Decidability and Complexity in Automatic Monoids

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Developments in Language Theory 2004

Automatic monoids

Idea: Multiplication with generators can be defined by automata.

Let $\mathcal{M} = (M, \circ)$ be a finitely generated monoid.
Then, $\mathcal{M}$ is automatic, if there exists a finite generating set Γ for $\mathcal{M}$ with:

- There exists a regular language $L \subseteq \Gamma^*$ such that
- the canonical morphism $\Gamma^* \to \mathcal{M}$ restricted to L is a bijection and
- for every generator $a \in \Gamma$, the relation $\{(u, v) \in L \times L \mid h(u) \circ a = h(v)\}$ is synchronized rational.

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Synchronized Rational Relations

A binary synchronized rational relation: In order to accept a pair $(u, v) \in \Sigma^* \times \Sigma^*$ a two-tape automaton operates as follows:

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

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q_0

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
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q_1

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

A binary synchronized rational relation: In order to accept a pair $(u, v) \in \Sigma^* \times \Sigma^*$ a two-tape automaton operates as follows:

q_2

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

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						q_m			
v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	#	$\cdots$	#
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

A binary synchronized rational relation: In order to accept a pair $(u, v) \in \Sigma^* \times \Sigma^*$ a two-tape automaton operates as follows:

q_{m+1}

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	#	$\cdots$	#
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

A binary synchronized rational relation: In order to accept a pair $(u, v) \in \Sigma^* \times \Sigma^*$ a two-tape automaton operates as follows:

q_n

v	b_0	b_1	b_2	$\cdots$	b_{m-1}	b_m	$\#$	$\cdots$	$\#$
u	a_0	a_1	a_2	$\cdots$	a_{m-1}	a_m	a_{m+1}	$\cdots$	a_n

Word problems

Let $\mathcal{M}$ be a monoid, finitely generated by the set Γ.

The word problem for $\mathcal{M}$ is the following computational problem:
INPUT: Two words $u, v \in \Gamma^*$
QUESTION: Do u and v represent the same monoid element of the monoid $\mathcal{M}$?

Well-known: For every automatic monoid, the word problem can be solved in quadratic time.

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Complexity of the word problem

Theorem

There exists a fixed automatic monoid with a P-complete word problem.

Proof: Let M be a Turing-machine with a P-complete acceptance problem.

W.l.o.g. assume that:

- The tape is $\# \Box\Box \cdots \Box \$$ when M terminates.
- M operates in a zick-zack way:

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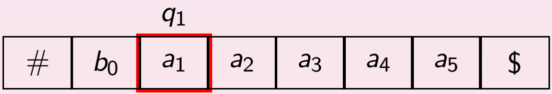

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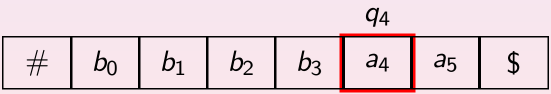

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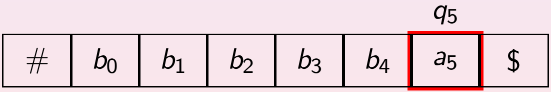

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							q_6
$\#$	b_0	b_1	b_2	b_3	b_4	b_5	$\$$

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		q_{10}					
$\#$	b_0	b_1	c_2	c_3	c_4	c_5	$\$$

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- M operates in a zick-zack way:

	q_{11}						
$\#$	b_0	c_1	c_2	c_3	c_4	c_5	$\$$

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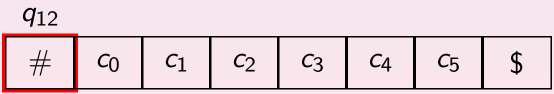

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W.l.o.g. assume that:

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q_{12}

$\#$	c_0	c_1	c_2	c_3	c_4	c_5	$\$$

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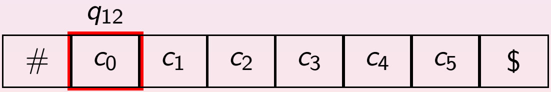

- M makes precisely $p(n)$ complete left-right-transversals for an input of size n.

Complexity of the word problem

We simulate M with a semi-Thue system (string rewriting system) R over an alphabet Γ such that:

- R is terminating and confluent. $\Rightarrow \ \mathrm{IRR}(R) \rightleftharpoons \Gamma^*/R$ bijectively
- For every $a \in \Gamma$, the relation $\{(u, v) \mid u, v \in \mathrm{IRR}(R), ua \xrightarrow{*}_R v\}$ is synchronized rational.

Complexity of the word problem

q_0

#	a_0	a_1	a_2	a_3	a_4	a_5	\$

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$$\# \; \boxed{q_0 \; a_0} \; a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; \$ \; \$ \; \$ \; \cdots$$

Complexity of the word problem

q_1

#	b_0	a_1	a_2	a_3	a_4	a_5	\$

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$$\# \quad \overline{b}_0 \quad q_1 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \$ \quad \$ \quad \$ \quad \cdots$$

Complexity of the word problem

q_1

#	b_0	a_1	a_2	a_3	a_4	a_5	\$

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$$\# \quad \overline{b}_0 \quad \boxed{q_1 \quad a_1} \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \$ \quad \$ \quad \$ \quad \cdots$$

Complexity of the word problem

q_2

$\#$	b_0	b_1	a_2	a_3	a_4	a_5	$\$$

We simulate M with a semi-Thue system (string rewriting system) R over an alphabet Γ such that:

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$\# \quad \overline{b}_0 \quad \overline{b}_1 \quad q_2 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \$ \quad \$ \quad \$ \quad \cdots$

Complexity of the word problem

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$$\# \quad \overline{b}_0 \quad \overline{b}_1 \quad \boxed{q_2 \quad a_2} \quad a_3 \quad a_4 \quad a_5 \quad \$ \quad \$ \quad \$ \quad \cdots$$

Complexity of the word problem

							q_6
$\#$	b_0	b_1	b_2	b_3	b_4	b_5	$\$$

We simulate M with a semi-Thue system (string rewriting system) R over an alphabet Γ such that:

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$$\# \quad \overline{b_0} \quad \overline{b_1} \quad \overline{b_2} \quad \overline{b_3} \quad \overline{b_4} \quad \overline{b_5} \quad \boxed{q_6 \quad \$} \quad \$ \quad \$ \quad \cdots$$

Complexity of the word problem

q_6

#	b_0	b_1	b_2	b_3	b_4	b_5	\$

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$$\# \quad \overline{b}_0 \quad \overline{b}_1 \quad \overline{b}_2 \quad \overline{b}_3 \quad \overline{b}_4 \quad \overline{b}_5 \quad \boxed{\overline{q}_6} \quad \$ \quad \$ \quad \cdots$$

Complexity of the word problem

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Complexity of the word problem

q_7

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Complexity of the word problem

q_8

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$$\# \quad \overline{b}_0 \quad \overline{b}_1 \quad \overline{b}_2 \quad \overline{b}_3 \quad \boxed{\overline{q}_8 \quad c_4} \quad c_5 \quad \$ \quad \$ \quad \cdots$$

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Complexity of the word problem

q_{12}

#	c_0	c_1	c_2	c_3	c_4	c_5	\$

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The semi-Thue system R:

$q\,a \to \overline{b}\,p$ if $\boxed{\begin{matrix} q & \\ a & * \end{matrix} \Rightarrow_M \begin{matrix} & p \\ b & * \end{matrix}}$ $\qquad$ $\overline{a}\,\overline{q} \to \overline{p}\,b$ if $\boxed{\begin{matrix} & q \\ * & a \end{matrix} \Rightarrow_M \begin{matrix} p & \\ * & b \end{matrix}}$

$q\,\$ \to \overline{q}$ for all states q $\qquad$ $\#\,\overline{q} \to \#\,q$ for all states q

Then we have:

- An input w of length n is accepted by the machine M if and only if

$$\# q_0 w \Box^{n-p(n)} \$^{p(n)} \overset{*}{\leftrightarrow}_R \# q_f \Box^{p(n)}$$

  Thus the word problem for Γ^*/R is P-complete.

- For every symbol a, the relation $\{(u,v) \mid u,v \in \mathrm{IRR}(R), ua \overset{*}{\to}_R v\}$ is synchronized rational.

Complexity of the word problem

The semi-Thue system R:

$$q\,a \to \overline{b}\,p \quad \text{if } \boxed{\begin{smallmatrix} q & \\ a & * \end{smallmatrix} \Rightarrow_M \begin{smallmatrix} & p \\ b & * \end{smallmatrix}} \qquad \overline{a}\,\overline{q} \to \overline{p}\,b \quad \text{if } \boxed{\begin{smallmatrix} & q \\ * & a \end{smallmatrix} \Rightarrow_M \begin{smallmatrix} p & \\ * & b \end{smallmatrix}}$$

$$q\,\$ \to \overline{q} \quad \text{for all states } q \qquad \#\,\overline{q} \to \#\,q \text{ for all states } q$$

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Complexity of the word problem

Open problem: Is there an automatic group with a P-complete word problem.

Important subclass of automatic groups: hyperbolic groups.

Cai, 1982: For every hyperbolic group, the word problem belongs to NC^2.

Theorem

For every hyperbolic group, the word problem belongs to $LOGCFL \subseteq NC^2$.

Proof: Every hyperbolic group can be presented as Γ^*/R for a semi-Thue system R that is length-reducing and confluent on ε:
$u \xleftrightarrow{*}_R \varepsilon \iff u \xrightarrow{*}_R \varepsilon$.
The language $\{u \in \Gamma^* \mid u \xrightarrow{*}_R \varepsilon\}$ is growing context-sensitive and hence belongs to LOGCFL (Dahlhaus, Warmuth).

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Cayley-graphs

Let $\mathcal{M}$ be a finitely generated monoid.
Let Γ be a finite generating set of $\mathcal{M}$.
Then, the Cayley-graph of $\mathcal{M}$ w.r.t. is the following edge-labeled graph:

$$\mathcal{C}(\mathcal{M}, \Gamma) = (\mathcal{M}, (\{(u, ua) \mid u \in \mathcal{M}\})_{a \in \Gamma}$$

The Cayley-graph of an automatic monoid is an automatic graph (in the sense of Khoussainov, Nerode).

Consequence: The first-order theory of the Cayley-graph of an automatic monoid is decidable.

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First-order logic

Let $G = (V, (E_a)_{a \in \Gamma})$ be an edge-labeled graph.
Let Ω be an infinite set of variables ranging over V.
The set of all first-order formulas over G is defined as follows:

- $x = y$ and $E_a(x, y)$ are FO-formulas, where $x, y \in \Omega$ and $a \in \Gamma$
- If ϕ and ψ are FO-formulas then also

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \exists x : \phi, \quad \forall x : \phi$$

  are FO-formulas.

A first-order sentence is a first-order formula without free variables.
The first-order theory of G is the set of all first-order sentences that are true in G.

First-order theory of the Cayley-graph

Recall: The first-order theory of the Cayley-graph of an automatic monoid is decidable.

A problem is elementary decidable if it can be solved in time $\mathcal{O}(2^{\cdot^{\cdot^{\cdot^{2^n}}}})$, where the height of this tower of exponents is constant.

Theorem

There exists a fixed automatic monoid $\mathcal{M}$ such that the first-order theory of the Cayley-graph of $\mathcal{M}$ is not elementary decidable.

Proof: Construct a fixed automatic monoid $\mathcal{M}$ such that the the theory of all finite words can be reduced to the first-order theory of $\mathcal{M}$.

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A finitely generated monoid $\mathcal{M}$ has finite geometric type if for some constant c, every $x \in \mathcal{M}$ has degree at most c in the Cayley-graph of $\mathcal{M}$.

Example: Cancellative monoids are of finite geometric type.

Theorem

Let $\mathcal{M}$ be an automatic of finite geometric type. Then the first-order theory of the Cayley-graph of $\mathcal{M}$ is in $\mathrm{DSPACE}(2^{2^{2^{\mathcal{O}(n)}}})$.

Proof: The Cayley-graph of an automatic of finite geometric type is an automatic graph of bounded degree.
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