we define a few properties that an expression may or may not have. We will use a suffix notation for these properties, i.e., if $E \in \mathbb{E}$ and ϕ is a property of expressions, then we write $E\phi$ instead of $\phi(E)$. We define the properties σ, $\perp$, $\Uparrow$, and $\Uparrow\!\!\!\perp$, respectively, as follows:

- $E\,\sigma$ (E is stable) if and only if $E \xrightarrow{\tau}$ does not hold.
- $E\perp$ if and only if $E \longrightarrow$ does not hold.
- $E \Uparrow$ (E is divergent) if there are expressions E_i for $i \in \mathbb{N}$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} E_2 \cdots$.
- $E\Uparrow\!\!\!\perp$ if ($E \Uparrow$ or there exists F with $E \Longrightarrow F\perp$).

Note also that for $\phi \in \{\Uparrow, \Uparrow\!\!\!\!\!-\}$, $E \Longrightarrow F\phi$ implies already $E\phi$.

In the remainder of this section we state a few useful properties of the above notions. We will use these properties quite extensively (later sometimes also without explicit reference) in the further discussion. In all lemmas let $E, F, G, H \in \mathbb{E}$ and $a \in \mathbb{A}$.

The following two statements can be also found in [8, Lemma 4] (without the Δ-operator, which does not complicate the situation). Lemma 1 can by shown by induction on the height of the derivation tree for $G \xrightarrow{a} H$, whereas Lemma 2 can be proved by induction on the structure of the expression G.

Lemma 1 *If $G \xrightarrow{a} H$, then $G\{E/X\} \xrightarrow{a} H\{E/X\}$.*

Lemma 2 *If $E \xrightarrow{a} F$ and the variable X is totally unguarded in G then $G\{E/X\} \xrightarrow{a} F$.*

Lemma 3 *Let $G\{E/X\} \xrightarrow{a} F$ be derivable by a derivation tree of height n. Then one of the following two cases holds:*

(1) X is totally unguarded in G and $E \xrightarrow{a} F$, which can be derived by a derivation tree of height at most n.
(2) $G \xrightarrow{a} H$ and $F \equiv H\{E/X\}$. Furthermore if X is guarded in G and $a = \tau$ then X is also guarded in H.

Proof. Induction on n: The case that G has the form $Y \in \mathbb{V}$, $a.G'$, or $G_1 + G_2$ is clear. If $G \equiv \Delta(G')$ and $\Delta(G'\{E/X\}) \xrightarrow{a} F$ then either $F \equiv \Delta(G'\{E/X\})$ and $a = \tau$, and we obtain the second case, or $G'\{E/X\} \xrightarrow{a} F$, which can be derived by a derivation tree of height at most $n - 1$. By induction, either

- X is totally unguarded in G', i.e., totally unguarded in G, and $E \xrightarrow{a} F$ by a derivation tree of height at most $n - 1$ or
- $G' \xrightarrow{a} H$ (and thus $G \xrightarrow{a} H$), $F \equiv H\{E/X\}$, and if X is guarded in G', i.e., guarded in G, and $a = \tau$ then X is also guarded in H.

Finally assume that $G \equiv recY.G'$. The case $X \equiv Y$ is clear. Thus assume that $X \not\equiv Y$. By renaming the bounded variable Y if necessary, we can assume that $Y \notin \mathbb{V}(E)$. Thus $recY.G'\{E/X\} \xrightarrow{a} F$ implies

$$(G'\{recY.G'/Y\})\{E/X\} \equiv (G'\{E/X\})\{recY.G'\{E/X\}/Y\} \xrightarrow{a} F$$

by a derivation tree of height at most $n-1$. The induction hypothesis implies that either

(1) X is totally unguarded in $G'\{recY.G'/Y\}$ (i.e., totally unguarded in G) and $E \xrightarrow{a} F$, which can be derived by a derivation tree of height at most $n - 1$, or

(2) $G'\{recY.G'/Y\} \xrightarrow{a} H$ (i.e., $G \xrightarrow{a} H$) and $F \equiv H\{E/X\}$. Furthermore if X is guarded in $G'\{recY.G'/Y\}$ (i.e., guarded in G) and $a = \tau$ then X is also guarded in H. $\square$

The preceding three lemmas easily imply the next two lemmas:

Lemma 4 *Let $\phi \in \{\sigma, \bot\}$. Then $G\{E/X\}\phi$ if and only if $G\phi$ and (X is weakly guarded in G or $E\phi$).*

Lemma 5 *$G\{E/X\} \Uparrow$ if and only if $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $E \Uparrow$).*

For $\phi = \bot$, the statement of Lemma 4 can be easily sharpened to: $G\{E/X\}\bot$ if and only if $G\bot$ and ($X \notin \mathbb{V}(G)$ or $E\bot$).

Lemma 6 *$recX.G \xrightarrow{a} E$ if and only if there exists $H \in \mathbb{E}$ with $G \xrightarrow{a} H$ and $E \equiv H\{recX.G/X\}$.*

Proof. First assume that $G \xrightarrow{a} H$ and $E \equiv H\{recX.G/X\}$. Then Lemma 1 implies $G\{recX.G/X\} \xrightarrow{a} E$, i.e, $recX.G \xrightarrow{a} E$. For the other direction assume that $recX.G \xrightarrow{a} E$ can be derived by a derivation tree of height n but there does not exist a derivation tree for this transition of height $< n$. Then $G\{recX.G/X\} \xrightarrow{a} E$ can be derived by a derivation tree of height $n-1$. By Lemma 3 either $G \xrightarrow{a} H$ and $E \equiv H\{recX.G/X\}$ for some H, or X is totally unguarded in G and $recX.G \xrightarrow{a} E$ can be derived by a derivation tree of height $\leq n-1$. But the second alternative contradicts the choice of n. $\square$

Lemma 7 *$recX.G \Uparrow$ if and only if $G \Uparrow$ or ($G \overset{\tau}{\Longrightarrow} H$ and X is totally unguarded in H).*

Proof. If $G \Uparrow$ then, by Lemma 5, also $G\{recX.G/X\} \Uparrow$, i.e., $recX.G \Uparrow$. If $G \overset{\tau}{\Longrightarrow} H$ and X is totally unguarded in H then $G\{recX.G/X\} \overset{\tau}{\Longrightarrow} H\{recX.G/X\}$, i.e, $recX.G \xrightarrow{\tau} E \Longrightarrow H\{recX.G/X\}$ for some E. Since X is totally unguarded in H, Lemma 2 implies

$$recX.G \xrightarrow{\tau} E \Longrightarrow H\{recX.G/X\} \xrightarrow{\tau} E,$$

i.e., $recX.G \Uparrow$. Finally assume that $recX.G \Uparrow$. Then $recX.G \xrightarrow{\tau} E$ and $E \Uparrow$. By Lemma 6 we have $G \xrightarrow{\tau} G'$ and $E \equiv G'\{recX.G/X\} \Uparrow$ for some G'. By Lemma 5 either $G' \Uparrow$ (and thus $G \Uparrow$), or $G' \Longrightarrow H$ (and thus $G \overset{\tau}{\Longrightarrow} H$) for some H such that X is totally unguarded in H. $\square$

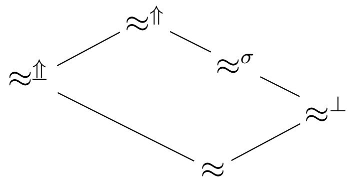


Fig. 2. Inclusions between the relations $\approx^{\phi}$

3 The bisimulations

A symmetric relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ is a *weak bisimulation* (WB) if for all $P, Q, P' \in \mathbb{P}$ and $a \in \mathbb{A}$ the following holds:

$$\text{if } (P,Q) \in \mathcal{R} \text{ and } P \xrightarrow{a} P', \text{ then } \exists Q' : Q \overset{\widehat{a}}{\Longrightarrow} Q' \text{ and } (P',Q') \in \mathcal{R}$$

Since we restrict to symmetric relations, we do not have to mention the symmetric simulation condition. We say that a symmetric relation $\mathcal{R}$ *preserves a property* ϕ if for all $P, Q, P' \in \mathbb{P}$ the following holds:

$$\text{if } (P,Q) \in \mathcal{R},\ P \Longrightarrow P', \text{ and } P'\phi, \text{ then } \exists Q' \in \mathbb{P} : Q \Longrightarrow Q' \text{ and } Q'\phi.$$

It follows immediately that a WB $\mathcal{R}$ preserves a property ϕ, if and only if the following holds:

$$\text{if } (P,Q) \in \mathcal{R} \text{ and } P\phi, \text{ then } \exists Q' \in \mathbb{P} : Q \Longrightarrow Q' \text{ and } Q'\phi.$$

We will always make use of this simple observation. Finally, a WB $\mathcal{R}$ is a WB$^{\phi}$ if it preserves ϕ. A WB$^{\sigma}$ (resp. WB$^{\perp}$, WB$^{\Uparrow}$, WB$^{\underline{\Uparrow}}$) is also called a *stable* (resp. *completed*, *divergent*, *divergent stable*) WB. In order to simplify notations, we will use WB$^{\varepsilon}$ as a synonym for WB in the following. In the sequel, ϕ will range over $\{\sigma, \perp, \Uparrow, \underline{\Uparrow}, \varepsilon\}$, unless stated otherwise. The relation $\approx^{\phi} \subseteq \mathbb{P} \times \mathbb{P}$ is defined as the union of all WB$^{\phi}$; it is easily seen to be itself a WB$^{\phi}$ as well as an equivalence relation.

Theorem 1 [7] *The equivalences $\approx^{\phi}$ are ordered by inclusion according to the lattice in Figure 2. The lower relation contains the upper if and only if both are connected by a line.*

The relation $\approx^{\phi}$ is not a congruence with respect to the $+$-operator (which is a well known deficiency), and for $\phi \in \{\Uparrow, \underline{\Uparrow}, \sigma, \perp\}$ also not a congruence with respect to the Δ-operator. For instance $\tau.0 \approx^{\Uparrow} 0$, but $\Delta(\tau.0) \not\approx^{\Uparrow} \Delta(0)$. To obtain the coarsest congruence on $\mathbb{P}$ that is contained in $\approx^{\phi}$, we define each $\simeq^{\phi}$ to be the relation that contains exactly the pairs $(P,Q) \in \mathbb{P} \times \mathbb{P}$ that satisfy the following *root conditions*:

- if $P \xrightarrow{a} P'$, then $Q \overset{a}{\Longrightarrow} Q'$ and $P' \approx^{\phi} Q'$ for some Q'
- if $Q \xrightarrow{a} Q'$, then $P \overset{a}{\Longrightarrow} P'$ and $P' \approx^{\phi} Q'$ for some P'

So far we defined the relations $\approx^\phi$ and $\simeq^\phi$ only on $\mathbb{P}$. We lift these relations from $\mathbb{P}$ to $\mathbb{E}$ as usual: Let $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ and $E, F \in \mathbb{E}$. Let $\vec{X} = (X_1, \ldots, X_n)$ be a sequence of variables that contains all variables from $\mathbb{V}(E) \cup \mathbb{V}(F)$. Then $(E, F) \in \mathcal{R}$ if for all $\vec{P} = (P_1, \ldots, P_n)$ with $P_i \in \mathbb{P}$ we have $(E\{\vec{P}/\vec{X}\}, F\{\vec{P}/\vec{X}\}) \in \mathcal{R}$.

Theorem 2 *The relation $\simeq^\phi$ is the coarsest congruence contained in $\approx^\phi$ with respect to the operators of $\mathbb{E}$. Furthermore the inclusions listed in Theorem 1 carry over from $\approx^\phi$ to $\simeq^\phi$.*

For the proof of this theorem we need the following lemma:

Lemma 8 *Let $P, Q \in \mathbb{P}$. If $P + R \approx^\phi Q + R$ for all $R \in \mathbb{P}$, then $P \simeq^\phi Q$.*

Proof. Assume that $P + R \approx^\phi Q + R$ for all $R \in \mathbb{P}$ and $P \not\simeq^\phi Q$. By the definition of $\simeq^\phi$, there is some $a \in \mathbb{A}$ such that w.l.o.g. $P \xrightarrow{a} P'$ but whenever $Q \stackrel{a}{\Longrightarrow} Q'$, then $P' \not\approx^\phi Q'$. Choose $R \equiv b.0$ where $b \in \mathbb{A}$ does not occur in P nor in Q [6]. Clearly, $P + R \xrightarrow{a} P'$, and since $P + R \approx^\phi Q + R$ there is some Q' such that $Q + R \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $P' \approx^\phi Q'$. However, if $a = \tau$ and $Q' \equiv Q + R$, then $P' \not\approx^\phi Q'$ since Q' may do a b-transition whereas P' does not have this possibility. Otherwise $Q \stackrel{a}{\Longrightarrow} Q'$ (since $R \stackrel{a}{\Longrightarrow} Q'$ is impossible because $b \neq a$), so again $P' \not\approx^\phi Q'$ and we conclude in either case by contradiction. $\Box$

Proof of Theorem 2. Due to the way we have lifted relations from $\mathbb{P}$ to $\mathbb{E}$, it suffices to prove the theorem for expressions from $\mathbb{P}$. That the inclusions from Figure 2 also hold for the relations $\simeq^\phi$ is easy to check. The inclusion $\simeq^\phi \subseteq \approx^\phi$ can be verified by proving that the relation $\simeq^\phi \cup \approx^\phi$ is a WB$^\phi$, which is easy to see by inspecting the root conditions. Also preservation of the property ϕ is straight-forward, let us only consider the case $\phi = \Uparrow$. Thus assume that $P{\Uparrow}$ and $P \simeq^\phi Q$ (the case $P \approx^\phi Q$ is clear). Then either $P\bot$ or $P \xrightarrow{\tau} P'{\Uparrow}$ for some P'. In the first case, $P \simeq^{\Uparrow} Q$ implies $Q\bot$. In the second case, $P \simeq^{\Uparrow} Q$ implies $Q \stackrel{\tau}{\Longrightarrow} Q'$ and $P' \approx^{\Uparrow} Q'$ for some Q'. Since $P'{\Uparrow}$ we obtain $Q'{\Uparrow}$ and thus $Q{\Uparrow}$.

It remains to show that $\simeq^\phi$ is the coarsest congruence with respect to the operators of $\mathbb{E}$. Congruence with respect to '+' and action prefix is clear (for the congruence with respect to action prefix we have to use the inclusion $\simeq^\phi \subseteq \approx^\phi$). Congruence with respect to Δ can be seen as follows. Assume that $P \simeq^\phi Q$. First note that $\{(\Delta(P), \Delta(Q)), (\Delta(Q), \Delta(P))\} \cup \approx^\phi$ is a WB$^\phi$. Thus we have $\Delta(P) \approx^\phi \Delta(Q)$. From this we deduce easily that the pair $(\Delta(P), \Delta(Q))$ satisfies also the root conditions, i.e., $\Delta(P) \simeq^\phi \Delta(Q)$. The

[6] For simplicity we assume that $\mathbb{A}$ is infinite and make use of the *fresh atom principle* when choosing a fresh $b \in \mathbb{A}$. We will do so in several other occasions throughout the paper. This is nevertheless not necessary. See, e.g. [8].

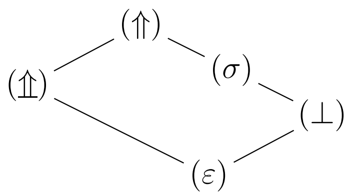


Fig. 3. Implications between the distinguishing axioms

congruence proof with respect to the recursion operator (the only hard part) is shifted to Appendix A.

It remains to argue that $\simeq^\phi$ is in fact the coarsest congruence contained in $\approx^\phi$. Assume that $\mathcal{R} \subseteq \approx^\phi$ is a congruence with respect to the operators of $\mathbb{E}$. Let $(P, Q) \in \mathcal{R}$. Thus for all $R \in \mathbb{P}$ we have $(P + R, Q + R) \in \mathcal{R}$, i.e., $P + R \approx^\phi Q + R$. By Lemma 8 we have $P \simeq^\phi Q$. Thus $\mathcal{R} \subseteq \simeq^\phi$. $\square$

4 Axioms

This section introduces a lattice of axioms characterising the congruences from the previous section. For $\phi \in \{\sigma, \perp, \Uparrow, \varepsilon\}$, the axioms for $\simeq^\phi$ are those in Table 1 plus axiom (ϕ) from Table 2. The axioms for $\simeq^{⇮}$ are those in Table 1 plus the axioms $(\Uparrow)$ and $(⇮)$ from Table 2. We write $E =^\phi F$ if $E = F$ can be derived by application of the axioms for $\simeq^\phi$.

The axioms $(S1)-(S4)$, $(\tau1)-(\tau3)$, and $(rec1)-(rec4)$ are standard [13]. The axiom $(rec5)$ makes divergence explicit if introduced due to silent recursion; it defines the nature of the Δ-operator. Axiom $(rec6)$ states the redundancy of recursion on an unguarded variable in the context of divergence.

We discuss the distinguishing axioms in reverse order relative to the listing in Table 2. Axiom $(⇮)$ characterises the property of $\mathrm{WB}^{⇮}$ that divergence cannot be distinguished when terminating. Axiom (ε) represents Milner's 'fair' setting, where divergence is never distinguished. The remaining three axioms state that divergence cannot be distinguished if the expression can still perform an action to escape the divergence $(\perp)$, that it cannot be distinguished if the expression can perform a silent step to escape divergence (σ), and that two consecutive divergences cannot be properly distinguished $(\Uparrow)$. It is a simple exercise to verify the implications between the distinguishing axioms as summarized in the lattice in Figure 3. It nicely reflects the inclusions between the respective congruences. The upper axioms turn into derivable laws given the lower ones (plus the core axioms from Table 1) as axioms.

Axiom (ε) is the same as axiom **R4** in [14] (where $recX.(\tau.X + E)$ should be read as $\Delta(E)$), and has appeared in [4] in a slightly different form. A version of (σ) has appeared in [4] in the context of failure semantics, and in [10] in the context of weak Markovian bisimulation, only that Δ is a constant rather than a unary operation. Axiom ($\Uparrow$) has been suggested in [3] (also with Δ as a constant) although no model has been provided. To the best of our knowledge, the other axioms have not appeared in the literature.

$(S1)\ E + F = F + E$	$(\tau 1)\ a.\tau.E = a.E$
$(S2)\ E + (F + G) = (E + F) + G$	$(\tau 2)\ \tau.E + E = \tau.E$
$(S3)\ E + E = E$	$(\tau 3)\ a.(E + \tau.F) = a.(E + \tau.F) + a.F$
$(S4)\ E + 0 = E$	

$(rec1)$ if Y is not free in $recX.E$ then $recX.E = recY.(E\{Y/X\})$

$(rec2)$ $recX.E = E\{recX.E/X\}$

$(rec3)$ if X is guarded in E and $F = E\{F/X\}$ then $F = recX.E$

$(rec4)$ $recX.(X + E) = recX.E$

$(rec5)$ $recX.(\tau.(X + E) + F) = recX.\Delta(E + F)$

$(rec6)$ $recX.(\Delta(X + E) + F) = recX.\Delta(E + F)$

Table 1
Core axioms

$(\Uparrow)$	$\Delta(\Delta(E) + F) = \tau.(\Delta(E) + F)$
(σ)	$\Delta(\tau.E + F) = \tau.(\tau.E + F)$
$(\perp)$	$\Delta(a.E + F) = \tau.(a.E + F)$
(ε)	$\Delta(E) = \tau.E$
$(\Uparrow)$	$\Delta(0) = \tau.0$

Table 2
Distinguishing axioms

5 Soundness

This section is devoted to the soundness of the axioms for $\simeq^{\phi}$:

Theorem 3 *If $E, F \in \mathbb{E}$ and $E =^{\phi} F$ then $E \simeq^{\phi} F$.*

Recall that a strong bisimulation is a symmetric relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for all $(P, Q) \in \mathcal{R}$ the following condition holds (see also [12]):

$$\text{if } P \xrightarrow{a} P', \text{ then } \exists Q' : Q \xrightarrow{a} Q' \text{ and } (P', Q') \in \mathcal{R}$$

We write $P \sim Q$ if there exists some strong bisimulation $\mathcal{R}$ containing the pair (P, Q). The following lemma is easy to see.

Lemma 9 *It holds that $\sim \subseteq \simeq^{\Uparrow}$.*

Proof of Theorem 3. Due to the definition of $\simeq^{\phi}$ for expressions with free variables, it suffices to check the soundness of the axioms only for $\mathbb{P}$. First we check the core axioms from Table 1, which have to be verified for our finest congruence $\simeq^{\Uparrow}$:

- $(S1) - (S4)$, $(rec1)$, $(rec2)$, and $(rec4)$ are sound for $\sim$ [12].
- $(rec3)$: see Theorem 13 from Appendix B. [7]
- $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$: Soundness for $\simeq^{\Uparrow}$ can be shown analogously to the soundness for $\simeq$, see e.g. [13].
- $(rec5)$ and $(rec6)$: see Appendix C.

We continue with the distinguishing axioms from Table 2.

$(\Uparrow)$: We need to prove that $\Delta(\Delta(P) + Q) \simeq^{\Uparrow} \tau.(\Delta(P) + Q)$. The symmetric closure of the relation

$$\{\langle \Delta(\Delta(P) + Q), \tau.(\Delta(P) + Q)\rangle, \langle \Delta(\Delta(P) + Q), \Delta(P) + Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

can be shown to be a $\mathrm{WB}^{\Uparrow}$. Furthermore,

$$\langle \Delta(\Delta(P) + Q), \tau.(\Delta(P) + Q)\rangle$$

satisfies the root condition, thus $\Delta(\Delta(P) + Q) \simeq^{\Uparrow} \tau.(\Delta(P) + Q)$.

$(\mathbin{\rotatebox[origin=c]{90}{$\Rightarrow$}}\!\!\!\!\;\overline{})$: We need to prove that $\Delta(0) \simeq^{\mathbin{⇮}} \tau.0$. It is not difficult to prove that the symmetric closure of $\{\langle \Delta(0), \tau.0\rangle, \langle \Delta(0), 0\rangle\}$ is a $\mathrm{WB}^{⇮}$ and that $\langle \Delta(0), \tau.0\rangle$ satisfies the root condition.

[7] Note that $(rec3)$ is also sound for $\sim$ [12], but since $(rec3)$ has the form of an implication, this does not imply the soundness with respect to $\simeq^{\Uparrow}$.

(σ): We have to prove that $\Delta(\tau.P+Q) \simeq^\sigma \tau.(\tau.P+Q)$. It is not difficult to prove that the symmetric closure of

$$\{\langle \Delta(\tau.P+Q), \tau.(\tau.P+Q)\rangle, \langle \Delta(\tau.P+Q), \tau.P+Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB$^\sigma$ and that $\langle \Delta(\tau.P+Q), \tau.(\tau.P+Q)\rangle$ satisfies the root condition.

($\perp$): We have to prove that $\Delta(a.P+Q) \simeq^\perp \tau.(a.P+Q)$. The symmetric closure of

$$\{\langle \Delta(a.P+Q), \tau.(a.P+Q)\rangle, \langle \Delta(a.P+Q), a.P+Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB$^\perp$ and the pair $\langle \Delta(a.P+Q), \tau.(a.P+Q)\rangle$ satisfies the root condition.

(ε): We have to show that $\Delta(P) \simeq \tau.P$. The symmetric closure of

$$\{\langle \Delta(P), \tau.P\rangle, \langle \Delta(P), P\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB and $\langle \Delta(P), \tau.P\rangle$ satisfies the root condition. $\square$

6 Derived laws

A few laws, derivable with the axioms for $\simeq^\Uparrow$ (and thus for all $\simeq^\phi$) give further insight, and will be useful for the further discussion. Since we have already shown soundness, these laws are also valid for $\simeq^\Uparrow$ instead of $=^\Uparrow$.

Lemma 10 *The following laws can be derived:*

$$\begin{array}{lrcl}
(\Uparrow_1) & \Delta(E) & =^\Uparrow & \Delta(E)+E \\
(\Uparrow_2) & \Delta(E) & =^\Uparrow & \tau.\Delta(E)+E \\
(\Uparrow_3) & \Delta(E) & =^\Uparrow & \tau.\Delta(E) \\
(\mathit{rec}7) & \mathit{rec}X.(\tau.(X+E)+F) & =^\Uparrow & \mathit{rec}X.(\tau.X+E+F)
\end{array}$$

Proof. First we derive ($\Uparrow_2$) as follows, where $X \in \mathbb{V} \setminus \mathbb{V}(E)$:

$$\begin{aligned}
\Delta(E) &=^\Uparrow \mathit{rec}X.\Delta(E) && (\mathit{rec}2)\\
&=^\Uparrow \mathit{rec}X.\Delta(0+E) && (\mathrm{S4})\\
&=^\Uparrow \mathit{rec}X.(\tau.(X+0)+E) && (\mathit{rec}5)\\
&=^\Uparrow \tau.(\mathit{rec}X.(\tau.(X+0)+E)+0)+E && (\mathit{rec}2)\\
&=^\Uparrow \tau.(\mathit{rec}X.\Delta(0+E)+0)+E && (\mathit{rec}5)\\
&=^\Uparrow \tau.(\mathit{rec}X.\Delta(E))+E && (\mathrm{S4})\\
&=^\Uparrow \tau.\Delta(E)+E && (\mathit{rec}2)
\end{aligned}$$

Now $(\Uparrow_3)$ can be deduced as follows:

$$\begin{aligned}
\tau.\Delta(E) &=^{\Uparrow} \tau.\Delta(E) + \Delta(E) && (\tau 2)\\
&=^{\Uparrow} \tau.\Delta(E) + \tau.\Delta(E) + E && (\Uparrow_2)\\
&=^{\Uparrow} \tau.\Delta(E) + E && (\mathrm{S3})\\
&=^{\Uparrow} \Delta(E) && (\Uparrow_2)
\end{aligned}$$

Law $(\Uparrow_1)$ is a straight-forward consequence of $(\Uparrow_2)$ and (S3):

$$\Delta(E) =^{\Uparrow} \tau.\Delta(E) + E =^{\Uparrow} \tau.\Delta(E) + E + E =^{\Uparrow} \Delta(E) + E$$

Finally for $(rec7)$ note that by $(rec5)$ both expressions can be transformed into $recX.\Delta(E+F)$. $\square$

Lemma 11 *If $E \xrightarrow{\tau}$, then there are $G, H \in \mathbb{E}$ with $E =^{\Uparrow} \tau.G + H$.*

Proof. We prove the lemma by induction on the structure of E. The case $E \equiv \tau.F$ is trivial. If $E \equiv \Delta(F)$, then with the derived law $(\Uparrow_2)$ we obtain $E \equiv \Delta(F) =^{\Uparrow} \tau.\Delta(F) + F$. If $E \equiv E_1 + E_2$, then w.l.o.g. we may assume that $E_1 \xrightarrow{\tau}$, which allows us to conclude inductively. Finally, if $E \equiv recX.F$, then Lemma 6 implies that $F \xrightarrow{\tau}$. Thus, by induction, we have $F =^{\Uparrow} \tau.G + H$. Hence, using axiom $(rec2)$ we obtain

$$E =^{\Uparrow} recX.(\tau.G{+}H) =^{\Uparrow} \tau.G\{recX.(\tau.G{+}H)/X\}{+}H\{recX.(\tau.G{+}H)/X\} \quad \square$$

Analogously to the previous lemma, we can prove the following lemma:

Lemma 12 *If $E \longrightarrow$, then there exist $G, H \in \mathbb{E}$ and $a \in \mathbb{A}$ with $E =^{\Uparrow} a.G + H$.*

Lemma 13 *If $E\bot$, then $E =^{\Uparrow} \sum_{Y \in \mathbb{V}(E)} Y$.*

Proof. The case $E \in \{0\} \cup \mathbb{V}$ is clear. The case $E \equiv E_1 + E_2$ can be dealt with inductively. Finally, if $E \equiv recX.F$, then Lemma 6 implies that $F\bot$. Thus, by induction, $F =^{\Uparrow} \sum_{Y \in \mathbb{V}(F)} Y$. If $X \notin \mathbb{V}(F)$, i.e., $\mathbb{V}(E) = \mathbb{V}(F)$, then $E =^{\Uparrow} recX.(\sum_{Y \in \mathbb{V}(F)} Y) =^{\Uparrow} \sum_{Y \in \mathbb{V}(E)} Y$ with axiom $(rec2)$. Otherwise, we have $\mathbb{V}(F) = \mathbb{V}(E) \cup \{X\}$. Then

$$E =^{\Uparrow} recX.(X + \sum_{Y \in \mathbb{V}(E)} Y) =^{\Uparrow} recX.(\sum_{Y \in \mathbb{V}(E)} Y) =^{\Uparrow} \sum_{Y \in \mathbb{V}(E)} Y$$

using axiom $(rec4)$ and $(rec2)$. $\square$

The next lemma only holds for $=^{\underline{\Uparrow}}$.

Lemma 14 *If $E \stackrel{\tau}{\Longrightarrow} P\bot$ for some $P \in \mathbb{P}$, then there are $G, H \in \mathbb{E}$ with $E =^{\Uparrow} \Delta(G) + H$.*

Proof. First note that $P\bot$ and $\mathbb{V}(P) = \emptyset$ imply $P =^{\Uparrow} 0$ by Lemma 13. The case $E \equiv \Delta(F)$ is clear. If $E \equiv \tau.F$, then $E \stackrel{\tau}{\longrightarrow} F \Longrightarrow P$. In case $F \equiv P$, we obtain $E =^{\Uparrow} \tau.0 =^{\Uparrow} \Delta(0)$. On the other hand, if $F \stackrel{\tau}{\Longrightarrow} P$, then by induction we get $F =^{\Uparrow} \Delta(G)+H$ for $G, H \in \mathbb{E}$. Hence, $E =^{\Uparrow} \tau.(\Delta(G)+H) =^{\Uparrow} \Delta(\Delta(G)+H)$. If $E \equiv E_1 + E_2$, then w.l.o.g. we may assume that $E_1 \stackrel{\tau}{\Longrightarrow} P$. Inductively we get $E_1 =^{\Uparrow} \Delta(G) + H$ for $G, H \in \mathbb{E}$. Thus, $E =^{\Uparrow} \Delta(G) + H + E_2$. Finally, assume that $E \equiv \mathit{rec}X.F$. Thus, $\mathit{rec}X.F \stackrel{\tau}{\Longrightarrow} P\bot$. Using Lemma 3 and 6, we obtain $F \stackrel{\tau}{\Longrightarrow} F'$ for some expression F' with $F'\{E/X\} \equiv P\bot$. Thus $F'\bot$ and $\mathbb{V}(F') = \emptyset$, i.e., $F' \in \mathbb{P}$ (note that $E \longrightarrow$). By induction we obtain $F =^{\Uparrow} \Delta(G) + H$ for $G, H \in \mathbb{E}$. Thus, $E =^{\Uparrow} \mathit{rec}X.(\Delta(G) + H) =^{\Uparrow} \Delta(G\{\mathit{rec}X.(\Delta(G) + H)/X\}) + H\{\mathit{rec}X.(\Delta(G) + H)/X\}$. $\square$

7 Completeness

In order to show completeness, i.e., that $E \simeq^{\phi} F$ implies $E =^{\phi} F$, we proceed along the lines of [13], except for the treatment of expressions from $\mathbb{E} \setminus \mathbb{P}$. We will work as much as possible in the setting of WB$^{\Uparrow}$, the finest setting.

We do not consider $\phi = \varepsilon$ in the sequel because by using axiom (ε), for every $E \in \mathbb{E}$ we find an E' such that E' does not contain the Δ-operator and $E =^{\varepsilon} E'$. This allows us to apply Milner's result [13] that in the absence of the Δ-operator the axioms from Table 1 together with ($\mathit{rec}7$) and Milner's law $\mathit{rec}X(\tau.X + E) = \mathit{rec}X(\tau.E)$ are complete for $\simeq^{\varepsilon}$. The latter law follows immediately from ($\mathit{rec}5$) and (ε).

7.1 A road map through the completeness proof

As already mentioned, our completeness proof proceeds along the lines of [13]. A first step is achieved in Section 7.2. We show that every expression can be transformed into a guarded expression using the axioms for $\simeq^{\Uparrow}$ (Theorem 4). This allows to concentrate on guarded expressions in the rest of the proof. The presence of the Δ-operator makes the proof of Theorem 4 slightly more complicated than the corresponding proof from [13].

Section 7.3 introduces the main technical tool for the completeness proof: equation systems. These are basically recursive definitions of formal process variables, which allow to eliminate the *rec*-operator. Analogously to process expressions we define the notion of a guarded equations system. To simplify the further reasoning, we have to use a very restricted form of equation systems that we call standard equation systems. We also introduce a technical condition on standard equation systems called saturatedness. Theorem 5 states that for every guarded expression E we can find a guarded and saturated standard equations system $\mathcal{E}$ such that E provably satisfies $\mathcal{E}$. The latter means that if we substitute the formal process variables of $\mathcal{E}$ with concrete expressions (where E is one of these expressions), then for every equation the left and right-hand side of the equation can be transformed into each other using the axioms for $\simeq^{\phi}$. Section 7.3 finishes with a further result, stating that if two expressions E and F both provably satisfy the same guarded equation system then $E =^{\phi} F$ (Theorem 6). For this result we can reuse the proof of the corresponding result from [13].

To finish the completeness proof, we need one further result, which is the main technical difficulty: In Section 7.4 we show that if P and Q are guarded expressions without free variables such that $P \simeq^{\phi} Q$ and both P and Q provably satisfy a guarded and saturated standard equation system, then we can find a single guarded equation system $\mathcal{E}$ such that both P and Q provably satisfy $\mathcal{E}$ (Theorem 7). For the proof of this result it is crucial that we restrict to expressions without free variables, which differs from the corresponding proof in [13]. Completeness for expressions without free variables follows easily from Theorem 4–7, see the proof of Theorem 8.

In Section 8 we extend completeness to expressions with free variables. Here, we will deviate from Milner's strategy. For $\sigma, \Uparrow$, and $\Updownarrow$, we are able to deduce the completeness for arbitrary expressions from the completeness for expressions without free variables, by analyzing the corresponding axioms. For $\phi = \perp$ we are only able to achieve completeness by introducing an additional axiom.

7.2 Reducing to guarded expressions

As in [13] the first step consists in transforming every expression into a guarded one:

Theorem 4 *Let $E \in \mathbb{E}$. There exists a guarded F with $E =^{\Uparrow} F$ (and thus $\mathbb{V}(E) = \mathbb{V}(F)$).*

For the proof we need the following simple lemma.

Lemma 15 *If X is unguarded in $E \in \mathbb{E}$ then $E =^{\Uparrow} E + X$.*

Proof. We prove the lemma by induction on the structure of E. Since X is unguarded in E we only have to consider the following cases.

$E \equiv X$: By axiom (S3) we have $X =^{\Uparrow} X + X$.

$E \equiv \tau.F$: We have

$$
\begin{aligned}
E \equiv \tau.F &=^{\Uparrow} \tau.F + F && (\tau 2)\\
&=^{\Uparrow} \tau.F + F + X && \text{(induction hypothesis)}\\
&=^{\Uparrow} E + X && (\tau 2)
\end{aligned}
$$

$E \equiv \Delta(F)$: With the derived law ($\Uparrow_1$) from Lemma 10, we can conclude analogously to case 2.

$E \equiv E_1 + E_2$: W.l.o.g. assume that X is unguarded in E_2. The induction hypothesis implies $E_2 =^{\Uparrow} E_2 + X$. Thus $E \equiv E_1 + E_2 =^{\Uparrow} E_1 + E_2 + X \equiv E + X$.

$E \equiv recY.F$: Since X must be free in E we have $X \not\equiv Y$. The induction hypothesis implies $F =^{\Uparrow} F + X$. Thus $F\{recY.F/Y\} =^{\Uparrow} F\{recY.F/Y\} + X$. Axiom ($rec2$) implies $recY.F =^{\Uparrow} recY.F + X$. □

Proof of Theorem 4. The proof follows [13]. We prove the theorem by an induction on the structure of the expression E. Only the case $E \equiv recX.E'$ is interesting. For this case we prove the following stronger statement (†).

Let $E \in \mathbb{E}$. Then there exists a guarded F such that
- X is guarded in F,
- there does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F,[8] and
- $recX.E =^{\Uparrow} recX.F$.

We prove (†) by an induction on the nesting depth $d(E)$ of recursions in E. We have for instance $d(recX.(a.recY.(a.X + b.Y) + a.(recX.(recX.(\tau))))) = 3$. First we consider the following case (‡):

There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(E)$ which lies within a subexpression $recZ.G$ of E.

This case also covers the induction base $d(E) = 0$. So assume that E satisfies (‡). It remains to remove all unguarded occurrences of X in E. Since E satisfies (‡) we know that no unguarded occurrence of X in E lies within a recursion. If X is guarded in E we are ready. So assume that X occurs unguarded in E. We now list several reduction steps which when iteratively applied to E terminate with an expression F that satisfies (†). During this reduction process we either eliminate an unguarded occurrence of X or we reduce the number of τ-guards

[8] A specific free occurrence of Y in F is called unguarded if this occurrence does not lie within a subexpression $a.F'$ with $a \neq \tau$.

and Δ-operators that proceed an unguarded occurrence of X. Since, as already remarked, no unguarded occurrence of X in E lies within a recursion, every unguarded occurrence of X in E can only lie within the scope of τ-prefixes, $+$, and Δ-operators. Thus, one of the following four cases must apply.

Case 1. $E \equiv \tau.(X+E')+F'$ (if E' (F') is in fact empty, then we may add a 0-expression for E' (F') in order to obtain the desired form): With the derivable law ($rec7$), we get

$$recX.E \equiv recX(\tau.(X+E')+F') =^{\Uparrow} recX.(\tau.X+E'+F').$$

We continue with the expression $\tau.X+E'+F'$.

Case 2. $E \equiv \tau.E'+F'$, where X is unguarded in E', but X is weakly guarded in E': Lemma 15 implies $E' =^{\Uparrow} X+E'$. Thus $E =^{\Uparrow} \tau.(X+E')+F'$. By case 1 we can continue with the expression $\tau.X+E'+F'$.

Case 3. $E \equiv \Delta(X+E')+F'$: With ($rec5$), ($rec6$), and ($rec7$) we get

$$\begin{aligned} recX.(\Delta(X+E')+F') &=^{\Uparrow} recX.(\Delta(E'+F')) \\ &=^{\Uparrow} recX.(\tau.(X+E')+F') \\ &=^{\Uparrow} recX.(\tau.X+E'+F'). \end{aligned}$$

We continue with the expression $\tau.X+E'+F'$.

Case 4. $E \equiv \Delta(E')+F'$, where X is unguarded in E', but X is weakly guarded in E': Again by Lemma 15 we have $E' =^{\Uparrow} X+E'$. Thus $E =^{\Uparrow} \Delta(X+E')+F'$. An application of case 3 gives the expression $\tau.X+E'+F'$.

By iterating these four reduction steps we finally arrive at an expression, where all unguarded occurrences of X in E occur in the form $E \equiv X+\cdots$ or $E \equiv \tau.X+\ldots$. Furthermore by axiom (S3) and ($\tau 2$) we may assume that there exists at most one occurrence of this form. Thus it remains to consider the following two cases:

Case 5. $E \equiv X+E'$: By axiom ($rec4$) we have

$$recX.E \equiv recX.(X+E') =^{\Uparrow} recX.E'.$$

Case 6. $E \equiv \tau.X+E'$: By axiom ($rec5$) we have

$$recX.E \equiv recX.(\tau.X+E') =^{\Uparrow} recX.(\Delta(E')).$$

Note that X is guarded in $\Delta(E')$ if X is guarded in E'. This concludes the consideration of case (‡).

It remains to consider the cases that are not covered by (‡). For this let us choose a subexpression $recX'.E'$ of E such that this subexpression does not lie within another recursion, thus $recX'.E'$ is an outermost recursion. Since $d(E') < d(E)$, the induction hypothesis implies that there exists an expression F with the following properties:

- X' is guarded in F.
- There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F.
- $recX'.E' =^{\Uparrow} recX'.F$

It follows that in the expression $F\{recX'.F/X'\}$ there does not exist an unguarded occurrence of any variable which lies within a recursion. Axiom (rec2) allows us to replace $recX'.E'$ within E by $F\{recX'.F/X'\}$. If we perform this step for every outermost recursion of E, we obtain an expression that satisfies (‡). This concludes the proof. □

Before we proceed with proving completeness, let us first derive a useful consequence of Theorem 4.

Lemma 16 *If $E \Uparrow$, then there are $G, H \in \mathbb{E}$ with $E =^{\Uparrow} \Delta(G) + H$.*

Proof. We prove the lemma by induction on the structure of E. By Theorem 4 we may assume that E is guarded . The case $E \equiv \Delta(F)$ is trivial. If $E \equiv \tau.F$ then we must have $F \Uparrow$. Thus, inductively, we obtain $F =^{\Uparrow} \Delta(G) + H$ for expressions G and H. Hence, $E =^{\Uparrow} \tau.(\Delta(G) + H) =^{\Uparrow} \Delta(\Delta(G) + H)$ with axiom ($\Uparrow$). If $E \equiv E_1 + E_2$ then w.l.o.g. we may assume that $E_1 \Uparrow$, which allows to conclude inductively. Finally, if $E \equiv recX.F$, then by Lemma 7, either $F \Uparrow$ or $F \stackrel{\tau}{\Longrightarrow} F'$ for some F' such that X is totally unguarded in F'. But in the latter case, X would be unguarded in F and hence $recX.F$ would be unguarded. Thus $F \Uparrow$. By induction, we obtain $F =^{\Uparrow} \Delta(G) + H$ for expressions G and H. Thus, $F =^{\Uparrow} recX.(\Delta(G) + H) =^{\Uparrow} \Delta(G\{recX.(\Delta(G) + H)/X\}) + H\{recX.(\Delta(G) + H)/X\}$. □

7.3 Equation systems

The basic ingredient of our completeness proof are equations systems. Let $V \subseteq \mathbb{V}$ be a set of variables and let $\vec{X} = (X_1, \ldots, X_n)$ be a sequence of variables, where $X_i \notin V$. An *equation system over the free variables V and the formal variables $\vec{X}$* is a set of equations $\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\}$ such that $E_i \in \mathbb{E}$ and $\mathbb{V}(E_i) \subseteq \{X_1, \ldots, X_n\} \cup V$ for $1 \leq i \leq n$. Let $\vec{F} = (F_1, \ldots, F_n)$ be an ordered sequence of expressions. Then $\vec{F}$ ϕ-provably satisfies the equation system $\mathcal{E}$ if $F_i =^{\phi} E_i\{\vec{F}/\vec{X}\}$ for all $1 \leq i \leq n$. An expression F ϕ-provably satisfies $\mathcal{E}$ if there exists a sequence of expressions $(F_1, \ldots, F_n)$, which ϕ-provably satisfies $\mathcal{E}$ and such that $F \equiv F_1$. We say that $\mathcal{E}$ is *guarded* if there exists a linear order $\prec$ on the variables $\{X_1, \ldots, X_n\}$ such that whenever the variable X_j is unguarded in the expression E_i then $X_j \prec X_i$. The equation system $\mathcal{E}$ is called a *standard equation system (SES)* over the free variables V and the formal variables $(X_1, \ldots, X_n)$ if there exists a partition $\{X_1, \ldots, X_n\} = \Omega^\Sigma \cup \Omega^\Delta$ (with $\Omega^\Sigma \cap \Omega^\Delta = \emptyset$) such that for every $1 \leq i \leq n$,

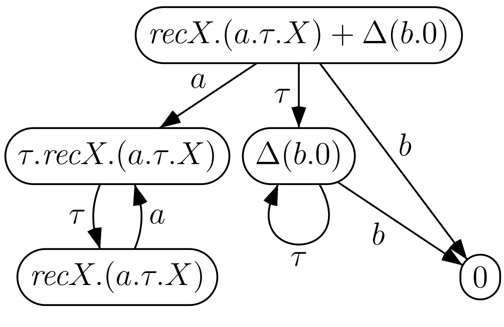
- if $X_i \in \Omega^\Sigma$, then E_i is a sum of expressions $a.X_j$ ($a \in \mathbb{A}$, $1 \leq j \leq n$) and variables $Y \in V$, and
- if $X_i \in \Omega^\Delta$, then $E_i \equiv \Delta(E_i')$, where E_i' is a sum of expressions $a.X_j$ ($a \in \mathbb{A}$, $1 \leq j \leq n$) and variables $Y \in V$.

W.l.o.g. we may assume that $X_1 \in \Omega^\Sigma$. If $X_1 \in \Omega^\Delta$, then we can introduce a new formal variable X_0 and add the equation $X_0 = \tau.X_1$ — this is justified by the derived law ($\Uparrow_3$). We write $X_i \xrightarrow{a}_{\mathcal{E}} X_j$ if $E_i \xrightarrow{a} X_j$. With $\Longrightarrow_{\mathcal{E}}$ we denote the reflexive and transitive closure of $\xrightarrow{\tau}_{\mathcal{E}}$. The relations $\stackrel{a}{\Longrightarrow}_{\mathcal{E}}$ and $\stackrel{\widehat{a}}{\Longrightarrow}_{\mathcal{E}}$ are derived from these relations as usual. If the SES $\mathcal{E}$ is clear from the context, then we will omit the subscript $\mathcal{E}$ in these relations. Note that $\mathcal{E}$ is guarded if and only if the relation $\xrightarrow{\tau}_{\mathcal{E}}$ is acyclic. Finally we say that the SES $\mathcal{E}$ is *saturated* if for all $1 \leq i, j \leq n$ and $Y \in V$ we have:

(1) If $X_i \stackrel{a}{\Longrightarrow} X_j$, then also $X_i \xrightarrow{a} X_j$.
(2) If $X_i \Longrightarrow X_j$ and Y occurs in E_j, then Y occurs already in E_i.

Let us consider an example:

Example 1 *Let $E \equiv recX.(a.\tau.X) + \Delta(b.0)$, which is guarded. The part of the transition system that is rooted at E looks as follows:*



By introducing for every node a formal variable X_1, we see that E $\Uparrow$-provably satisfies the following guarded SES $\mathcal{E}$:

$$
\begin{aligned}
X_1 &= a.X_2 + b.X_3 + \tau.X_4 \\
X_2 &= \tau.X_5 \\
X_3 &= 0 \\
X_4 &= \Delta(b.X_3) \\
X_5 &= a.X_2
\end{aligned}
$$

Note that $\mathcal{E}$ is not saturated. But $\mathcal{E}$ also satisfies the following guarded and saturated SES:

$$
\begin{aligned}
X_1 &= a.X_2 + b.X_3 + \tau.X_4 + a.X_5 \\
X_2 &= \tau.X_5 + a.X_2 \\
X_3 &= 0 \\
X_4 &= \Delta(b.X_3) \\
X_5 &= a.X_2
\end{aligned}
$$

(end of example.)

It is worth to highlight that the required partitioning of formal variables into Ω^Σ and Ω^Δ induces standard equation systems with a distinguished structure. This structure is crucial in order to carry over Milner's saturation property in the presence of the Δ-operator:

Theorem 5 *Every guarded $E \in \mathbb{E}$ ϕ-provably satisfies a guarded and saturated SES over the free variables $\mathbb{V}(E)$.*

Proof. First we prove by induction on the structure of the expression E that E ϕ-provably satisfies a guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(E)$ and the formal variables $(X_1, \ldots, X_m)$. Furthermore for the inductive proof we need the following property (§):

> If $Y \in \mathbb{V}(E)$ is guarded in E then there does not exist i such that $X_1 \Longrightarrow X_i$ and Y occurs in the expression E_i, where $X_i = E_i$ is an equation of $\mathcal{E}$.

$E \equiv 0$ or $E \in \mathbb{V}$: trivial

$E \equiv a.F$: By induction F ϕ-provably satisfies a guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(F)$ and the formal variables $(X_1, \ldots, X_m)$. Then $a.F$ ϕ-provably satisfies the guarded SES $\{X_0 = a.X_1\} \cup \mathcal{E}$ over the free variables $\mathbb{V}(E)$ and the formal variables $(X_0, \ldots, X_m)$. Furthermore this new SES satisfies (§) if $\mathcal{E}$ satisfies (§).

$E \equiv \Delta(F)$: Again let $\mathcal{E}$ be a guarded SES over the free variables $\mathbb{V}(F)$ and the formal variables $(X_1, \ldots, X_m)$ that is ϕ-provably satisfied by F. Assume that the equation $X_1 = E_1$ belongs to $\mathcal{E}$, where w.l.o.g. $X_1 \in \Omega^\Sigma$. Then $\Delta(F)$ ϕ-provably satisfies the guarded SES $\{X_0 = \Delta(E_1)\} \cup \mathcal{E}$ over $(X_0, \ldots, X_m)$. Furthermore this new SES satisfies (§) if $\mathcal{E}$ satisfies (§).

$E \equiv F + G$: Assume that F (resp. G) ϕ-provably satisfies the guarded SES $\mathcal{E}$ (resp. $\mathcal{F}$) over the free variables $\mathbb{V}(F)$ (resp. $\mathbb{V}(G)$) and the formal variables $(X_1, \ldots, X_m)$ (resp. $(Y_1, \ldots, Y_n)$), where w.l.o.g. $\{X_1, \ldots, X_m\} \cap \{Y_1, \ldots, Y_n\} = \emptyset$. Assume that the equations $X_1 = F_1$ and $Y_1 = G_1$ belong to $\mathcal{E}$ and $\mathcal{F}$, respectively, where w.l.o.g. $X_1, Y_1 \in \Omega^\Sigma$. Then $F+G$ ϕ-provably satisfies the guarded

SES $\{Z = F_1+G_1\}\cup\mathcal{E}\cup\mathcal{F}$ over the formal variables $(Z, X_1, \ldots, X_m, Y_1, \ldots, Y_n)$. Furthermore this new SES satisfies (§) if $\mathcal{E}$ and $\mathcal{F}$ satisfy (§).

$E \equiv \mathit{rec}X_0.F$, where X_0 is guarded in F: The case $X_0 \notin \mathbb{V}(F)$ is trivial, thus assume that $X_0 \in \mathbb{V}(F)$. Let F ϕ-provably satisfy the guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(F)$ and the formal variables $(X_1, \ldots, X_m)$. Assume that $\mathcal{E}$ satisfies (§) and assume that the equation $X_1 = E_1$ belongs to $\mathcal{E}$, where w.l.o.g. $X_1 \in \Omega^\Sigma$. By replacing every right hand side E_i of an equation of $\mathcal{E}$ by $E_i\{E_1/X_0\}$ we obtain a new SES $\mathcal{F}$. Note that due to (§), the free variable X_0 does not appear as a summand in E_1, hence X_0 does not occur in the SES $\mathcal{F}$. Then $\mathit{rec}X_0.F$ ϕ-provably satisfies the SES $\{X_0 = E_1\} \cup \mathcal{F}$ over the formal variables $(X_0, \ldots, X_m)$ and the free variables $\mathbb{V}(E) = \mathbb{V}(F)\backslash\{X_0\}$. Moreover, since $\mathcal{E}$ satisfies (§), this new SES is guarded and satisfies again (§).

It remains to transform a guarded SES $\mathcal{E}$, which is ϕ-provably satisfied by an expression E, into a guarded and saturated SES, which is also ϕ-provably satisfied by E. We only show, how the first condition of the definition of a saturated SES can be enforced by induction on the length of the transition sequence $X_i \stackrel{a}{\Longrightarrow} X_j$, the second condition on free variables can be enforced similarly. First assume that for $\mathcal{E}$ we have $X_i \stackrel{\tau}{\longrightarrow} X_k \stackrel{a}{\Longrightarrow} X_j$ for some k. By induction we may assume that already $X_i \stackrel{\tau}{\longrightarrow} X_k \stackrel{a}{\longrightarrow} X_j$. First assume that $X_k \in \Omega^\Sigma$. Let $X_k = a.X_j + E_k$ be the equation defining X_k. The equation defining X_i is either of the form $X_i = \tau.X_k + E_i$ or of the form $X_i = \Delta(\tau.X_k + E_i)$. In both cases we can use axiom ($\tau 2$) in order to replace this equation by $X_i = \tau.X_k+a.X_j+E_k+E_i$ or $X_i = \Delta(\tau.X_k+a.X_j+E_k+E_i)$, respectively. Now assume that $X_k \in \Omega^\Delta$ and let $X_k = \Delta(a.X_j + E_k)$ be the equation defining X_k. Since $\Delta(a.X_j + E_k) =^{\Uparrow} \Delta(a.X_j + E_k) + a.X_j + E_k$ by the law ($\Uparrow_1$) from Lemma 10 we can argue as in case $X_k \in \Omega^\Sigma$.

It remains to consider the case that $\mathcal{E}$ satisfies $X_i \stackrel{a}{\longrightarrow}\Longrightarrow X_k \stackrel{\tau}{\longrightarrow} X_j$ for some k. By induction we may assume that already $X_i \stackrel{a}{\longrightarrow} X_k \stackrel{\tau}{\longrightarrow} X_j$. First assume that $X_k \in \Omega^\Sigma$ and let $X_k = \tau.X_j + E_k$ be the equation defining X_k. The equation defining X_i is either of the form $X_i = a.X_k + E_i$ or of the form $X_i = \Delta(a.X_k+E_i)$. We can use axiom ($\tau 3$) in order to replace this equation by $X_i = a.X_k+a.X_j+E_i$ or $X_i = \Delta(a.X_k+a.X_j+E_i)$, respectively. Now assume that $X_k \in \Omega^\Delta$ and let $X_k = \Delta(\tau.X_j + E_k)$ be the equation defining X_k. Since $\Delta(\tau.X_j+E_k) =^{\Uparrow} \Delta(\tau.X_j+E_k)+\tau.X_j+E_k$ by the law ($\Uparrow_1$) from Lemma 10 we can argue as in case $X_k \in \Omega^\Sigma$. The resulting SES is still ϕ-provably satisfied by E, it is guarded, and it satisfies $X_i \stackrel{a}{\longrightarrow} X_j$. $\Box$

Using axiom ($\mathit{rec}3$), the following theorem can be shown analogously to [13].

Theorem 6 *Let $E, F \in \mathbb{E}$ and let $\mathcal{E}$ be a guarded (not necessarily standard) equation system such that both E and F ϕ-provably satisfy $\mathcal{E}$. Then $E =^\phi F$.*

In this section we will restrict to expressions from $\mathbb{P}$. The main technical result of this section is

Theorem 7 *Let $P, Q \in \mathbb{P}$ such that $P \simeq^\phi Q$. Furthermore P (resp. Q) ϕ-provably satisfies the guarded and saturated SES $\mathcal{E}_1 = \{X_i = E_i \mid 1 \leq i \leq m\}$ (resp. $\mathcal{E}_2 = \{Y_j = F_j \mid 1 \leq j \leq n\}$). Then there exists a guarded equation system $\mathcal{E}$ such that both P and Q ϕ-provably satisfy $\mathcal{E}$.*

Let us postpone the proof of Theorem 7 for a moment and first see how completeness for $\mathbb{P}$ can be deduced:

Theorem 8 *If $P, Q \in \mathbb{P}$ and $P \simeq^\phi Q$, then $P =^\phi Q$.*

Proof. By Theorem 4, both P and Q can be turned into guarded expressions $P', Q' \in \mathbb{P}$ via the axioms for $\simeq^\Uparrow$. Due to soundness, we still have $P' \simeq^\phi Q'$. By Theorem 5, P' (resp. Q') ϕ-provably satisfies a guarded and saturated SES $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) without free variables. By Theorem 7 there is some guarded equation system $\mathcal{E}$ which is ϕ-provably satisfied by P' and Q'. Theorem 6 gives $P' =^\phi Q'$, and hence $P =^\phi Q$, concluding the proof. $\square$

In order to prove Theorem 7, we need the following lemmas.

Lemma 17 *Let $\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\}$ be a guarded SES without free variables, which is ϕ-provably satisfied by $(P_1, \ldots, P_n)$. If $P_i \approx^\phi P \stackrel{\widehat{a}}{\Longrightarrow} Q$, then $X_i \stackrel{\widehat{a}}{\Longrightarrow} X_k$ and $Q \approx^\phi P_k$ for some k.*

Proof. Since $\mathcal{E}$ is guarded there exists a linear order $\prec$ on the formal variables $\{X_1, \ldots, X_n\}$ such that $X_i \stackrel{\tau}{\longrightarrow} X_k$ implies $X_k \prec X_i$. We prove the lemma by an induction along the order $\prec$.

Let us first consider the case $a = \tau$, i.e, $P \Longrightarrow Q$. The case $P \equiv Q$ is trivial. Thus assume that $P \stackrel{\tau}{\longrightarrow} R \Longrightarrow Q$. Then $P \approx^\phi P_i \simeq^\phi E_i\{\vec{P}/\vec{X}\}$ implies $E_i\{\vec{P}/\vec{X}\} \Longrightarrow R'$ and $R \approx^\phi R'$ for some R'. If $E_i\{\vec{P}/\vec{X}\} \equiv R'$ then we have $R \approx^\phi R' \simeq^\phi P_i$. Since the transition sequence $R \Longrightarrow Q$ is shorter than the original sequence $P \stackrel{\tau}{\Longrightarrow} Q$, we can conclude by an induction on the length of the transition sequence. Thus assume that $E_i\{\vec{P}/\vec{X}\} \stackrel{\tau}{\longrightarrow} P' \Longrightarrow R'$, where $P' \not\equiv E_i\{\vec{P}/\vec{X}\}$. We obtain $P' \equiv P_j$ and $X_i \stackrel{\tau}{\longrightarrow} X_j$ for some j. Since $X_j \prec X_i$ and $P_j \Longrightarrow R'$ we obtain inductively $X_j \Longrightarrow X_\ell$ and $R' \approx^\phi P_\ell$ for some ℓ. Finally $R \Longrightarrow Q$, $R \approx^\phi R' \approx^\phi P_\ell$ and $X_\ell \prec X_i$ implies inductively $X_\ell \Longrightarrow X_k$ and $Q \approx^\phi P_k$ for some k.

Now assume that $a \neq \tau$, i.e, $P \stackrel{a}{\Longrightarrow} Q$. Since $P \approx^{\phi} P_i \simeq^{\phi} E_i\{\vec{P}/\vec{X}\}$ we get $E_i\{\vec{P}/\vec{X}\} \stackrel{a}{\Longrightarrow} R$ and $Q \approx^{\phi} R$ for some R. If $E_i\{\vec{P}/\vec{X}\} \stackrel{a}{\longrightarrow} P_j \Longrightarrow R$ for some j, then $X_i \stackrel{a}{\longrightarrow} X_j \Longrightarrow X_k$ and $P_k \approx^{\phi} R \approx^{\phi} Q$ for some k by the previous paragraph. On the other hand, if $E_i\{\vec{P}/\vec{X}\} \stackrel{\tau}{\longrightarrow} P' \stackrel{a}{\Longrightarrow} R$ with $E_i\{\vec{P}/\vec{X}\} \not\equiv P'$, then $P' \equiv P_j$ and $X_i \stackrel{\tau}{\longrightarrow} X_j$ for some j. Thus $X_j \prec X_i$ and by induction we get $X_j \stackrel{a}{\Longrightarrow} X_k$ and $P_k \approx^{\phi} R \approx^{\phi} Q$ for some k. $\square$

For the further consideration it is useful to define for $\phi \in \{\Uparrow, \Updownarrow, \sigma, \perp\}$ the property ϕ^* on $\mathbb{P}$ by

$$P\phi^* \text{ if and only if } \begin{cases} P\phi & \text{if } \phi \in \{\Uparrow, \Updownarrow\} \\ \text{not } P\phi & \text{if } \phi \in \{\sigma, \perp\} \end{cases}$$

Lemma 18 *If $P\phi^*$, then $\tau.P =^{\phi} \Delta(P)$.*

Proof. We distinguish on the value of ϕ.

$\phi = \Uparrow$: Then $P \Uparrow$. Thus, by Lemma 16 we obtain $P =^{\Uparrow} \Delta(Q)+R$ for expressions Q, R. Hence, using axiom $(\Uparrow)$, $\tau.P =^{\Uparrow} \tau.(\Delta(Q)+R) =^{\Uparrow} \Delta(\Delta(Q)+R) =^{\Uparrow} \Delta(P)$.

$\phi = \sigma$: Then $P \stackrel{\tau}{\longrightarrow}$. Thus, by Lemma 11 there are $Q, R \in \mathbb{P}$ with $P =^{\Uparrow} \tau.Q+R$. Hence, using axiom (σ), $\tau.P =^{\Uparrow} \tau.(\tau.Q+R) =^{\sigma} \Delta(\tau.Q+R) =^{\Uparrow} \Delta(P)$.

$\phi = \perp$: Analogously to the case $\phi = \sigma$ using Lemma 12 and axiom $(\perp)$.

$\phi = \Updownarrow$: Then $P\Updownarrow$. By Lemma 13, 14, and 16 we obtain either $P =^{\Updownarrow} 0$ or $P =^{\Updownarrow} \Delta(Q) + R$ for some $Q, R \in \mathbb{P}$. In the latter case we can argue as in case $\phi = \Uparrow$. If $P =^{\Updownarrow} 0$, then using axiom $(\Updownarrow)$, $\tau.P =^{\Updownarrow} \tau.0 =^{\Updownarrow} \Delta(0) =^{\Updownarrow} \Delta(P)$. $\square$

Now we are able to prove Theorem 7.

Proof of Theorem 7. Assume that $\mathcal{E}_1$ is ϕ-provably satisfied by the expressions $(P_1, \ldots, P_m) \in \mathbb{P}$, where $P \equiv P_1$, and that $\mathcal{E}_2$ is ϕ-provably satisfied by the expressions $(Q_1, \ldots, Q_n) \in \mathbb{P}$, where $Q \equiv Q_1$. Thus $P_i =^{\phi} E_i\{\vec{P}/\vec{X}\}$ and $Q_j =^{\phi} F_j\{\vec{Q}/\vec{Y}\}$, and hence by soundness also $P_i \simeq^{\phi} E_i\{\vec{P}/\vec{X}\}$ and $Q_j \simeq^{\phi} F_j\{\vec{Q}/\vec{Y}\}$. Since $P, Q \in \mathbb{P}$, both $\mathcal{E}_1$ and $\mathcal{E}_2$ do not have free variables. Moreover, w.l.o.g. $X_1, Y_1 \in \Omega^{\Sigma}$.

Claim 1. If $P_i \approx^{\phi} Q_j$, then the following implications hold:

(1) If $X_i \stackrel{a}{\longrightarrow} X_k$, then either ($a = \tau$ and $P_k \approx^{\phi} Q_j$) or $Y_j \stackrel{a}{\longrightarrow} Y_\ell$ and $P_k \approx^{\phi} Q_\ell$ for some ℓ.
(2) If $Y_j \stackrel{a}{\longrightarrow} Y_\ell$, then either ($a = \tau$ and $P_i \approx^{\phi} Q_\ell$) or $X_i \stackrel{a}{\longrightarrow} X_k$ and $P_k \approx^{\phi} Q_\ell$ for some k.

By symmetry it suffices to show the first statement. Assume that $X_i \xrightarrow{a} X_k$. Thus $E_i\{\vec{P}/\vec{X}\} \xrightarrow{a} P_k$. Since $E_i\{\vec{P}/\vec{X}\} \simeq^\phi P_i \approx^\phi Q_j$, we have $Q_j \stackrel{\widehat{a}}{\Longrightarrow} R$ for some R with $P_k \approx^\phi R$. By Lemma 17 we obtain $Y_j \stackrel{\widehat{a}}{\Longrightarrow} Y_\ell$ and $P_k \approx^\phi R \approx^\phi Q_\ell$ for some ℓ. Since $\mathcal{E}_2$ is saturated we obtain the conclusion of Claim 1.

The following claim can be shown analogously:

Claim 2. If $P_i \simeq^\phi Q_j$, then the following implications hold:

(1) If $X_i \xrightarrow{a} X_k$, then $Y_j \xrightarrow{a} Y_\ell$ and $P_k \approx^\phi Q_\ell$ for some ℓ.
(2) If $Y_j \xrightarrow{a} Y_\ell$, then $X_i \xrightarrow{a} X_k$ and $P_k \approx^\phi Q_\ell$ for some k.

Let $I = \{(i,j) \mid 1 \le i \le m, 1 \le j \le n, P_i \approx^\phi Q_j\}$. For every $(i,j) \in I$ let $Z_{i,j}$ be a new variable and let $\vec{Z} = (Z_{i,j})_{(i,j)\in I}$. Furthermore, for $(i,j) \in I$ we define

$$\begin{aligned} H_{i,j} \equiv & \textstyle\sum\{a.Z_{k,\ell} \mid X_i \xrightarrow{a} X_k,\, Y_j \xrightarrow{a} Y_\ell,\, P_k \approx^\phi Q_\ell\} + \\ & \textstyle\sum\{\tau.Z_{k,j} \mid X_i \xrightarrow{\tau} X_k,\, \neg\exists\ell\,(Y_j \xrightarrow{\tau} Y_\ell \,\wedge\, P_k \approx^\phi Q_\ell)\} + \\ & \textstyle\sum\{\tau.Z_{i,\ell} \mid Y_j \xrightarrow{\tau} Y_\ell,\, \neg\exists k\,(X_i \xrightarrow{\tau} X_k \,\wedge\, P_k \approx^\phi Q_\ell)\} \\ G_{i,j} \equiv & \begin{cases} H_{i,j} & \text{if } X_i, Y_j \in \Omega^\Sigma \text{ and} \\ \Delta(H_{i,j}) & \text{if } X_i \in \Omega^\Delta \text{ or } Y_j \in \Omega^\Delta. \end{cases} \end{aligned}$$

Now the equation system $\mathcal{E}$ over the formal variables $\vec{Z}$ contains for every $(i,j) \in I$ the equation $Z_{i,j} = G_{i,j}$. From the guardedness of $\mathcal{E}_1$ and $\mathcal{E}_2$ it follows easily that also $\mathcal{E}$ is guarded. Before we continue with the proof, let us consider an example:

Example 2 *Let $P = b.\Delta(a.0 + a.\tau.0) + c.0$ and $Q = b.\Delta(a.0) + c.0 + c.\tau.0$. Clearly, we have $P \simeq^\Uparrow Q$. The expression P $\Uparrow$-provably satisfies the following guarded and saturated SES:*

$$\begin{aligned} X_1 &= b.X_2 + c.X_3 \\ X_2 &= \Delta(a.X_3 + a.X_4) \\ X_3 &= 0 \\ X_4 &= \tau.X_3 \end{aligned}$$

To see this, choose $P_1 = P$, $P_2 = \Delta(a.0 + a.\tau.0)$, $P_3 = 0$, and $P_4 = \tau.0$. The expression Q $\Uparrow$-provably satisfies the following guarded and saturated SES:

$$\begin{aligned} Y_1 &= b.Y_2 + c.Y_3 + c.Y_4 \\ Y_2 &= \Delta(a.Y_3) \\ Y_3 &= 0 \\ Y_4 &= \tau.Y_3 \end{aligned}$$

This can be seen by choosing $Q_1 = Q$, $Q_2 = \Delta(a.\tau.0)$, $Q_3 = 0$, and $Q_4 = \tau.0$. We obtain $I = \{(1,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$ and thus

$$\begin{aligned} G_{1,1} &= b.Z_{2,2} + c.Z_{3,3} + c.Z_{3,4} \\ G_{2,2} &= \Delta(a.Z_{3,3} + a.Z_{4,3}) \\ G_{3,3} &= 0 \\ G_{3,4} &= G_{4,3} = G_{4,4} = \tau.Z_{3,3} \end{aligned}$$

(end of example.)

We will show that P ϕ-provably satisfies $\mathcal{E}$; that also Q ϕ-provably satisfies $\mathcal{E}$ can be shown analogously. For this we define for every $(i,j) \in I$ an expression $R_{i,j}$. Let us fix $(i,j) \in I$, thus $P_i \approx^\phi Q_j$. For our further considerations the following to complementary cases will be crucial.

$$\forall \ell, a \left\{ Y_j \xrightarrow{a} Y_\ell \;\Rightarrow\; \exists k \,(X_i \xrightarrow{a} X_k \;\wedge\; P_k \approx^\phi Q_\ell) \right\} \tag{1}$$

$$\exists \ell \left\{ Y_j \xrightarrow{\tau} Y_\ell \;\wedge\; \neg\exists k (X_i \xrightarrow{\tau} X_k \;\wedge\; P_k \approx^\phi Q_\ell) \right\} \tag{2}$$

We define $R_{i,j}$ by

$$R_{i,j} \equiv \begin{cases} P_i & \text{if } X_i \in \Omega^\Delta \text{ or } (X_i, Y_j \in \Omega^\Sigma \text{ and } (1)) \\ \tau.P_i & \text{if } (X_i \in \Omega^\Sigma \text{ and } Y_j \in \Omega^\Delta) \text{ or } (X_i, Y_j \in \Omega^\Sigma \text{ and } (2)) \end{cases}$$

Let $\vec{R} = (R_{i,j})_{(i,j)\in I}$. Note that $R_{1,1} \equiv P_1 \equiv P$ by $P_1 \simeq^\phi Q_1$ and Claim 2.2 (recall that $X_1, Y_1 \in \Omega^\Sigma$). Hence, it remains to prove $R_{i,j} =^\phi G_{i,j}\{\vec{R}/\vec{Z}\}$.

Case 1. $X_i \in \Omega^\Sigma$: Using Claim 1 and axiom (τ1) and (S1)–(S3) we obtain

$$H_{i,j}\{\vec{R}/\vec{Z}\} =^\Uparrow \begin{cases} E_i\{\vec{P}/\vec{X}\} =^\phi P_i & \text{if } (1) \\ E_i\{\vec{P}/\vec{X}\} + \tau.P_i =^\phi P_i + \tau.P_i =^\Uparrow \tau.P_i & \text{if } (2). \end{cases}$$

This step is analogous to [13]. If moreover $Y_j \in \Omega^\Sigma$, then the definition of $R_{i,j}$ implies $R_{i,j} =^\phi H_{i,j}\{\vec{R}/\vec{Z}\} \equiv G_{i,j}\{\vec{R}/\vec{Z}\}$. On the other hand, if $Y_j \in \Omega^\Delta$, then $Q_j \simeq^\phi F_j\{\vec{Q}/\vec{Y}\} \equiv \Delta(Q')$ for some Q'. We have to show that $\tau.P_i \equiv R_{i,j} =^\Uparrow \Delta(H_{i,j}\{\vec{R}/\vec{Z}\})$. If (2), then $\Delta(H_{i,j}\{\vec{R}/\vec{Z}\}) =^\phi \Delta(\tau.P_i)$. But $(\tau.P_i)\phi^*$. For $\phi \in \{\sigma, \bot\}$ this is clear, whereas for $\phi \in \{\Uparrow, \Updownarrow\}$, $P_i \approx^\phi Q_j \simeq^\phi \Delta(Q')$ implies $P_i\phi^*$ (and thus $(\tau.P_i)\phi^*$). Hence, with Lemma 18 we obtain $\Delta(\tau.P_i) =^\phi \tau.\tau.P_i =^\Uparrow \tau.P_i$. If (1), then $\Delta(H_{i,j}\{\vec{R}/\vec{Z}\}) =^\phi \Delta(P_i)$. We claim that $P_i\phi^*$, which implies again $\Delta(P_i) =^\phi \tau.P_i$ with Lemma 18. If $\phi \in \{\Uparrow, \Updownarrow\}$, then we already argued above that $P_i\phi^*$. For $\phi = \sigma$ let us assume that $P_i\sigma$ in order to deduce a contradiction. We have $P_i \approx^\sigma Q_j \simeq^\sigma \Delta(Q')$. Since $P_i\sigma$, $\Delta(Q')$ has to move silently into a stable expression. Thus, $Q' \xrightarrow{\tau}$, i.e., $Y_j \xrightarrow{\tau} Y_\ell$ for some ℓ. Then (1) implies that also $X_i \xrightarrow{\tau} X_k$ for some k. Thus $P_i \simeq^\sigma E_i\{\vec{P}/\vec{X}\} \xrightarrow{\tau}$, i.e., $P_i \xrightarrow{\tau}$, a contradiction to $P_i\sigma$. If $\phi = \bot$, then we can argue analogously.

Case 2. $X_i \in \Omega^\Delta$: Then $E_i \equiv \Delta(E_i')$ for a sum E_i'. Analogously to Case 1 we have

$$H_{i,j}\{\vec{R}/\vec{Z}\} =^{\Uparrow} \begin{cases} E_i'\{\vec{P}/\vec{X}\} & \text{if (1)} \\ E_i'\{\vec{P}/\vec{X}\} + \tau.P_i & \text{if (2).} \end{cases}$$

Let $P' \equiv E_i'\{\vec{P}/\vec{X}\}$, thus $P_i =^\phi \Delta(P')$. We have to check $P_i \equiv R_{i,j} =^\phi \Delta(H_{i,j}\{\vec{R}/\vec{Z}\})$. If (1), then $\Delta(H_{i,j}\{\vec{R}/\vec{Z}\}) =^{\Uparrow} \Delta(P') =^\phi P_i$. On the other hand, if (2), then

$$\Delta(H_{i,j}\{\vec{R}/\vec{Z}\}) =^{\Uparrow} \Delta(P' + \tau.P_i) =^\phi$$
$$\Delta(P' + \tau.\Delta(P')) =^{\Uparrow} \Delta(\Delta(P')) =^{\Uparrow} \tau.\Delta(P') =^\phi \Delta(P') =^\phi P_i.$$

This concludes the proof of Theorem 7 and hence of Theorem 8. $\Box$

8 Completeness for open expressions

In the previous section, we proved completeness only for expressions without free variables. Recall that for arbitrary expressions $E, F \in \mathbb{E}$, we have $E \simeq^\phi F$ if and only if for all $\vec{P}$ we have $E\{\vec{P}/\vec{X}\} \simeq^\phi F\{\vec{P}/\vec{X}\}$. Here $\vec{X}$ is a sequence of variables that contains all variables from $\mathbb{V}(E) \cup \mathbb{V}(F)$, and $\vec{P}$ is an arbitrary sequence over $\mathbb{P}$ of the same length. In order to prove completeness for the whole set $\mathbb{E}$, we will argue in a purely syntactical way by investigating our axioms.

Definition 1 *A* proof tree *for $E = F$ is a rooted tree T. The root of T is labeled with $E = F$, and every leaf v of T is either labeled with an identity $A = A$ or with an instance of an axioms for $\simeq^\phi$. If v is an internal node of T, then either*

(1) v is labeled with an identity $G = recX.H$, and v has exactly one child u which is labeled with the identity $G = H\{G/X\}$, where X is guarded in H, or
(2) v is labeled with an identity $G_1 = G_3$, and v has exactly two children v_1 and v_2 that are labeled with identities $G_1 = G_2$ and $G_2 = G_3$, respectively.
(3) v is labeled with an identity $G_1 + H_1 = G_2 + H_2$, and v has exactly two children v_1 and v_2 that are labeled with the identities $G_1 = G_2$ and $H_1 = H_2$, respectively, and similarly for the other operators.

It is easy to see that $E =^\phi F$ if and only if there exists a proof tree for $E = F$. This allows to show statements by induction on the size of proof trees.

For our further considerations let us fix a variable X and an action $a \in \mathbb{A} \setminus \{\tau\}$. For an expression $E \in \mathbb{E}$ let $\widehat{E}$ be the expression that results from E by replacing every subterm of the form $a.F$ by X. In case E contains a subterm of the form $recX.G$ such that G contains a subterm of the form $a.F$, then we first have to rename the bounded variable X in $recX.G$ in order to avoid new variable bindings.

Lemma 19 *If $E =^{\phi} F$ and $\phi \neq \perp$, then also $\widehat{E} =^{\phi} \widehat{F}$.*

Proof. We prove the lemma by induction on the size of a proof tree T for $E = F$. If T consists of a single node, then there are two cases:

- If $E \equiv F$, then also $\widehat{E} \equiv \widehat{F}$.
- Let $E = F$ be an instance of an axiom for $\simeq^{\phi}$. In most cases, $\widehat{E} = \widehat{F}$ is again an axiom for $\simeq^{\phi}$. We only consider those axioms that explicitly deal with the $a.(\,)$-operator:
  ($\tau 1$): $E \equiv a.\tau.G$ and $F = a.G$. Then $\widehat{E} \equiv X \equiv \widehat{F}$.
  ($\tau 3$): $E \equiv a.(G + \tau.H)$ and $F \equiv a.(G + \tau.H) + a.H$. Then $\widehat{E} \equiv X =^{\Uparrow} X + X \equiv \widehat{F}$ with axiom ($S3$).

Note that we cannot deal with axiom ($\perp$): We have $\Delta(a.0) =^{\perp} \tau.a.0$ but $\Delta(X) =^{\perp} \tau.X$ does not hold, since $\tau.0 \not\simeq^{\perp} \Delta(0)$. This is the reason for excluding $\phi = \perp$ in the lemma.

For the inductive step we have to consider the three cases listed in Definition 1.

(1) The root of T is labeled with $E \equiv G = recY.H \equiv F$ (w.l.o.g $X \neq Y$) and its child node is labeled with $G = H\{G/Y\}$ where Y is guarded in H. Then by induction $\widehat{G} =^{\phi} \widehat{H\{G/Y\}} \equiv \widehat{H}\{\widehat{G}/Y\}$. Since Y is also guarded in $\widehat{H}$, we obtain $\widehat{G} =^{\phi} recY.\widehat{H} \equiv \widehat{recY.H}$.
(2) The root of T has two child nodes that are labeled with identities $E = G$ and $G = F$, respectively. Then by induction $\widehat{E} =^{\phi} \widehat{G}$ and $\widehat{G} =^{\phi} \widehat{F}$, which implies $\widehat{E} =^{\phi} \widehat{F}$.
(3) The root of T is labeled with $E \equiv G_1 + H_1 = G_2 + H_2 \equiv F$ and its child nodes are labeled with $G_1 = G_2$ and $H_1 = H_2$. Then by induction $\widehat{G_1} =^{\phi} \widehat{G_2}$ and $\widehat{H_1} =^{\phi} \widehat{H_2}$, i.e., $\widehat{E} \equiv \widehat{G_1} + \widehat{H_1} =^{\phi} \widehat{G_2} + \widehat{H_2} \equiv \widehat{F}$. For the other operators we can argue analogously. $\square$

Lemma 20 *Let $\phi \neq \perp$ and $E, F \in \mathbb{E}$. If $a \in \mathbb{A} \backslash \{\tau\}$ does neither occur in E nor in F, then $E\{a.0/X\} =^{\phi} F\{a.0/X\}$ implies $E =^{\phi} F$.*

Proof. Assume that $E\{a.0/X\} =^{\phi} F\{a.0/X\}$. Since a does neither occur in E nor in F, we have $\widehat{E\{a.0/X\}} \equiv E$ and $\widehat{F\{a.0/X\}} \equiv F$. Thus, Lemma 19 implies $E =^{\phi} F$. $\square$

Theorem 9 *Let $\phi \neq \perp$ and $E, F \in \mathbb{E}$. If $E \simeq^{\phi} F$ then $E =^{\phi} F$.*

Proof. Let $E \simeq^{\phi} F$. We prove by induction on $|\mathbb{V}(E) \cup \mathbb{V}(F)|$ that $E =^{\phi} F$. If $\mathbb{V}(E) \cup \mathbb{V}(F) = \emptyset$, then in fact $E, F \in \mathbb{P}$ and $E =^{\phi} F$ by Theorem 8. Thus let $X \in \mathbb{V}(E) \cup \mathbb{V}(F)$. Since $E \simeq^{\phi} F$, we have $E\{a.0/X\} \simeq^{\phi} F\{a.0/X\}$, where $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F. Thus, by induction $E\{a.0/X\} =^{\phi} F\{a.0/X\}$ and hence $E =^{\phi} F$ by Lemma 20. □

In order to obtain a complete axiomatisation of $\simeq^{\perp}$ for open expressions, we have to introduce the following additional axiom $(\perp')$

$(\perp')$ If $E\{0/X\} = F\{0/X\}$ and $E\{a.0/X\} = F\{a.0/X\}$ where $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F, then $E = F$.

This new axiom is indeed sound for $\simeq^{\perp}$:

Theorem 10 *If $E\{0/X\} \simeq^{\perp} F\{0/X\}$ and $E\{a.0/X\} \simeq^{\perp} F\{a.0/X\}$, where $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F, then $E \simeq^{\perp} F$.*

Proof. Let $E\{0/X\} \simeq^{\perp} F\{0/X\}$ and $E\{a.0/X\} \simeq^{\perp} F\{a.0/X\}$ where $a \in \mathbb{A} \setminus \{\tau\}$ does neither occur in E nor in F. We have to show that $E \simeq^{\perp} F$. Due to the definition of $\simeq^{\perp}$ for expressions with free variables, it suffices to consider the case that $\mathbb{V}(E) \cup \mathbb{V}(F) = \{X\}$. Thus we have to show that $E\{P/X\} \simeq^{\perp} F\{P/X\}$ for all $P \in \mathbb{P}$. Fix an arbitrary $P \in \mathbb{P}$. We distinguish the following two cases.

Case 1. $P \longrightarrow$, i.e., not $P\perp$: We first claim that the symmetric closure of

$$\mathcal{R} = \mathrm{Id}_{\mathbb{P}} \cup \{\langle G\{P/X\}, H\{P/X\}\rangle \mid a \text{ does not occur in } G \text{ or } H \\ \text{and } G\{a.0/X\} \approx^{\perp} H\{a.0/X\}\}$$

is a $\mathrm{WB}^{\perp}$. Let us start with showing that $\mathcal{R}$ is a WB. Consider a pair $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$ and assume that $G\{P/X\} \xrightarrow{b} P'$. By Lemma 3 we can distinguish the following two cases:

Case i. $G \xrightarrow{b} G'$ and $P' \equiv G'\{P/X\}$. Thus, $G\{a.0/X\} \xrightarrow{b} G'\{a.0/X\}$. Since $G\{a.0/X\} \approx^{\perp} H\{a.0/X\}$, we have $H\{a.0/X\} \xRightarrow{\widehat{b}} Q$ and $G'\{a.0/X\} \approx^{\perp} Q$ for some Q. Note that $G \xrightarrow{b} G'$ implies $b \neq a \neq \tau$. Hence, by Lemma 3 we have $H \xRightarrow{\widehat{b}} H'$ and $G'\{a.0/X\} \approx^{\perp} Q \equiv H'\{a.0/X\}$. Thus, $H\{P/X\} \xRightarrow{\widehat{b}} H'\{P/X\}$ and $\langle G'\{P/X\}, H'\{P/X\}\rangle \in \mathcal{R}$ (note also that since a does not occur in G, $G \xrightarrow{b} G'$ implies that a does not occur in G' as well, and similarly for H, H').

Case ii. $P \xrightarrow{b} P'$ and X is totally unguarded in G. Thus, $G\{a.0/X\} \xrightarrow{a} 0$. Since $G\{a.0/X\} \approx^{\perp} H\{a.0/X\}$, we obtain $H\{a.0/X\} \xRightarrow{a}$. Using Lemma 3 and the fact that $a \in \mathbb{A} \setminus \{\tau\}$ does not occur in H it follows $H \Longrightarrow H'$ for some H' such that X is totally unguarded in H'. Hence, $H\{P/X\} \Longrightarrow$

$H'\{P/X\} \xrightarrow{b} P'$.

It remains to show that $\mathcal{R}$ preserves $\perp$. Let $G\{P/X\}\perp$. Lemma 4 implies that $G\perp$ and that X is weakly guarded in G (recall that $P \longrightarrow$). Thus, $G\{a.0/X\}\perp$, from which we obtain $H\{a.0/X\} \Longrightarrow H'\{a.0/X\}\perp$ for some H' with $H \Longrightarrow H'\perp$. Hence, X must be weakly guarded in H' (otherwise $H'\{a.0/X\} \xrightarrow{a}$), and thus, $H\{P/X\} \Longrightarrow H'\{P/X\}\perp$. This concludes the proof that $\mathcal{R}$ is a $\mathrm{WB}^{\perp}$. Hence $\mathcal{R} \subseteq \approx^{\perp}$.

Now let $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$ be a pair such that not only $G\{a.0/X\} \approx^{\perp} H\{a.0/X\}$ but $G\{a.0/X\} \simeq^{\perp} H\{a.0/X\}$. The same arguments, which have shown that $\mathcal{R}$ is a WB, together with $\mathcal{R} \subseteq \approx^{\perp}$ show that $\langle G\{P/X\}, H\{P/X\}\rangle$ satisfies the root conditions. Thus $G\{P/X\} \simeq^{\perp} H\{P/X\}$. In particular, we obtain $E\{P/X\} \simeq^{\perp} F\{P/X\}$. This finishes the proof for Case 1.

Case 2. $P\perp$. We first claim that the symmetric closure of

$$\mathcal{R} = \{\langle G\{P/X\}, H\{P/X\}\rangle \mid G\{0/X\} \simeq^{\perp} H\{0/X\}\}$$

is a $\mathrm{WB}^{\perp}$. Analogously to case 1 we can show that $\mathcal{R}$ is a WB (note that Case ii cannot occur, since $P\perp$). In order to show that $\mathcal{R}$ preserves $\perp$, consider a pair $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$ with $G\{P/X\}\perp$. Then $G\perp$. Hence also $G\{0/X\}\perp$, which implies $H\{0/X\} \Longrightarrow H'\{0/X\}\perp$ for some H' with $H \Longrightarrow H'$. Since we assumed $P\perp$, it follows $H\{P/X\} \Longrightarrow H'\{P/X\}\perp$.

This finishes the proof that $\mathcal{R}$ is a $\mathrm{WB}^{\perp}$. The subsequent reasoning is completely analogous to Case 1. $\square$

If we add the axiom $(\perp')$ to the standard axioms for $\simeq^{\perp}$, then we can prove completeness in the same way as in the proof of Theorem 9. Thus, we obtain:

Theorem 11 *Let $E, F \in \mathbb{E}$. If $E \simeq^{\perp} F$ then $E =^{\perp} F$ can be derived by the standard axioms for $\simeq^{\perp}$ plus the axiom $(\perp')$.*

9 Conclusion

This paper has developed sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalences. We have not covered the weak bisimulation preorders $\mathrm{WB}^{\downarrow}$ and $\mathrm{WB}^{\downdownarrows}$ considered in [7]. We claim however that adding the axiom $\Delta(E) \leq E + F$ to the axioms of $\mathrm{WB}^{\Uparrow}$ (resp. $\mathrm{WB}^{\uparrow}$) is enough to obtain completeness of $\mathrm{WB}^{\downdownarrows}$ (resp. $\mathrm{WB}^{\downarrow}$). Note that $\mathrm{WB}^{\downarrow}$ is axiomatised in [15], so only $\mathrm{WB}^{\downdownarrows}$ needs further work.

The axiomatisation developed in this paper opens the way towards a complete equational characterisation of the bisimulation fragment of the linear time – branching time spectrum with silent moves. On the technical side, it is an open problem whether the auxiliary axiom ($\perp'$) is indeed necessary for achieving completeness of open expressions for $\simeq^{\perp}$. We have tried – but not succeeded – to circumvent this somewhat unsatisfactory specific treatment via moving to alternative forms of defining bisimulation on open expressions.

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Appendix

A Congruence with respect to *rec*

This section is devoted to the congruence property of $\simeq^\phi$ with respect to recursion. Let $\mathcal{S} \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation. We say that $\mathcal{S}$ is an *observational congruence up to* $\approx^\phi$ if $\mathcal{S}$ preserves ϕ and:

$$(\dagger)\quad \begin{array}{l} \text{if } (P,Q) \in \mathcal{S} \text{ and } P \stackrel{a}{\longrightarrow} P' \text{ for } P,P',Q \in \mathbb{P} \text{ and } a \in \mathbb{A}\,, \\ \text{then } Q \stackrel{a}{\Longrightarrow} Q' \text{ and } P' \;\mathcal{S}\; R \approx^\phi Q' \text{ for some } R,Q' \in \mathbb{P} \end{array}$$

Lemma 21 *If $\mathcal{S}$ is an observational congruence up to $\approx^\phi$, then $\mathcal{S} \subseteq \simeq^\phi$.*

Proof. First, note that $(\dagger)$ implies that $\mathcal{S} \circ \approx^\phi$ is a WB. Moreover, since both $\mathcal{S}$ and $\approx^\phi$ preserve ϕ, also $\mathcal{S} \circ \approx^\phi$ preserves ϕ. Thus, $\mathcal{S} \circ \approx^\phi$ is a WB$^\phi$ and $\mathcal{S} \circ \approx^\phi \subseteq \approx^\phi$.

Now assume that $(P,Q) \in \mathcal{S}$ and $P \stackrel{a}{\longrightarrow} P'$. Then $(\dagger)$ and $\mathcal{S} \circ \approx^\phi \subseteq \approx^\phi$ implies that $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \approx^\phi Q'$ for some $Q' \in \mathbb{P}$. Furthermore, since $\mathcal{S}$ and $\approx^\phi$ are symmetric, also the symmetric root condition holds. Thus $P \simeq^\phi Q$. $\Box$

In order to prove $recX.E \simeq^\phi recX.F$ if $E \simeq^\phi F$, it is sufficient to construct an observational congruence up to $\approx^\phi$ containing the pair $(recX.E, recX.F)$. This will be done in Lemma 23. We will need the following statement.

Lemma 22 *Let $E,F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \simeq^\phi F$, i.e, $E\{P/X\} \simeq^\phi F\{P/X\}$ for all $P \in \mathbb{P}$. Then the following holds:*

(1) If $E \stackrel{\tau}{\Longrightarrow} E'$, where X is totally unguarded in E', then also $F \stackrel{\tau}{\Longrightarrow} F'$ for some F' such that X is totally unguarded in F'.
(2) If $\phi = \Uparrow$ and $E \Uparrow$, then also $F \Uparrow$.
(3) If $\phi = \Uparrow\!\!\!\!\perp$ and $E \Uparrow$, then either $F \Uparrow$ or $F \stackrel{\tau}{\Longrightarrow} F' \perp$ for some F' such that X is weakly guarded in F'.

Proof. We will only prove the last statement, the other statements can be shown similarly. Thus, assume that $\phi = \Uparrow\!\!\!\!\perp$ and $E \Uparrow$. Let $a \in \mathbb{A} \setminus \{\tau\}$. Since $E \Uparrow$, also $E\{a.0/X\} \Uparrow$ by Lemma 5. Then $E\{a.0/X\} \simeq^{\Uparrow\!\!\!\perp} F\{a.0/X\}$ implies that either $F\{a.0/X\} \Uparrow$ or $F\{a.0/X\} \stackrel{\tau}{\Longrightarrow} Q\bot$ for some Q. If $F\{a.0/X\} \Uparrow$, then Lemma 5 implies $F \Uparrow$. If $F\{a.0/X\} \Longrightarrow Q\bot$, then $a \neq \tau$ and Lemma 3 imply $Q \equiv G\{a.0/X\}$ and $F \stackrel{\tau}{\Longrightarrow} G$ for some G. Finally, since $G\{a.0/X\}\bot$, X must be weakly guarded in G by Lemma 4. $\Box$

Lemma 23 *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \simeq^\phi F$. Furthermore, let*

$$\mathcal{R} = \{\langle G\{recX.E/X\}, G\{recX.F/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\}.$$

Then $\mathcal{S} = (\mathcal{R} \cup \mathcal{R}^{-1})$ is an observational congruence up to $\approx^\phi$.

Proof. Let us first check condition (†). By symmetry it suffices to consider a pair $\langle G\{recX.E/X\}, G\{recX.F/X\}\rangle$. We proceed by an induction on the height of the derivation tree for the transition $G\{recX.E/X\} \xrightarrow{a} P$.

$G \equiv 0$ or $G \equiv a.H$ for some $a \in \mathbb{A}$: trivial

$G \equiv X$: Assume that $recX.E \xrightarrow{a} P$. Thus $E\{recX.E/X\} \xrightarrow{a} P$, which can be derived by a smaller derivation tree. Thus, the induction hypothesis implies $E\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q'$ and $P\,\mathcal{S}\,R \approx^\phi Q'$ for some R, Q'. Since $E \simeq^\phi F$, we have $E\{recX.F/X\} \simeq^\phi F\{recX.F/X\}$. This implies $F\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ and $Q' \approx^\phi Q$ for some Q and thus finally $recX.F \stackrel{a}{\Longrightarrow} Q$ and $P\,\mathcal{S}\,R \approx^\phi Q' \approx^\phi Q$.

$G \equiv \Delta(H)$: Assume that $\Delta(H\{recX.E/X\}) \xrightarrow{a} P$. The case $a = \tau$ and $P \equiv \Delta(H\{recX.E/X\})$ is trivial. On the other hand, if $H\{recX.E/X\} \xrightarrow{a} P$, then by induction $H\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \approx^\phi Q$. Thus also $\Delta(H\{recX.F/X\}) \xrightarrow{a} Q$.

$G \equiv H_1 + H_2$: Assume that $H_1\{recX.E/X\} + H_2\{recX.E/X\} \xrightarrow{a} P$. W.l.o.g. we have $H_1\{recX.E/X\} \xrightarrow{a} P$. By induction $H_1\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \approx^\phi Q$. Thus $H_1\{recX.F/X\} + H_2\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$.

$G \equiv recY.H$: If $X = Y$, then $G\{recX.E/X\} \equiv G \equiv G\{recX.F/X\}$. Thus assume that $X \neq Y$. Since $recX.E, recX.F \in \mathbb{P}$, it follows

$$(recY.H)\{recX.E/X\} \equiv recY.H\{recX.E/X\} \text{ and}$$
$$(recY.H)\{recX.F/X\} \equiv recY.H\{recX.F/X\}.$$

Assume that $recY.H\{recX.E/X\} \xrightarrow{a} P$. Hence

$$(H\{recX.E/X\})\{recY.H\{recX.E/X\}/Y\} \equiv$$
$$(H\{recY.H/Y\})\{recX.E/X\} \xrightarrow{a} P$$

by a smaller derivation tree. By induction $(H\{recY.H/Y\})\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \approx^\phi Q$, which implies $recY.H\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$.

It remains to show that $\mathcal{S}$ preserves ϕ. Since we already know that $\mathcal{S}$ satisfies (†), it suffices to show that $G\{recX.E/X\}\phi$ implies $G\{recX.F/X\} \Longrightarrow Q\phi$ for some Q. For $\phi = \sigma$ and $\phi = \perp$ this is trivial due to (†).

Let $\phi = \Uparrow$ and $G\{recX.E/X\} \Uparrow$. We have to show that also $G\{recX.F/X\} \Uparrow$. First consider the case $G \equiv X$, i.e., let $recX.E \Uparrow$. By Lemma 7 either $E \Uparrow$ or $E \stackrel{\tau}{\Longrightarrow} E'$ for some E' such that X is totally unguarded in E'. If $E \Uparrow$ then $E \simeq^{\Uparrow} F$ and Lemma 22(2) imply $F \Uparrow$, thus also $recX.F \Uparrow$ by Lemma 7. Similarly, if $E \stackrel{\tau}{\Longrightarrow} E'$, where X is totally unguarded in E', then $E \simeq^{\Uparrow} F$ and Lemma 22(1) imply $F \stackrel{\tau}{\Longrightarrow} F'$ for some F' such that X is totally unguarded in F'. Thus $recX.F \Uparrow$ by Lemma 7.

Now assume that G is arbitrary and that $G\{recX.E/X\} \Uparrow$. By Lemma 5 either $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$). If $G \Uparrow$ then also $G\{recX.F/X\} \Uparrow$. Thus, assume that $G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$. Since $recX.E \Uparrow$, from the previous paragraph we obtain $recX.F \Uparrow$. Finally $G \Longrightarrow H$ and X totally unguarded in H imply $G\{recX.F/X\} \Uparrow$ by Lemma 5.

Now assume that $\phi = \text{⇑\!\!\!\perp}$ and $G\{recX.E/X\}\text{⇑\!\!\!\perp}$. If $G\{recX.E/X\} \Longrightarrow P\perp$, then (†) implies $G\{recX.F/X\} \Longrightarrow Q$ for some Q with $P\,\mathcal{S}\,R \approx^{\text{⇑\!\!\!\perp}} Q$. Since $P\perp$, we also have $R\perp$. Hence, $Q\text{⇑\!\!\!\perp}$. Now assume that $G\{recX.E/X\} \Uparrow$. Again we first consider the case $G \equiv X$, i.e., $recX.E \Uparrow$. As for $\phi = \Uparrow$, we have either $E \Uparrow$ or $E \stackrel{\tau}{\Longrightarrow} E'$ for some E' such that X is totally unguarded in E'. In the latter case we can conclude analogously to $\phi = \Uparrow$ that $recX.F \Uparrow$. Thus, assume that $E \Uparrow$. Then $E \simeq^{\text{⇑\!\!\!\perp}} F$ and Lemma 22(3) imply either $F \Uparrow$ or $F \stackrel{\tau}{\Longrightarrow} F'\perp$ for some F' such that X is weakly guarded in F'. If $F \Uparrow$, then also $recX.F \Uparrow$ by Lemma 7. If $F \stackrel{\tau}{\Longrightarrow} F'\perp$ for some F' such that X is weakly guarded in F', then $F\{recX.F/X\} \stackrel{\tau}{\Longrightarrow} F'\{recX.F/X\}$, i.e, $recX.F \stackrel{\tau}{\Longrightarrow} F'\{recX.F/X\}$, by Lemma 1. Furthermore, since $F'\perp$ and X is weakly guarded in F', we have $F'\{recX.F/X\}\perp$ by Lemma 4.

If G is arbitrary and $G\{recX.E/X\} \Uparrow$, then analogously to the case $\phi = \Uparrow$ either $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$). If $G \Uparrow$, then also $G\{recX.F/X\} \Uparrow$. Thus, assume that $G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$. From the previous paragraph we obtain either $recX.F \Uparrow$ or $recX.F \stackrel{\tau}{\Longrightarrow} Q\perp$ for some Q. If $recX.F \Uparrow$, then $G\{recX.F/X\} \Uparrow$ by Lemma 5. On the other hand, if $recX.F \stackrel{\tau}{\Longrightarrow} Q\perp$, then, since X is totally unguarded in H, $G\{recX.F/X\} \Longrightarrow H\{recX.F/X\} \stackrel{\tau}{\Longrightarrow} Q\perp$ by Lemma 1 and Lemma 2. $\Box$

Eventually, we have all the means to derive that $\simeq^{\phi}$ is a congruence with respect to *rec*.

Corollary 1 If $E, F \in \mathbb{E}$, then $E \simeq^{\phi} F$ implies $recX.E \simeq^{\phi} recX.F$.

Proof. Due to the definition of $\simeq^\phi$ for expressions with free variables, it suffices to consider only those $E, F \in \mathbb{E}$ where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Assume that $E \simeq^\phi F$ holds. Then the relation $\mathcal{S}$ appearing in Lemma 23 is an observational congruence up to $\approx^\phi$. Choosing $G \equiv X$ implies $\langle recX.E, recX.F\rangle \in \mathcal{S}$ and thus $recX.E \simeq^\phi recX.F$ by Lemma 21. $\square$

B Unique solution of guarded equations

Abusing notation relative to Appendix A, we shall for this section redefine the notion of an observational congruence up to $\approx^\phi$ as follows: A symmetric relation $\mathcal{S} \subseteq \mathbb{P} \times \mathbb{P}$ is an observational congruence up to $\approx^\phi$ if $\mathcal{S}$ preserves ϕ and:

(†) if $(P, Q) \in \mathcal{S}$ and $P \stackrel{a}{\Longrightarrow} P'$ for $P, P', Q \in \mathbb{P}$, then $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \approx^\phi R_1 \;\mathcal{S}\; R_2 \approx^\phi Q'$ for some $Q', R_1, R_2 \in \mathbb{P}$

Lemma 24 *If $\mathcal{S}$ is an observational congruence up to $\approx^\phi$, then $\mathcal{S} \subseteq \simeq^\phi$.*

Proof. First we prove that $\approx^\phi \circ\mathcal{S}\circ \approx^\phi$ is a WB$^\phi$, the rest of the proof is analogous to the proof of Lemma 21. Preservation of ϕ by $\approx^\phi \circ\mathcal{S}\circ\approx^\phi$ is clear. In order to show that $\approx^\phi \circ\mathcal{S}\circ\approx^\phi$ is a WB, assume that $P \approx^\phi R_1 \;\mathcal{S}\; R_2 \approx^\phi Q$ and $P \stackrel{a}{\longrightarrow} P'$. Then there exists an R_1' with $R_1 \stackrel{\widehat{a}}{\Longrightarrow} R_1'$ and $P' \approx^\phi R_1'$. The case $a = \tau$ and $R_1 \equiv R_1'$ is clear. Thus, let us assume that $R_1 \stackrel{a}{\Longrightarrow} R_1'$. Then (†) implies $R_2 \stackrel{a}{\Longrightarrow} R_2'$ and $R_1' \approx^\phi \circ\mathcal{S}\circ \approx^\phi R_2'$ for some R_2'. Finally $R_2 \stackrel{a}{\Longrightarrow} R_2'$ and $R_2 \approx^\phi Q$ implies $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $R_2' \approx^\phi Q'$ for some Q'. $\square$

Lemma 25 *Assume that*

- $E \in \mathbb{E}$, $P, Q \in \mathbb{P}$,
- X *is guarded in* E, $\mathbb{V}(E) \subseteq \{X\}$,
- $P \simeq^\phi E\{P/X\}$ *and* $Q \simeq^\phi E\{Q/X\}$.

Then the symmetric relation

$$\mathcal{S} = \{\langle G\{P/X\}, G\{Q/X\}\rangle, \langle G\{Q/X\}, G\{P/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\}$$

is an observational congruence up to $\approx^\phi$.

Proof. The proof for the property (†) is the same as for condition ($*$) in [14, p 159]. It remains to show that $\mathcal{S}$ preserves ϕ. By symmetry it suffices to consider a pair $\langle G\{P/X\}, G\{Q/X\}\rangle$ with $\mathbb{V}(G) \subseteq \{X\}$.

First assume that $\phi = \Uparrow$ and $G\{P/X\} \Uparrow$. Since $G\{P/X\} \simeq^\Uparrow G\{E/X\}\{P/X\}$, also $G\{E/X\}\{P/X\} \Uparrow$. Since X is guarded in $G\{E/X\}$, Lemma 5 implies

$G\{E/X\} \Uparrow$. Thus $G\{E/X\}\{Q/X\} \Uparrow$ and $G\{Q/X\} \Uparrow$.

Now assume that $\phi \in \{\sigma, \perp\}$ and $G\{P/X\} \Longrightarrow P'\phi$. Then there exists R with $G\{E/X\}\{P/X\} \Longrightarrow R\phi$. Since X is guarded in $G\{E/X\}$, we can trace by Lemma 1 and Lemma 3 the transition sequence $G\{E/X\}\{P/X\} \Longrightarrow R$ and obtain an H with $R \equiv H\{P/X\}$, $G\{E/X\}\{Q/X\} \Longrightarrow H\{Q/X\}$, and X guarded in H. Since $H\{P/X\}\phi$ and X is guarded in H this implies with Lemma 4 $H\{Q/X\}\phi$. Finally $G\{E/X\}\{Q/X\} \Longrightarrow H\{Q/X\}\phi$ implies $G\{Q/X\} \Longrightarrow Q'\phi$ for some Q'.

Finally assume that $\phi = \Uparrow\!\!\!\Uparrow$ and $G\{P/X\}\Uparrow\!\!\!\Uparrow$. Thus $G\{E/X\}\{P/X\}\Uparrow\!\!\!\Uparrow$, i.e., $G\{E/X\}\{P/X\} \Uparrow$ or $G\{E/X\}\{P/X\} \Longrightarrow R\perp$. By using the arguments from the previous two paragraphs, we obtain $G\{E/X\}\{Q/X\}\Uparrow\!\!\!\Uparrow$, which finally gives us $G\{Q/X\}\Uparrow\!\!\!\Uparrow$. □

Theorem 12 *Assume that*

- $E \in \mathbb{E}$, $P, Q \in \mathbb{P}$,
- X *is guarded in* E, $\mathbb{V}(E) \subseteq \{X\}$,
- $P \simeq^\phi E\{P/X\}$, *and* $Q \simeq^\phi E\{Q/X\}$.

Then $P \simeq^\phi Q$.

Proof. We obtain the conclusion by choosing $G \equiv X$ in the relation $\mathcal{S}$ defined in Lemma 25. □

Theorem 13 *Assume that*

- $E \in \mathbb{E}$, $P \in \mathbb{P}$,
- X *is guarded in* E, $\mathbb{V}(E) \subseteq \{X\}$, *and*
- $P \simeq^\phi E\{P/X\}$.

Then $P \simeq^\phi recX.E$.

Proof. We have $recX.E \simeq^\phi E\{recX.E/X\}$, thus we can apply Theorem 12 with $Q \equiv recX.E$. □

Using the definition of $\simeq^\phi$ for expressions with free variables, Theorem 12 and Theorem 13 hold for arbitrary expressions from $\mathbb{E}$.

C Soundness of (*rec*5) and (*rec*6)

For (*rec*5) we have to prove that

$$P \equiv recX.(\tau.(X + E) + F) \simeq^{\Uparrow} recX.(\Delta(E + F)) \equiv Q,$$

where w.l.o.g. $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Define the following relations:

$$\begin{aligned}
\mathcal{R}_0 &= \{\langle G\{P/X\}, G\{Q/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\} \\
\mathcal{R}_1 &= \{\langle P + E\{P/X\},\ \Delta(E\{Q/X\} + F\{Q/X\})\rangle\} \\
\mathcal{R} &= \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}
\end{aligned}$$

Thus $\mathcal{R}$ is symmetric. We will show that $(R_1, R_2) \in \mathcal{R}$ and $R_1 \xrightarrow{a} R_1'$ implies $R_2 \overset{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. This implies also that $\mathcal{R}$ preserves $\Uparrow$ and that $\mathcal{R}$ is a WB$^{\Uparrow}$, i.e., $\mathcal{R} \subseteq \approx^{\phi}$. Hence, also the root conditions are satisfied for all pairs in $\mathcal{R}$, thus $\mathcal{R} \subseteq \simeq^{\Uparrow}$. If we choose $G \equiv X$ in $\mathcal{R}_0$ this implies $P \simeq^{\Uparrow} Q$.

In order to prove that $(R_1, R_2) \in \mathcal{R}$ and $R_1 \xrightarrow{a} R_1'$ imply $R_2 \overset{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2', we first consider the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. We treat this case by induction on the size of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$ using a case distinction on the expression G in $\mathcal{R}_0$. Most cases are straight-forward, we only consider the two cases resulting from $G \equiv X$:

Case 1. $R_1 \equiv P \xrightarrow{a} R_1'$ and $R_2 \equiv Q$: Thus $\mathit{rec}X.(\tau.(X + E) + F) \xrightarrow{a} R_1'$, i.e, $\tau.(P + E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_1'$ by a smaller derivation tree. There are two cases:

Case 1.1. $a = \tau$ and $R_1' \equiv P + E\{P/X\}$: We have

$$Q \equiv \mathit{rec}X.(\Delta(E + F)) \xrightarrow{\tau} \Delta(E\{Q/X\} + F\{Q/X\})$$

and $\langle P + E\{P/X\},\ \Delta(E\{Q/X\} + F\{Q/X\})\rangle \in \mathcal{R}$.

Case 1.2. $F\{P/X\} \xrightarrow{a} R_1'$. By induction we obtain $F\{Q/X\} \overset{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $Q \xrightarrow{\tau} \Delta(E\{Q/X\} + F\{Q/X\}) \overset{a}{\Longrightarrow} R_2'$.

Case 2. $R_1 \equiv Q \xrightarrow{a} R_1'$ and $R_2 \equiv P$: Thus $\mathit{rec}X.(\Delta(E + F)) \xrightarrow{a} R_1'$, i.e, $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_1'$ by a smaller derivation tree. There are three cases:

Case 2.1. $a = \tau$ and $R_1' \equiv \Delta(E\{Q/X\} + F\{Q/X\})$. Then $P \xrightarrow{\tau} P + E\{P/X\}$ and $\langle \Delta(E\{Q/X\} + F\{Q/X\}), P + E\{P/X\}\rangle \in \mathcal{R}$.

Case 2.2. $E\{Q/X\} \xrightarrow{a} R_1'$. By induction we obtain $E\{P/X\} \overset{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P \xrightarrow{\tau} P + E\{P/X\} \overset{a}{\Longrightarrow} R_2'$.

Case 2.2. $F\{Q/X\} \xrightarrow{a} R_1'$. By induction we obtain $F\{P/X\} \overset{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus also $P \overset{a}{\Longrightarrow} R_2'$. This concludes the consideration of the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

It remains to consider the case $(R_1, R_2) \in \mathcal{R}_1 \cup \mathcal{R}_1^{-1}$. For this we will make use of the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. There are two cases:

Case 1. $R_1 \equiv P + E\{P/X\}$ and $R_2 \equiv \Delta(E\{Q/X\} + F\{Q/X\})$. Thus we have $P + E\{P/X\} \xrightarrow{a} R_1'$, and we can distinguish the following two cases:

Case 1.1. $P \xrightarrow{a} R_1'$. Since $\langle P, Q\rangle \in \mathcal{R}_0$, we have $Q \xRightarrow{a} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $\Delta(E\{Q/X\} + F\{Q/X\}) \xRightarrow{a} R_2'$.

Case 1.2. $E\{P/X\} \xrightarrow{a} R_1'$. Since $\langle E\{P/X\}, E\{Q/X\}\rangle \in \mathcal{R}_0$, we obtain $E\{Q/X\} \xRightarrow{a} R_2'$ and thus $\Delta(E\{Q/X\} + F\{Q/X\}) \xRightarrow{a} R_2'$ for some R_2' with $(R_1', R_2') \in \mathcal{R}$.

Case 2. $R_1 \equiv \Delta(E\{Q/X\} + F\{Q/X\})$ and $R_2 \equiv P + E\{P/X\}$. We can distinguish the following three cases:

Case 2.1. $a = \tau$ and $R_1' \equiv \Delta(E\{Q/X\} + F\{Q/X\})$. Since $P \xrightarrow{\tau} P + E\{P/X\}$, we have $P + E\{P/X\} \xrightarrow{\tau} P + E\{P/X\}$.

Case 2.2. $E\{Q/X\} \xrightarrow{a} R_1'$. Since $\langle E\{Q/X\}, E\{P/X\}\rangle \in \mathcal{R}_0$, we obtain $E\{P/X\} \xRightarrow{a} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P + E\{P/X\} \xRightarrow{a} R_2'$.

Case 2.3. $F\{Q/X\} \xrightarrow{a} R_1'$. Since $\langle F\{Q/X\}, F\{P/X\}\rangle \in \mathcal{R}_0$, we obtain $F\{P/X\} \xRightarrow{a} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P \xRightarrow{a} R_2'$ and hence $P + E\{P/X\} \xRightarrow{a} R_2'$. This concludes the correctness proof of (*rec*5).

For (*rec*6) we have to prove that

$$P \equiv \mathit{rec}X.(\Delta(X + E) + F) \simeq^{\Uparrow} \mathit{rec}X.(\Delta(E + F)) \equiv Q,$$

where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. We proceed analogously to (*rec*5). Define

$$\begin{aligned}
\mathcal{R}_0 &= \{\langle G\{P/X\}, G\{Q/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\},\\
\mathcal{R}_1 &= \{\langle \Delta(P + E\{P/X\}),\ \Delta(E\{Q/X\} + F\{Q/X\})\rangle\}, \text{ and}\\
\mathcal{R} &= \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}.
\end{aligned}$$

Then it can be shown that $(R_1, R_2) \in \mathcal{R}$ and $R_1 \xrightarrow{a} R_1'$ imply $R_2 \xRightarrow{a} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. The proof of this is analogous to those for (*rec*5) and left to the reader.