Logical Aspects of Cayley-Graphs: The Monoid Case

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1 Introduction

Cayley-graphs of groups are a fundamental tool in combinatorial group theory [28, 29] and serve as link between other fields like toplogy, graph theory, and automata theory, see, e.g., the results in [34, 35]. The concept of Cayley-graphs can be easily generalized from groups to monoids: the monoid elements are the vertices and edges result from right-multiplication with generators of the monoid. So far, Cayley-graphs of monoids received less attention than Cayley-graphs of groups; their combinatorial properties were studied in [20–22, 51] and in [44, 45], Cayley-graphs of automatic monoids are investigated.

In a previous a paper [25] we have investigated the logical aspects of Cayley-graphs of groups. Building on the seminal results of Muller and Schupp [34, 35], we have shown that the Cayley-graph of a group G has a decidable MSO-theory if and only if G is context-free (or equivalently, virtually-free). We have shown furthermore that the Cayley-graph of a group G has a decidable first-order theory if and only if G has a decidable word problem.

The results mentioned in the previous paragraph do not carry over to monoids, see, e.g., Proposition 3.3. Our main results about the monoid case state the preservation of the decidability of the first-order (resp. monadic second-order) theory under some well-known algebraic constructions. In order to obtain these results, we use a construction that works for arbitrary relational structures: In [50], Walukiewicz proved that the MSO-theory of the tree-like unfolding (see Section 5.1) of a structure can be reduced to the MSO-theory of the original structure; the original statement goes back to [42, 43, 46]. Using this deep result, we prove in Section 6 that the class of finitely generated monoids, whose Cayley-graphs have decidable MSO-theories, is closed under finite free products (Theorem 4.1(2)). The same result also holds for first-order theories, in fact we will prove a more general preservation theorem in this case. For this, we generalize tree-like unfoldings. This leads us to the notion of a factorized unfolding: the tree-like unfolding of a structure $\mathcal{A}$ consists of the set of words over the set of elements of $\mathcal{A}$. This set of words is equipped with the natural tree structure. Hence the successors of any element of the tree can be identified with the

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elements of $\mathcal{A}$ and can therefore naturally be endowed with the structure of $\mathcal{A}$. Basically, a factorized unfolding is the quotient of this structure with respect to Mazurkiewicz's trace equivalence, in fact, it is a generalization of this quotient (see Section 5.2). In general, the MSO-theory of a factorized unfolding may be undecidable, even in case the underlying structure has a decidable MSO-theory. On the other hand, the first-order theory of a factorized unfolding can be reduced to the first-order theory of the underlying structure (Theorem 5.6). Section 7 is devoted to the proof of this result. It uses a technique of Ferrante and Rackoff [15] and a thorough analysis of factorized unfoldings using ideas from the theory of Mazurkiewicz traces. Based on this result, we will prove in Section 6 that the class of finitely generated monoids, whose Cayley-graphs have decidable first-order theories, is closed under finite graph products (Theorem 4.1(1)). The graph product is a well-known construction in mathematics, see, e.g., [17, 18, 48].

Our results on first-order theories of Cayley-graphs should be also compared with the classical results about first-order theories of monoids: the first-order theory of a monoid $\mathcal{M}$ contains all true first-order statements about $\mathcal{M}$ that are built over the signature containing the monoid operation and all monoid elements as constants. The first-order theory of the Cayley-graph of $\mathcal{M}$ can be seen as a fragment of the whole first-order theory of $\mathcal{M}$ in the sense that only equations of the form $xa = y$, with x and y variables and $a \in \mathcal{M}$ are allowed. In this context we should mention the classical results of Makanin, stating that the existential first-order theory of a free monoid [30] or free group [31] is decidable. In [11] it was shown that under some algebraic restrictions, the decidability of the existential first-order theory is preserved under graph products.

Some of the results of this paper can be also found in the extended abstract [26].

2 Preliminaries

For a binary relation $\to$ on some set, we denote by $\xrightarrow{*}$ the reflexive and transitive closure of $\to$. Let A be an alphabet (finite or infinite). The empty word over A is denoted by ε. For $s \in A^*$ let $|s|$ denote the length of the word s. The set of all $a \in A$ that occur in s is $\mathrm{alph}(s)$. For $s, t \in A^*$ we write $s \preceq t$ if s is a prefix of t.

Relational structures and logic The notion of a structure (or model) is defined as usual in logic, see, e.g., [19]. Here we only consider *relational structures*. Sometimes, we will also use constants, but a constant c can be always replaced by the unary relation $\{c\}$. Let us fix a relational structure $\mathcal{A} = (A, (R_i)_{i \in J})$, where $R_i \subseteq A^{n_i}$, $i \in J$. The *signature of* $\mathcal{A}$ contains the equality symbol $=$, and for every $i \in J$ it contains a relation symbol of arity n_i that we denote without risk of confusion by R_i as well. For $B \subseteq A$ we define the restriction $\mathcal{A}{\restriction}B = (B, (R_i \cap B^{n_i})_{i \in J})$, it is a structure over the same signature as $\mathcal{A}$. Let $\mathcal{A} \backslash B = \mathcal{A}{\restriction}(A \backslash B)$. Given further relations R_j, $j \in K$, $J \cap K = \emptyset$, we also write $(\mathcal{A}, (R_i)_{i \in K})$ for the structure $(A, (R_i)_{i \in J \cup K})$.

Next, let us introduce *monadic second-order logic (MSO-logic)*. Let $\mathbb{V}_1$ be a countably infinite set of *first-order variables* which range over elements of the universe A. First-order variables are denoted x, y, z, x', etc. Let $\mathbb{V}_2$ be a countably infinite set of *second-*

order variables which range over subsets of A. Variables from $\mathbb{V}_2$ are denoted X, Y, Z, X', etc. *MSO-formulas* over the signature of $\mathcal{A}$ are constructed from the atomic formulas $R_i(x_1, \dots, x_{n_i})$, $x = y$, and $x \in X$ (where $i \in J$, $x_1, \dots, x_{n_i}, x, y \in \mathbb{V}_1$, and $X \in \mathbb{V}_2$) using the Boolean connectives $\neg, \wedge$, and $\vee$, and quantifications over variables from $\mathbb{V}_1$ and $\mathbb{V}_2$. The notion of a free occurrence of a variable is defined as usual. A formula without free occurrences of variables is called an *MSO-sentence*. If $\varphi(x_1, \dots, x_n, X_1, \dots, X_m)$ is an MSO-formula such that at most the first-order variables among $x_1, \dots, x_n$ and the second-order variables among $X_1, \dots, X_m$ occur freely in φ, and $a_1, \dots, a_n \in A$, $A_1, \dots, A_m \subseteq A$, then $\mathcal{A} \models \varphi(a_1, \dots, a_n, A_1, \dots, A_m)$ means that φ evaluates to true in $\mathcal{A}$ if the free variable x_i (resp. X_j) evaluates to a_i (resp. A_j). The *MSO-theory* of $\mathcal{A}$, denoted by $\mathrm{MSOTh}(\mathcal{A})$, is the set of all MSO-sentences φ such that $\mathcal{A} \models \varphi$.

A *first-order formula* over the signature of $\mathcal{A}$ is an MSO-formula that does not contain any occurrences of second-order variables. In particular, first-order formulas do not contain atomic subformulas of the form $x \in X$. The *quantifier-depth* of a first-order formula φ is the maximal number of nested quantifiers in φ. The *first-order theory* $\mathrm{FOTh}(\mathcal{A})$ of $\mathcal{A}$ is the set of all first-order sentences φ such that $\mathcal{A} \models \varphi$. With $\Sigma_n(\mathcal{A})$ (resp. $\Pi_n(\mathcal{A})$) (where $n \geq 0$) we denote the set of all sentences in $\mathrm{FOTh}(\mathcal{A})$ of the form $B_1 B_2 \cdots B_n : \varphi$, where φ is a Boolean formula, B_i for i odd is a nonempty block of existential (resp. universal) quantifiers and B_i for i even is a nonempty block of universal (resp. existential) quantifiers.

An important method for proving the decidability of logical theories are interpretations. Let $\mathcal{B}$ be another relational structure with universe B. Then we say that $\mathcal{A}$ is *MSO-interpretable* (resp. *first-order interpretable*) in $\mathcal{B}$ if there exist MSO formulas (resp. first-order formulas) $\psi(x)$ and $\phi_i(\widetilde{x}_i)$ ($i \in J$, $\widetilde{x}_i$ is a tuple of first-order variables of length n_i) over the signature of $\mathcal{B}$ such that the structure $(\psi(x)^{\mathcal{B}}, (\phi_i(\widetilde{x}_i)^{\mathcal{B}})_{i \in J})$ is isomorphic to $\mathcal{A}$. Here $\psi(x)^{\mathcal{B}} = \{b \in B \mid \mathcal{B} \models \psi(b)\}$ and $\phi_i(\widetilde{x}_i)^{\mathcal{B}} = \{\widetilde{c} \in B^{n_i} \mid \mathcal{B} \models \phi_i(\widetilde{c})\}$. It is easy to see that if $\mathcal{A}$ is MSO-interpretable (resp. first-order interpretable) in $\mathcal{B}$ and $\mathrm{MSOTh}(\mathcal{B})$ (resp. $\mathrm{FOTh}(\mathcal{B})$) is decidable, then also $\mathrm{MSOTh}(\mathcal{A})$ (resp. $\mathrm{FOTh}(\mathcal{A})$) is decidable.

Undirected graphs An *undirected graph* is a relational structure $G = (V, E)$, where V is called the set of nodes and $E \subseteq V \times V$ is a symmetric and irreflexive edge relation (thus, undirected graphs do not have self loops). A *path* of length $n \geq 0$ in G between $u \in V$ and $v \in V$ is a sequence $[v_0, v_1, \dots, v_n]$ of nodes such that $v_0 = u$, $v_n = v$, and $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. We write $d_G(u, v)$ for the distance between the nodes $u, v \in V$, i.e., $d_G(u, v)$ is the minimal length of a path between u and v. The *r-sphere, centered at $v \in V$*, is $S_G(r, v) = \{u \in V \mid d_G(v, u) \leq r\}$. For a k-tuple $\widetilde{v} = (v_1, \dots, v_k) \in V^k$ we define $S_G(r, \widetilde{v}) = \bigcup_{i=1}^{k} S_G(r, v_i)$.

Word problems Let $\mathcal{M}$ be a finitely generated monoid and let Γ be a finite generating set for $\mathcal{M}$, i.e., there exists a surjective monoid homomorphism $h : \Gamma^* \to \mathcal{M}$. The *word problem* for $\mathcal{M}$ with respect to Γ is the set $W(\mathcal{M}, \Gamma) = \{(u, v) \in \Gamma^* \times \Gamma^* \mid h(u) = h(v)\}$. The following fact is well-known:

Theorem 2.1. *Let $\mathcal{M}$ be a finitely generated monoid and let Γ_1 and Γ_2 be two finite generating sets for $\mathcal{M}$. Then $W(\mathcal{M}, \Gamma_1)$ is logspace reducible to $W(\mathcal{M}, \Gamma_2)$.*[1]

Thus, the computational complexity of the word problem does not depend on the underlying set of generators.

Mazurkiewicz traces A detailed introduction to the theory of Mazurkiewicz traces can be found in [12]. An *independence alphabet* is a pair (A, I), where A is a possibly infinite set and $I \subseteq A \times A$ is symmetric and irreflexive (thus, it is an undirected graph). The relation I is known as the *independence relation*, its complement $D = (A \times A) \setminus I$ is the *dependence relation*. The pair (A, D) is called a *dependence alphabet*. For $a \in A$, we let $I(a) = \{b \in A \mid (a, b) \in I\}$ and $D(a) = \{b \in A \mid (a, b) \in D\} = A \setminus I(a)$. Let $\equiv_I$ be the smallest congruence on A^* that contains all pairs (ab, ba) with $(a, b) \in I$. The *trace monoid (free partially commutative monoid)* $\mathbb{M}(A, I)$ associated to (A, I) is the quotient monoid $A^*/{\equiv_I}$; its elements are called *traces*. Trace monoids will be one of the few examples of not necessarily finitely generated monoids in this work. Extreme cases are *free monoids* (if $D = A \times A$) and *free commutative monoids* (if $D = \{(a, a) \mid a \in A\}$). Trace monoids were first investigated in [8]. Mazurkiewicz [32] introduced them into computer science.

Let us fix a trace monoid $\mathbb{M} = \mathbb{M}(A, I)$. The trace represented by the word $s \in A^*$ is denoted by $[s]_I$. The neutral element of $\mathbb{M}$ is the empty trace $[\varepsilon]_I$, briefly ε. An element $a \in A$ will be identified with the trace $[a]_I$.

Let $t = [s]_I \in \mathbb{M}$. We define $|t| = |s|$ (the length of t) and $\mathrm{alph}(t) = \mathrm{alph}(s)$. For two traces $t, u \in \mathbb{M}$ we write $(t, u) \in I$ if $\mathrm{alph}(t) \times \mathrm{alph}(u) \subseteq I$. The trace $t \in \mathbb{M}$ can be visualized by its *dependence graph* D_t. To define D_t, choose an arbitrary word $w = a_1 a_2 \cdots a_n$, $a_i \in A$, with $t = [w]_I$ and define $D_t = (\{1, \dots, n\}, E, \lambda)$, where $E = \{(i, j) \mid i < j, (a_i, a_j) \in D\}$ and $\lambda(i) = a_i$. If we identify isomorphic dependence graphs, then this definition is independent of the chosen word representing t. Moreover, the mapping $t \mapsto D_t$ is injective. As a consequence of the representation of traces by dependence graphs, one obtains Levi's Lemma for traces, see, e.g., [12, p 74], which is one of the fundamental facts in trace theory. The formal statement is as follows.

Lemma 2.2. *Let $u_1, \dots, u_m, v_1, \dots, v_n \in \mathbb{M}$. Then $u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$ if and only if there exist $w_{i,j} \in \mathbb{M}$* $(1 \le i \le m,\ 1 \le j \le n)$ such that

- $u_i = w_{i,1} w_{i,2} \cdots w_{i,n}$ for every $1 \le i \le m$,
- $v_j = w_{1,j} w_{2,j} \cdots w_{m,j}$ for every $1 \le j \le n$, and
- $(w_{i,j}, w_{k,\ell}) \in I$ if $1 \le i < k \le m$ and $n \ge j > \ell \ge 1$.

The situation in the lemma will be visualized by a diagram of the following kind. The i–th column corresponds to u_i, the j–th row corresponds to v_j, and the intersection of the i–th column and the j–th row represents $w_{i,j}$. Furthermore $w_{i,j}$ and $w_{k,\ell}$ are independent if one of them is left-above the other one.

[1] See, e.g., [40] for the notion of logspace reducibility.

v_n	$w_{1,n}$	$w_{2,n}$	$w_{3,n}$	$\dots$	$w_{m,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_3	$w_{1,3}$	$w_{2,3}$	$w_{3,3}$	$\dots$	$w_{m,3}$
v_2	$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$\dots$	$w_{m,2}$
v_1	$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$\dots$	$w_{m,1}$
	u_1	u_2	u_3	$\dots$	u_m

A consequence of Levi's Lemma is that trace monoids are cancellative, i.e., $usv = utv$ implies $s = t$ for all traces $s, t, u, v \in \mathbb{M}$.

A subset $L \subseteq \mathbb{M}$ is *recognizable* if there exists a finite monoid S and a monoid homomorphism $h : \mathbb{M} \to S$, which may be assumed to be surjective, such that $L = h^{-1}(h(L))$. The class of recognizable subsets of $\mathbb{M}$ is a Boolean algebra. In case A is finite, it is easy to see that L is recognizable if and only if the language $\{u \in A^* \mid [u]_I \in L\}$ is a regular subset of A^*. Thus, every finite subset of $\mathbb{M}$ is recognizable. Moreover, the recognizable subsets of $\mathbb{M}$ are closed under products [39].

We end this section with a brief discussion of *trace rewriting systems*, which generalize semi-Thue systems [5] from words to traces. Formally, a trace rewriting system over $\mathbb{M}$ is a subset $R \subseteq \mathbb{M} \times \mathbb{M}$. Its *domain* is the set $\mathrm{dom}(R) = \{s \in \mathbb{M} \mid \exists t \in \mathbb{M} : (s, t) \in R\}$ and its *range* is defined dually by $\mathrm{ran}(R) = \{t \in \mathbb{M} \mid \exists s \in \mathbb{M} : (s, t) \in R\}$. We define the *one-step rewrite relation* $\to_R$ on $\mathbb{M}$ as follows: $s \to_R t$ if there exist $u, v \in \mathbb{M}$ and $(\ell, r) \in R$ with $s = u\ell v$ and $t = urv$. The *Thue congruence* $\overset{*}{\leftrightarrow}_R$ is the smallest equivalence relation on $\mathbb{M}$ that contains $\to_R$; it is easily seen to be a congruence on the trace monoid $\mathbb{M}$. Thus, we can define the quotient monoid $\mathbb{M}/_{\overset{*}{\leftrightarrow}_R}$, briefly $\mathbb{M}/_R$. In case $I = \emptyset$, i.e., $\mathbb{M} = A^*$, R is called a *semi-Thue system* over A.

The set $\mathrm{RED}(R)$ is the set of all traces $t \in \mathbb{M}$ such that $t \to_R s$ for some s. The set of *irreducible traces* (with respect to R) is $\mathrm{IRR}(R) = \mathbb{M} \backslash \mathrm{RED}(R)$. The sytem R is *terminating* if there does not exist an infinite chain $s_1 \to_R s_2 \to_R s_3 \to_R \cdots$ in $\mathbb{M}$, it is *length-reducing* if $|s| > |t|$ for all $(s, t) \in R$ and, finally, it is *confluent* if for all $s, t, u \in \mathbb{M}$ with $t \; {}_R\!\overset{*}{\leftarrow} s \overset{*}{\to}_R u$ there exists $v \in \mathbb{M}$ with $t \overset{*}{\to}_R v \; {}_R\!\overset{*}{\leftarrow} u$. It is well-known that R is confluent if and only if R is *Church-Rosser*, i.e., for all $s, t \in \mathbb{M}$, if $s \overset{*}{\leftrightarrow}_R t$, then $s \overset{*}{\to}_R u \; {}_R\!\overset{*}{\leftarrow} t$ for some $u \in \mathbb{M}$, see [5, p 12]. Moreover, if R is terminating and confluent, then for every $s \in \mathbb{M}$ there exists a unique *normal form* $\mathrm{NF}_R(s) \in \mathrm{IRR}(R)$ such that $s \overset{*}{\to}_R \mathrm{NF}_R(s)$ and $s \overset{*}{\leftrightarrow}_R t$ if and only if $\mathrm{NF}_R(s) = \mathrm{NF}_R(t)$. Thus, if A and R are both finite and R is moreover terminating and confluent, then the word problem for $\mathbb{M}/_R$ is decidable. In general, it is undecidable whether a finite length-reducing trace rewriting system is confluent, see [36]. This is in sharp contrast to semi-Thue systems, and makes confluence proofs challenging.

3 Cayley-graphs

In this section we introduce the main concept of this work — Cayley-graphs of monoids, and prove some basic results on these graphs.

Let $\mathcal{M} = (M, \circ, 1)$ be a finitely generated monoid with identity 1 and let Γ be a finite generating set for $\mathcal{M}$. The *Cayley-graph* of $\mathcal{M}$ with respect to Γ is the following relational

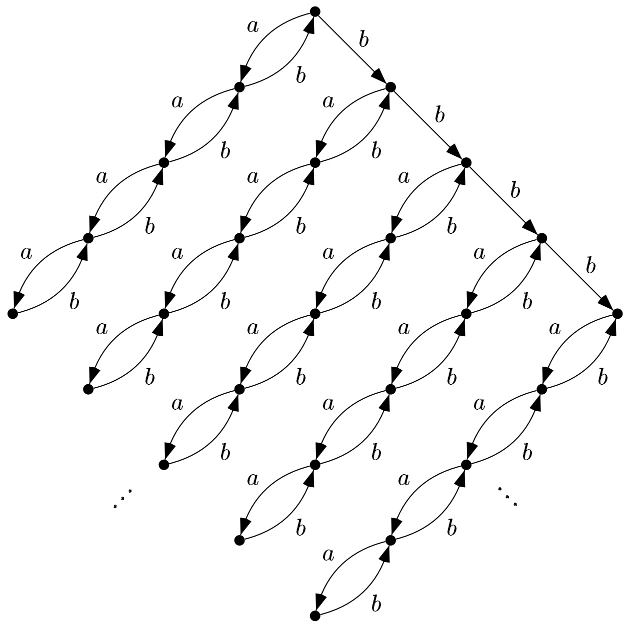

Fig. 1.

structure:

$$\mathcal{C}(\mathcal{M}, \Gamma) = (M, (\{(u,v) \mid u \circ a = v\})_{a \in \Gamma})$$

It is a directed graph, where every edge has a label from Γ and $\{(u,v) \mid u \circ a = v\}$ is the set of a-labeled edges. We express the fact that there exists an a-labeled edge from x to y by writing $x \circ a = y$ or briefly $xa = y$. Since Γ generates $\mathcal{M}$, $\mathcal{C}(\mathcal{M}, \Gamma)$ is (weakly) connected.

Cayley-graphs of groups play an important role in combinatorial group theory [28], see also the surveys of Babai [1] and Schupp [41]. On the other hand, only a few papers deal with Cayley-graphs for monoids. Combinatorial aspects of Cayley-graphs of monoids are studied in [20–22, 51]. In [44, 45], Cayley-graphs of automatic monoids are investigated. The work of Calbrix and Knapik on Thue-specifications [7, 24] covers Cayley-graphs of monoids $\Gamma^*/_R$ with R terminating and confluent as a special case.

Figure 1 and 2 depict some typical Cayley-graphs. Figure 1 shows the Cayley-graph of $\{a,b\}^*/_{\{(ab,\varepsilon)\}}$, with respect to the generating set $\{a,b\}$. Figure 2 shows the Cayley-graph of $\{a,b\}^*/_{\{(ab,baa)\}}$ with respect to $\{a,b\}$. The concrete shape of the Cayley-graph $\mathcal{C}(\mathcal{M}, \Gamma)$ depends heavily on the chosen set of generators Γ. Nevertheless, and similarly to the word problem, the chosen generating set has no influence on the decidability (or complexity) of the first-order (resp. monadic second-order) theory of the Cayley-graph:

Proposition 3.1. *Let Γ_1 and Γ_2 be finite generating sets for the monoid $\mathcal{M}$. Then* $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}, \Gamma_1))$ *is logspace reducible to* $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}, \Gamma_2))$ *and the same holds for the Σ_n-, Π_n-, and MSO-theories.*

Proof. We prove the statement for the first-order theories. The same construction also applies for the other theories mentioned in the proposition. The only difficulty is to find

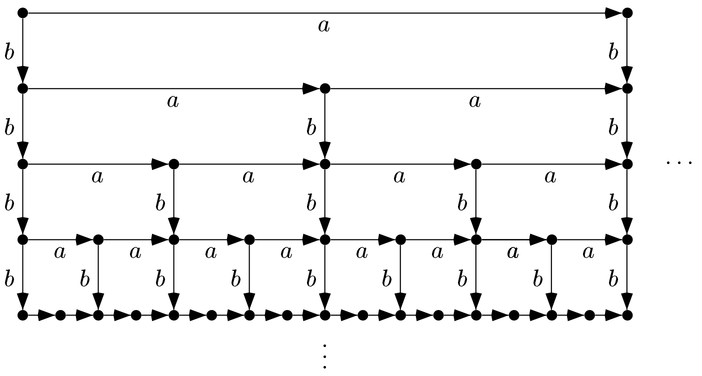

Fig. 2.

a logspace-reduction that preserves the quantifier alternation depth. Given a first-order sentence ϕ_1 over the signature of $\mathcal{C}(\mathcal{M}, \Gamma_1)$ we construct a first-order sentence ϕ_2 over the signature of $\mathcal{C}(\mathcal{M}, \Gamma_2)$ such that $\mathcal{C}(\mathcal{M}, \Gamma_1) \models \phi_1$ if and only if $\mathcal{C}(\mathcal{M}, \Gamma_2) \models \phi_2$ as follows: For $a \in \Gamma_1$, fix some letters $b_1^a, \ldots, b_{n_a}^a \in \Gamma_2$ such that a and $b_1^a \ldots b_{n_a}^a$ describe the same element of the monoid $\mathcal{M}$. Define the first-order formula $\theta(x, (x_k^a)_{\substack{a\in\Gamma_1\\1\le k\le n_a}})$ as

$$\theta(x, (x_k^a)_{\substack{a\in\Gamma_1\\1\le k\le n_a}}) \equiv \bigwedge_{a\in\Gamma_1} x b_1^a = x_1^a \ \wedge \bigwedge_{\substack{a\in\Gamma_1\\1<k\le n_a}} x_{k-1}^a b_k^a = x_k^a.$$

Then we replace simultaneously in ϕ_2 every quantification $\exists x : \psi$ (resp. $\forall x : \psi$) by

$$\exists x\, \exists_{\substack{a\in\Gamma_1\\1\le k\le n_a}} x_k^a : \theta(x, (x_k^a)_{\substack{a\in\Gamma_1\\1\le k\le n_a}}) \wedge \psi$$
$$\text{resp. } \forall x\, \forall_{\substack{a\in\Sigma_1\\1\le k\le n_a}} x_k^a : \theta(x, (x_k^a)_{\substack{a\in\Gamma_1\\1\le k\le n_a}}) \to \psi.$$

Here, the x_k^a are new variables that do not appear in ϕ_1. Finally, to obtain ϕ_2, we replace in the resulting sentence every occurrence of an atomic predicate $xa = y$ by $x_{n_a}^a = y$. It is easy to see that $\mathcal{C}(\mathcal{M}, \Gamma_1) \models \phi_1$ if and only if $\mathcal{C}(\mathcal{M}, \Gamma_2) \models \phi_2$. Moreover, if ϕ_1 is in prenex normal form and we transform ϕ_2 into prenex normal form (for which we only have to pull out all quantifiers), then the two sentences have the same quantifier alternation depth. Finally, note that the sentence ϕ_2 can be produced using only logarithmic space, an observation that completes the proof of this proposition. $\square$

Whenever the specific generating set Γ will be of no importance, we will briefly write $\mathcal{C}(\mathcal{M})$ instead of $\mathcal{C}(\mathcal{M}, \Gamma)$. The next proposition states a simple connection between the word problem and the first-order theory of the Cayley-graph.

Proposition 3.2. *Let $\mathcal{M}$ be a finitely generated monoid such that $\Sigma_1(\mathcal{C}(\mathcal{M}))$ is decidable. Then the word problem for $\mathcal{M}$ is decidable.*

Proof. Choose a finite generating set Γ for $\mathcal{M}$. Two given words $u = a_0a_1 \cdots a_{m-1}$ and $v = b_0b_1 \cdots b_{n-1}$, where $a_i, b_j \in \Gamma$, represent different elements in $\mathcal{M}$ if and only if there exists $x \in \mathcal{M}$ such that the (unique) paths in $\mathcal{C}(\mathcal{M}, \Gamma)$ starting in x and labeled by u and v, respectively, end in different nodes. This fact can be easily expressed by an Σ_1-sentence of first-order logic:

$$\exists x_0 \cdots \exists x_m \exists y_0 \cdots \exists y_n \left\{ \begin{array}{l} x_0 = y_0 \ \wedge \ x_m \neq y_n \ \wedge \\ \bigwedge\limits_{0 \leq i < m} x_i a_i = x_{i+1} \ \wedge \ \bigwedge\limits_{0 \leq i < n} y_i b_i = y_{i+1} \end{array} \right\}$$

$\square$

Note that Book [4] has shown that the word problem for a monoid $\Gamma^*/_R$, where Γ and R are finite and R is length-reducing and confluent, can be solved in linear time. Hence the following proposition shows that the converse implication of Proposition 3.2 becomes false.

Proposition 3.3. *There exists a fixed finite, length-reducing, and confluent semi-Thue system R over a finite alphabet Γ such that $\Sigma_1(\mathcal{C}(\Gamma^*/_R))$ is undecidable.*

Proof. By [37, Thm. 2.4] there exists a fixed finite, length-reducing, and confluent semi-Thue system R over Γ such that the common right-multiplier problem is undecidable for $\Gamma^*/_R$,[2] which is the following problem:
INPUT: Words $u, v \in \Gamma^*$
QUESTION: Does there exist $x \in \Gamma^*$ with $xu \overset{*}{\leftrightarrow}_R xv$?
But this is a Σ_1-property of the Cayley-graph of $\Gamma^*/_R$ that can be constructed effectively from u and v. This proves the proposition. $\square$

The next result was shown in [10]. A semi-Thue system R over Γ is *left-basic* if it satisfies the following two conditions:

- if $\ell \in \mathrm{dom}(R)$, $r \in \mathrm{ran}(R)$, and $r = u\ell v$, then $u = v = \varepsilon$.
- if $\ell \in \mathrm{dom}(R)$, $r \in \mathrm{ran}(R)$, $ur = \ell v$, and $|\ell| > |u|$, then $v = \varepsilon$.

A semi-Thue system R over a finite alphabet Γ is *regular* if R can be written as $R = \bigcup_{i=1}^{n} L_i \times R_i$ where both $L_i \subseteq \Gamma^*$ and $R_i \subseteq \Gamma^*$ are regular for $1 \leq i \leq n$.

Proposition 3.4. *Let R be a terminating, confluent, left-basic, and regular semi-Thue system R over a finite alphabet Γ. Then* $\mathrm{MSOTh}(\mathcal{C}(\Gamma^*/_R))$ *is decidable.*

4 Graph products

In this section we will introduce graph products of monoids. The graph product construction generalizes both the free product and the direct product. Graph products were

[2] In [37, Thm. 2.4] undecidability is stated for the common left-multiplier problem, but by reversing the words in R, undecidability is also obtained for the common right-multiplier problem.

introduced in [17]. Our main result will state that the decidability of the first-order theory (resp. MSO-theory) of the Cayley-graph is preserved under graph products (resp. free products). Other closure results for graph products can be found for instance in [18, 27, 48, 49].

Let (Σ, I_Σ) be a finite independence alphabet, i.e., Σ is finite, and let $\mathcal{M}_\sigma = (M_\sigma, \circ_\sigma, 1_\sigma)$ be a finitely generated monoid for every $\sigma \in \Sigma$. Let $A_\sigma = M_\sigma \setminus \{1_\sigma\}$, where w.l.o.g. $A_\sigma \cap A_\tau$ for $\sigma \neq \tau$. Define an independence alphabet (A, I) by

$$A = \bigcup_{\sigma \in \Sigma} A_\sigma \quad \text{and} \quad I = \bigcup_{(\sigma,\tau) \in I_\Sigma} A_\sigma \times A_\tau.$$

Let R_σ for $\sigma \in \Sigma$ and R be the following trace rewriting systems over $\mathbb{M}(A, I)$:

$$\begin{aligned} R_\sigma &= \{(ab, c) \mid a, b, c \in A_\sigma, a \circ_\sigma b = c\} \cup \{(ab, \varepsilon) \mid a, b \in A_\sigma, a \circ_\sigma b = 1_\sigma\} \\ R &= \bigcup_{\sigma \in \Sigma} R_\sigma. \end{aligned}$$

Then the *graph product* $\mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma \in \Sigma})$ is the quotient monoid $\mathbb{M}(A, I)/_R$. Special cases are the free product $*_{\sigma \in \Sigma} \mathcal{M}_\sigma$ (if $I_\Sigma = \emptyset$) and the *direct product* $\prod_{\sigma \in \Sigma} \mathcal{M}_\sigma$ (if $I_\Sigma = (\Sigma \times \Sigma) \setminus \{(\sigma, \sigma) \mid \sigma \in \Sigma\}$). If every $\mathcal{M}_\sigma$ is generated by Γ_σ, then from the definition of $\mathbb{P}$ it is obvious that $\mathbb{P}$ is generated by $\Gamma = \bigcup_{\sigma \in \Sigma} \Gamma_\sigma$. In fact, if the monoid $\mathcal{M}_\sigma$ is isomorphic to $\Gamma_\sigma^*/_{S_\sigma}$, then, using Tietze transformations, it is easy to see that $\mathbb{P}$ is isomorphic to $\Gamma^*/_S$, where $S = \bigcup_{\sigma \in \Sigma} S_\sigma \cup \{(ab, ba) \mid a \in \Gamma_\sigma, b \in \Gamma_\tau, (\sigma, \tau) \in I_\Sigma\}$.

The following theorem is the main result of this section. For technical reasons, we will add in the further discussion the neutral element of a monoid as a constant to the Cayley-graph. Thus, define the *rooted Cayley-graph* of a finitely generated monoid $\mathcal{M}$ as the rooted graph $(\mathcal{C}(\mathcal{M}), 1)$, where 1 is the neutral element of $\mathcal{M}$.

Theorem 4.1. *Let $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma \in \Sigma})$, where $\mathcal{M}_\sigma$ is finitely generated.*

(1) If $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ is decidable for all $\sigma \in \Sigma$, then also $\mathrm{FOTh}(\mathcal{C}(\mathbb{P}), 1)$ is decidable.
(2) If $I_\Sigma = \emptyset$ and $\mathrm{MSOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ is decidable for all $\sigma \in \Sigma$, then also $\mathrm{MSOTh}(\mathcal{C}(\mathbb{P}), 1)$ is decidable.

Before we go into the details of the proof of Theorem 4.1 (that can be found in Section 6) let us first state some consequences and limitations.

In [25] the authors have shown that for a finitely generated group $\mathcal{G}$, $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$ is decidable if and only if the word problem of $\mathcal{G}$ is decidable. Together with Statement (1) in Theorem 4.1 we obtain the following result:

Corollary 4.2. *Let $\mathcal{M}$ be a graph product of automatic monoids and groups with decidable word problems. Then $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}))$ is decidable.*

Statement (1) in Theorem 4.1 does not generalize to MSO-theories:

Proposition 4.3. *Let $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma \in \Sigma})$, where $\mathcal{M}_\sigma$ is nontrivial and finitely generated by Γ_σ. If $\mathrm{MSOTh}(\mathcal{C}(\mathbb{P}), 1)$ is decidable, then:*

- (Σ, I_Σ) *does not contain an induced cycle of length 4 (also called C4),*
- *if* $(\sigma, \tau) \in I_\Sigma$ *and* $\mathcal{M}_\sigma$ *is infinite, then* $\mathcal{M}_\tau$ *is finite,*
- *if* $(\sigma, \sigma_1), (\sigma, \sigma_2) \in I_\Sigma$, $\sigma_1 \neq \sigma_2$, *and* $\mathcal{M}_\sigma$ *is infinite, then* $(\sigma_1, \sigma_2) \in I_\Sigma$, *and*
- $\mathrm{MSOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ *is decidable for every* $\sigma \in \Sigma$.

Proof. If one of the first three conditions is not satisfied, then $\mathbb{P}$ contains a submonoid of the form $\mathcal{M}_1 \times \mathcal{M}_2$, where both $\mathcal{M}_1$ and $\mathcal{M}_2$ are infinite (note that we assume that every $\mathcal{M}_\sigma$ is nontrivial). Since $\mathcal{C}(\mathcal{M}_i)$ is infinite and every node has finite outdegree, we find an infinite path $a_{i,1} \to a_{i,2} \to \cdots$ in $\mathcal{C}(\mathcal{M}_i)$. In $\mathcal{M}_1 \times \mathcal{M}_2 \subseteq \mathbb{P}$, these two paths generate an infinite grid. Hence the MSO-theory of $\mathcal{C}(\mathbb{P})$ is undecidable, see, e.g., [16].

Next we show that $\mathrm{MSOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ is decidable for every $\sigma \in \Sigma$ in case $\mathrm{MSOTh}(\mathcal{C}(\mathbb{P}), 1)$ is decidable. Note that $\mathcal{M}_\sigma$ is the least subset of $(\mathcal{C}(\mathbb{P}), 1)$ containing 1 that is closed under a-successors for $a \in \Gamma_\sigma$; hence M_σ is MSO-definable in $(\mathcal{C}(\mathbb{P}), 1)$. Since an MSO sentence φ holds in $(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ if and only if its restriction to $\mathcal{M}_\sigma \subseteq \mathbb{P}$ holds in $(\mathcal{C}(\mathbb{P}), 1)$, the result follows. $\square$

In order to prove Theorem 4.1, we will introduce in the next section two general unfolding operations that work for arbitrary relational structures.

5 Unfoldings

5.1 Tree-like unfoldings

In [42] Semenov introduced the following construction, which he attributes to An. A. Muchnik and which generalizes a construction from [43, 46].

Definition 5.1. *Let* $\mathcal{A} = (A, (R_i)_{1 \le i \le \kappa})$ *be a relational structure with finitely many relations, where the relation* R_i *has arity* n_i. *On the set of finite words* A^*, *we define the following relations:*

$$\begin{aligned} \widehat{R_i} &= \{(ua_1, ua_2, \ldots, ua_{n_i}) \mid u \in A^*, (a_1, a_2, \ldots, a_{n_i}) \in R_i\} \\ \mathrm{suc} &= \{(u, ua) \mid u \in A^*, a \in A\} \\ \mathrm{cl} &= \{(ua, uaa) \mid u \in A^*, a \in A\}\ ^3 \end{aligned}$$

The relational structure $\widehat{\mathcal{A}} = (A^*, (\widehat{R_i})_{1 \le i \le \kappa}, \mathrm{suc}, \mathrm{cl})$ *is called the* tree-like unfolding *of* $\mathcal{A}$.

One can think of the structure $\widehat{\mathcal{A}}$ as a tree (A^*, suc) together with some additional relations. Any tuple of elements of A^* that appears in one of the additional relations is "local": the distance between any two entries in the tree (A^*, suc) is at most 2. The term tree-like unfolding comes from the fact that $\widehat{\mathcal{A}}$ is an extension of the tree (A^*, suc).

In [42], Semenov also sketched a proof of the following result, which he attributes to An. A. Muchnik. A complete proof was given by Walukiewicz [50].

[3] "cl" stands for "clone".

Theorem 5.2 (cf. [50]). $\mathrm{MSOTh}(\widehat{\mathcal{A}})$ *can be reduced to* $\mathrm{MSOTh}(\mathcal{A})$.

The relations of the tree-like unfolding are instances of a more general construction, which will be crucial for our notion of factorized unfoldings: Let φ be a first-order formula over the signature of $\mathcal{A}$ with $\sum_{i=1}^{n} k_i$ free variables, where $k_i \in \mathbb{N}$ ($k_i = 0$ is allowed). For words $u_i = a_{i,1}a_{i,2}\cdots a_{i,k_i}$ ($a_{i,j} \in A$) of length k_i ($1 \leq i \leq n$) we write $\mathcal{A} \models \varphi(u_1, u_2, \ldots, u_n)$ if

$$\mathcal{A} \models \varphi(a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, \ldots, a_{n,1}, \ldots, a_{n,k_n}).$$

An n-ary relation R over A^* is *k-suffix definable* in $\mathcal{A}$ if there are $k_1, \ldots, k_n \leq k$ and a first-order formula φ over the signature of $\mathcal{A}$ with $\sum_{i=1}^{n} k_i$ free variables such that

$$R = \{(uu_1,\ uu_2,\ \ldots, uu_n) \mid u, u_i \in A^*, |u_i| = k_i, \mathcal{A} \models \varphi(u_1, u_2, \ldots, u_n)\}.$$

Note that a formula φ with m free variables defines many different k-suffix definable relations, namely one for each partition of the number m.

All relations of $\widehat{\mathcal{A}}$ are 2-suffix definable in $\mathcal{A}$. On the other hand, there exist 2-suffix definable relations such that adding them to $\widehat{\mathcal{A}}$ makes Theorem 5.2 fail: To see this, let

$$\mathrm{cp} = \{(ua, uba) \mid u \in A^*, a, b \in A\},$$

which is 2-suffix definable in $\mathcal{A}$.[4] Recall that $\preceq$ denotes the prefix order on A^*; thus it is the reflexive transitive closure of the relation suc from $\widehat{\mathcal{A}}$ and therefore MSO definable in $\widehat{\mathcal{A}}$.

Proposition 5.3. *Let* $S = \{(n, n+1) \mid i \in \mathbb{N}\}$ *be the successor relation on* $\mathbb{N}$. *Then* $\mathrm{MSOTh}(\mathbb{N}, S)$ *is decidable but* $\mathrm{FOTh}(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp})$ *is undecidable.*

Proof. The decidability of $\mathrm{MSOTh}(\mathbb{N}, S)$ was shown by Büchi [6]. For the undecidability of $\mathrm{FOTh}(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp})$ recall that it is undecidable whether a given two-counter machine (with zero-tests), started with empty counters, finally terminates. Thus, let us fix a two-counter machine $\mathcal{CM}$ with initial state q_0 and final state $q_f \neq q_0$. We will construct a first-order sentence $\phi_{\mathcal{CM}}$ such that $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \phi_{\mathcal{CM}}$ if and only if $\mathcal{CM}$, started in the configuration $(q_0, 0, 0)$ terminates. We can assume that the state space Q of $\mathcal{CM}$ is $\{1, \ldots, \lambda\}$ for some $\lambda \in \mathbb{N}$. Then a computation of $\mathcal{CM}$ starting in $(q_0, 0, 0)$ can be encoded by a sequence of the form $q_0\, m_0\, n_0\, 0\, q_1\, m_1\, n_1\, 0 \cdots q_k\, m_k\, n_k \in \mathbb{N}^*$ such that $q_i \in Q$, $m_i, n_i \geq 1$, $m_0 = n_0 = 1$, and $(q_{i+1}, m_{i+1} - 1, n_{i+1} - 1)$ is a successor configuration of $(q_i, m_i - 1, n_i - 1)$ for $\mathcal{CM}$.

First, note that the relation suc from the tree-like unfolding is first-order definable in $(\mathbb{N}^*, \preceq)$. For a fixed $n \in \mathbb{N}$, it is also easy to write down a formula $\psi_n(x)$ such that $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \psi_n(w)$ if and only if $w = v\,n$ for some $v \in \mathbb{N}^*$: We start with $\psi_0(x) \equiv \neg\exists y : \widehat{S}(y, x) \ \wedge\ \exists z : \mathrm{suc}(z, x)$ and define inductively $\psi_n(x) \equiv \exists y : \psi_{n-1}(y) \ \wedge\ \widehat{S}(y, x)$. Next, using the prefix relation $\preceq$ and the formulas ψ_n for $0 \leq n \leq \lambda$, we can construct a first-order formula $\phi_0(x)$ such that $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \phi_0(w)$ if and only if $w \in \mathbb{N}^*$ has

[4] "cp" stands for "copy".

the form $q_0\, m_0\, n_0\, 0\, q_1\, m_1\, n_1\, 0 \cdots q_k\, m_k\, n_k$ with $q_i \in Q$, $m_i, n_i \geq 1$, $m_0 = n_0 = 1$, and $q_k = q_f$. Furthermore, we claim that there exists a formula $\phi_1(x)$ such that for every $w \in \mathbb{N}^*$ we have $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \phi_1(w)$ if and only if for every prefix $v\, p\, k\, \ell\, 0\, q\, m\, n \preceq w$ with $p, k, \ell, q, m, n \geq 1$, the configuration $(q, m-1, n-1)$ is a successor configuration of $(p, k-1, \ell-1)$, i.e., one of the finitely many transition rules of $\mathcal{CM}$ transforms the configuration $(p, k-1, \ell-1)$ into $(q, m-1, n-1)$. Let us consider one such transition rule, saying, e.g., that if $\mathcal{CM}$ is in state $q_1 \in Q$, then $\mathcal{CM}$ can move into state $q_2 \in Q$ and add 1 to the first counter. It suffices to construct a formula $\theta(x)$ such that for every word w of the form $v\, p\, m\, n\, 0\, q\, k\, \ell$ with $v \in \mathbb{N}^*$, we have $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \theta(w)$ if and only if $p = q_1$, $q = q_2$, $\ell = n$, and $k = m + 1$.

It is easy to express $p = q_1$ and $q = q_2$ using ψ_{q_1} and ψ_{q_2}. Thus, it remains to express $k = m+1$ (and $\ell = n$, which can be done analogously): This is the case if and only if there are $x_i, y_i \in \mathbb{N}^*$ for $0 \leq i \leq 4$ such that

$$\mathrm{suc}(x_4, w) \;\wedge\; y_4 = x_4; \wedge\, \widehat{S}(x_0, y_0) \;\wedge\; \bigwedge_{i=1}^{4} (\mathrm{suc}(x_{i-1}, x_i) \wedge \mathrm{cp}(y_{i-1}, y_i))\,.$$

This formula expresses that

(1) $x_0, x_1, x_2, x_3, x_4, w$ is a successor sequence, i.e., $x_4 = v\, p\, m\, n\, 0\, q\, k$, $x_3 = v\, p\, m\, n\, 0\, q$, $\ldots$, $x_0 = v\, p\, m$,
(2) $y_0 = v\, p\, (m+1)$ because of $\widehat{S}(x_0, y_0)$,
(3) $y_4 = x_4$, i.e., $y_4 = v\, p\, m\, n\, 0\, q\, k$,
(4) and y_{i-1} is obtained from y_i by deleting the penultimate letter, i.e., $y_3 = v\, p\, m\, n\, 0\, k$, $y_2 = v\, p\, m\, n\, k$, $y_1 = v\, p\, m\, k$, and $y_0 = v\, p\, k$.

Since $v\, p\, k = y_0 = v\, p\, (m+1)$ the formula expresses indeed $k = m + 1$. Other transitions can be dealt with similarly. Hence, we have $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp}) \models \exists x : \phi_0(x) \;\wedge\; \phi_1(x)$ if and only if $\mathcal{CM}$ reaches the final state q_f from the initial configuration $(q_0, 0, 0)$. This proves the theorem. □

Since $\preceq$ is MSO-definable in the presence of suc, the previous proposition implies that $\mathrm{MSOTh}(\mathbb{N}^*, \widehat{S}, \mathrm{suc}, \mathrm{cp})$ is undecidable. Thus, the presence of the relation cp makes Walukiewicz's result fail.

Recall that the underlying set of the tree-like unfolding of a structure $\mathcal{A}$ is the set of all finite words over the carrier set of $\mathcal{A}$. In factorized unfoldings that we introduce next, this underlying set consists of Mazurkiewicz traces.

5.2 Factorized unfoldings

Let (A, I) be an independence alphabet. For an n-ary relation R over A^*, we define its I-quotient

$$R/_I = \{([u_1]_I, \ldots, [u_n]_I) \mid (u_1, \ldots, u_n) \in R\}.$$

For instance, $\preceq/_I$ is the prefix order on traces.

Definition 5.4. *Let $\mathcal{A}$ be a relational structure with carrier set A. Let furthermore*

- *$I \subseteq A \times A$ be an independence relation that is first-order definable in $\mathcal{A}$,*
- *$\eta : \mathbb{M}(A, I) \to S$ be a monoid homomorphism into some finite monoid S such that $\eta^{-1}(q) \cap A$ is first-order definable in $\mathcal{A}$ for every $q \in S$, and*
- *R_i be a k_i-suffix definable relation in $\mathcal{A}$ for $1 \leq i \leq \kappa$.*

Then the structure $\mathcal{B} = (\mathbb{M}(A, I), (\eta^{-1}(q))_{q\in S}, (R_i/_I)_{1\leq i\leq\kappa})$ is a factorized unfolding of $\mathcal{A}$, *it is also called the* factorized unfolding of $\mathcal{A}$ corresponding to I, η, and $(R_i)_{1\leq i\leq\kappa}$.

Note that in contrast to the tree-like unfolding there are many different factorized unfoldings of $\mathcal{A}$.

The notion of a factorized unfolding is a proper generalization of the tree-like unfolding, also in case $I = \emptyset$ in Definition 5.4: By Proposition 5.3, the relation cp cannot be defined in the tree-like unfolding $\widehat{\mathcal{A}}$, but since it is 2-suffix definable it may be part of a factorized unfolding. On the other hand, for the relations $\eta^{-1}(q)$ in the above definition we have the following:

Lemma 5.5. *Let* $\mathrm{MSOTh}(\mathcal{A})$ *be decidable and $\eta : A^* \to S$ be a monoid morphism into a finite monoid S such that $\eta^{-1}(q) \cap A$ is MSO-definable in $\mathcal{A}$ for every $q \in S$. Then also* $\mathrm{MSOTh}(\widehat{\mathcal{A}}, (\eta^{-1}(q))_{q\in S})$ *is decidable.*

Proof. Since $P_q := \eta^{-1}(q) \cap A$ is MSO-definable in $\mathcal{A}$, the structure $\mathcal{B} = (\mathcal{A}, (P_q)_{q\in S})$ has a decidable MSO-theory. Hence, by Theorem 5.2, also $\mathrm{MSOTh}(\widehat{\mathcal{B}})$ is decidable and it suffices to prove that $\eta^{-1}(q) \subseteq A^*$ is MSO-definable in $\widehat{B}$. But $x \in \eta^{-1}(q)$ if there is a partition $(X_s)_{s\in S}$ of the universe A^* with $\varepsilon \in X_1$ (where 1 is the unit of the monoid S), $x \in X_q$ and, for all $(y, z) \in \mathrm{suc}$: if $y \in X_{s_1}$ and $z \in \widehat{P_s}$, then $z \in \widehat{P_{s_1 s}}$. □

The following theorem is our main result for factorized unfoldings.

Theorem 5.6. *Let $\mathcal{A}$ be a relational structure and let $\mathcal{B}$ be the factorized unfolding of $\mathcal{A}$ corresponding to I, η, and $(R_i)_{1\leq i\leq\kappa}$, where $\{I(a) \mid a \in A\} \subseteq 2^A$ is finite. Then any first-order sentence φ of quantifier alternation depth d over the signature of $\mathcal{B}$ can be transformed effectively into a sentence θ of quantifier alternation depth $d + O(1)$ and size $2^{2^{O(|\varphi|)}}$ over the signature of $\mathcal{A}$ such that $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \models \theta$.*

Remark 5.7. We postpone the lengthy proof of this theorem to Section 7. It will also show that not only the size of θ is bounded doubly exponential in the size of φ, but also the time needed to construct θ from φ is bounded doubly exponential in $|\varphi|$.

In case A is finite, a much simpler proof using automatic structures is given as Theorem 8.2. By Theorem 8.1 below, the finiteness of $\{I(a) \mid a \in A\}$ is necessary for Theorem 5.6 to hold.

Corollary 5.8. *Let $\mathcal{A}$ be a relational structure with a decidable first-order theory. Let $\mathcal{B}$ be the factorized unfolding of $\mathcal{A}$ corresponding to I, η, and $(R_i)_{1\leq i\leq\kappa}$, where $\{I(a) \mid a \in A\} \subseteq 2^A$ is finite. Then* $\mathrm{FOTh}(\mathcal{B})$ *is decidable.*

Remark 5.9. Let $(R_i)_{i\in\mathbb{N}}$ be a list of all relations that are k-suffix definable in $\mathcal{A}$ for some $k \in \mathbb{N}$. Moreover, let $(L_i)_{i\in\mathbb{N}}$ be a list of all subsets $\eta^{-1}(q) \subseteq \mathbb{M}(A, I)$ such that $\eta : \mathbb{M}(A, I) \to S$ is a homomorphism into a finite monoid S, $q \in S$, and $\eta^{-1}(p) \cap A$ is first-order definable in $\mathcal{A}$ for every $p \in S$. Note that every L_i is recognizable. Then also the first-order theory of $\mathcal{B} = (\mathbb{M}(A, I), (L_i)_{i\in\mathbb{N}}, (R_i/_I)_{i\in\mathbb{N}})$ is decidable. The important point is that any first-order sentence over the signature of $\mathcal{B}$ can mention only finitely many relations $R_i/_I$ and L_i. Thus, it suffices to work in a suitable reduct of $\mathcal{B}$ with only finitely many relations, which can be handled by Theorem 5.6 provided the lists above are recursive enumerations. This is no problem for the list $(R_i)_{i\in\mathbb{N}}$ since any such relation is uniquely given by a tuple $(k_1, k_2, \dots, k_n)$ of natural numbers and a first-order formula φ with $k_1 + \cdots + k_n$ free variables. For the languages L_i, we list all finite monoids S with distinguished element $q \in S$ and all tuples $(\varphi_p)_{p\in S}$ of first-order formulas with one free variable such that

(1) the sets $\{a \in A \mid \mathcal{A} \models \varphi_p(a)\}$ for $p \in S$ form a partition of A and
(2) if $\mathcal{A} \models \varphi_{p_1}(a) \wedge \varphi_{p_2}(b)$ and $(a, b) \in I$, then $p_1p_2 = p_2p_1$ in S.

These tuples can be enumerated since the first-order theory of $\mathcal{A}$ is decidable and I is first-order definable in $\mathcal{A}$. By the second requirement, any such tuple encodes an homomorphism η from $\mathbb{M}(A, I)$ into S satisfying $\eta^{-1}(p) \cap A = \{a \in A \mid \mathcal{A} \models \varphi_p(a)\}$. Hence from this enumeration of tuples, we obtain an effective enumeration of the sets L_i as required.

6 Proof of Theorem 4.1

Using Theorem 5.2 and Theorem 5.6, we will give a proof of Theorem 4.1 in this section. Let us fix a graph product $\mathbb{P} = \mathbb{P}(\Sigma, I_\Sigma, (\mathcal{M}_\sigma)_{\sigma\in\Sigma})$ for the further discussion. Define A_σ, A, I, R_σ, and R as in Section 4. The crucial fact for our further investigation is the following:

Lemma 6.1. *The trace rewriting system R over $\mathbb{M}(A, I)$ is confluent.*

Proof. Since R is terminating, it suffices by Newman's Lemma [38] to show that R is *locally confluent*, i.e., for all $s, s_1, s_2 \in \mathbb{M}(A, I)$ with $s_1 \,{}_R\!\leftarrow s \to_R s_2$ there exists $s' \in \mathbb{M}(A, I)$ with $s_1 \xrightarrow{*}_R s' \,{}_R\!\xleftarrow{*} s_2$.

Thus, assume that $s \to_R s_1$ and $s \to_R s_2$. Hence, $s = t_ia_ib_iu_i$ and $s_i = t_ir_iu_i$ for $i = 1, 2$, where $(a_ib_i, r_i) \in R$. Thus, $r_i \in A \cup \{\varepsilon\}$. By applying Levi's Lemma 2.2 to the identity $t_1a_1b_1u_1 = t_2a_2b_2u_2$, we obtain the following diagram:

$$\begin{array}{|c||c|c|c|}
\hline
u_2 & w_2 & q_1 & v_2 \\
\hline
a_2b_2 & p_2 & t & q_2 \\
\hline
t_2 & v_1 & p_1 & w_1 \\
\hline\hline
 & t_1 & a_1b_1 & u_1 \\
\end{array}$$

Thus, $(w_1, w_2) \in I$. For the further arguments it is easy to see that we may assume $v_1 = v_2 = \varepsilon$. Assume that $a_i, b_i \in A_{\sigma_i}$. Let us first consider the case $t \neq \varepsilon$. Thus, $\sigma_1 = \sigma_2 = \sigma$, $r_1, r_2 \in A_\sigma \cup \{\epsilon\}$, and $(c, w_i) \in I$ for all $c \in A_\sigma$ and $i = 1, 2$. Moreover, since $(p_1, p_2) \in I$ but both traces only contain symbols from A_σ, we have $p_1 = \varepsilon$ or $p_2 = \varepsilon$ and similarly $q_1 = \varepsilon$ or $q_2 = \varepsilon$. If $p_1 = p_2 = q_1 = q_2 = \varepsilon$ then $s_1 = s_2$. Otherwise, since a_1b_1 cannot be a proper factor of a_2b_2 and vice versa, we obtain up to symmetry the following diagram:

u_2	w_2	b_1	ε
a_2b_2	a_2	$b_2 = a_1$	ε
t_2	ε	ε	w_1
	t_1	a_1b_1	u_1

Thus, $s_1 = a_2w_2r_1w_1 = w_1a_2r_1w_2$ and $s_2 = w_1r_2w_2b_1 = w_1r_2b_1w_2$. Finally, by definition of the system R_σ it follows that a_2r_1 and r_2b_1 can be reduced to $a_2 \circ_\sigma b_2 \circ_\sigma b_1 = a_2 \circ_\sigma a_1 \circ_\sigma b_1$. This concludes the case $t \neq \varepsilon$.

Now assume that $t = \varepsilon$. Thus, we have the following diagram:

u_2	w_2	q_1	ε
a_2b_2	p_2	ε	q_2
t_2	ε	p_1	w_1
	t_1	a_1b_1	u_1

If also $p_1 = \varepsilon$, i.e.,

u_2	w_2	a_1b_1	ε
a_2b_2	p_2	ε	q_2
t_2	ε	ε	w_1
	t_1	a_1b_1	u_1

then $(w_1q_2, a_1) \in I$ implies $(w_1q_2, r_1) \in I$. We have to show that $s_1 = p_2w_2r_1w_1q_2$ and $s_2 = w_1r_2w_2a_1b_1$ can be reduced to the same trace. We have $s_2 \to_R w_1r_2w_2r_1$. Moreover with the independencies listed above, we obtain

$$s_1 = p_2w_2r_1w_1q_2 = p_2w_2w_1q_2r_1 = w_1p_2q_2w_2r_1 \to_R w_1r_2w_2r_1.$$

If one of the traces p_2, q_1, or q_2 is empty, then we can argue analogously. Thus, we may assume that p_1, p_2, q_1, and q_2 are nonempty. It follows $p_1 = a_1$, $q_1 = b_1$, $p_2 = a_2$, and $q_2 = b_2$. Then all traces from $\{w_1, w_2, a_1b_1, a_2b_2\}$ are pairwise independent, from which it follows again easily that s_1 and s_2 can be reduced to $w_1w_2r_1r_2$. $\square$

Since R is also terminating, the previous lemma implies that $\mathbb{P}$ is in one-to-one correspondence with $\mathrm{IRR}(R) \subseteq \mathbb{M}(A, I)$, which is the set of all traces that do not contain a factor of the form ab with $a, b \in A_\sigma$ for some $\sigma \in \Sigma$.

For the further consideration, assume that $\mathcal{M}_\sigma$ is finitely generated by $\Gamma_\sigma \subseteq A_\sigma$. Then $\mathbb{P}$ is finitely generated by $\Gamma = \bigcup_{\sigma\in\Sigma} \Gamma_\sigma$. Our next goal is to define the rooted Cayley-graph $(\mathcal{C}(\mathbb{P}, \Gamma), 1)$ within the trace monoid $\mathbb{M}(A, I)$. For $a \in \Gamma$, let us define the edge-relation

$$F_a = \{(s, t) \in \mathrm{IRR}(R) \times \mathrm{IRR}(R) \mid sa \xrightarrow{*}_R t\}.$$

Since $\mathbb{P} \cong \mathbb{M}(A, I)/_R$ and R is confluent and terminating, we obtain the following lemma:

Lemma 6.2. $(\mathrm{IRR}(R), (F_a)_{a\in\Gamma}, \varepsilon)$ *is isomorphic to* $(\mathcal{C}(\mathbb{P}, \Gamma), 1)$.

For a trace $t \in \mathbb{M}(A, I)$ let $\max(t) = \{a \in A \mid \exists u \in A^* : t = [ua]_I\}$.

Lemma 6.3. *For $s, t \in \mathrm{IRR}(R) \subseteq \mathbb{M}(A, I)$ and $a \in \Gamma_\sigma \subseteq A_\sigma$ we have $(s, t) \in F_a$ if and only if in $\mathbb{M}(A, I)$:*

- $t = sa$ (and thus $\max(s) \cap A_\sigma = \emptyset$), or
- $s = tb$ for $b \in A_\sigma$ and $b \circ_\sigma a = 1_\sigma$, or
- $s = ub$ for $u \in \mathrm{IRR}(R)$, $b \in A_\sigma$, $b \circ_\sigma a = c \neq 1_\sigma$, and $t = uc$.

Proof. If one of the three cases above holds, then it is easy to see that indeed $sa \xrightarrow{*}_R t$, i.e., $(s, t) \in F_a$. Now assume that $sa \xrightarrow{*}_R t \in \mathrm{IRR}(R)$. If $t \neq sa$, then $sa \rightarrow_R v \xrightarrow{*}_R t$ for some trace v. Thus, there exist $\tau \in \Sigma$, $(bb', r) \in R_\tau$, and $s_1, s_2 \in \mathbb{M}(A, I)$ such that $sa = s_1 bb' s_2$. By applying Levi's Lemma to this identity and using $s \in \mathrm{IRR}(R)$, we obtain the following diagram:

$$\begin{array}{|c||c|c|}
\hline
s_2 & s_2 & \varepsilon \\ \hline
bc & b & a = b' \\ \hline
s_1 & s_1 & \varepsilon \\ \hline\hline
 & s & a
\end{array}$$

Thus, $\tau = \sigma$ and $(a, s_2) \in I$, which implies also $(b, s_2) \in I$. Thus, $s = ub$, for $u = s_1 s_2$. If $r = \varepsilon$, i.e., $b \circ_\sigma a = 1_\sigma$, then $v = u \in \mathrm{IRR}(R)$. Thus, $t = u$ and $s = tb$, i.e., the second case from the lemma holds. On the other hand, if $r = b \circ_\sigma a = c \neq 1_\sigma$, then $v = uc$, which again belongs to $\mathrm{IRR}(R)$ (otherwise, since $b, c \in A_\sigma$, also $s = ub \in \mathrm{RED}(R)$). Hence $t = v = uc$. ☐

We now define a structure

$$\mathcal{A} = (A, (A_\sigma)_{\sigma \in \Sigma}, (E_a)_{a \in \Gamma}, (a)_{a \in \Gamma}), \tag{1}$$

where for $a \in \Gamma_\sigma$, $\sigma \in \Sigma$, E_a consists of all pairs $(x, y) \in A_\sigma \times A_\sigma$ such that $x \circ_\sigma a = y$ in $\mathcal{M}_\sigma$. Thus, $\mathcal{A}$ is the disjoint union of the restricted Cayley-graphs $\mathcal{C}(\mathcal{M}_\sigma, \Gamma_\sigma) \backslash \{1_\sigma\}$, where moreover every generator $a \in \Gamma_\sigma$ is added as a constant, and every $A_\sigma \subseteq A$ is added as a unary predicate. We will apply Theorem 5.2 and 5.6 to the structure $\mathcal{A}$. For this, we now define a suitable factorized unfolding of $\mathcal{A}$. First, note that the independence relation $I \subseteq A \times A$ is first-order definable in $\mathcal{A}$ using the unary predicates A_σ and that $\{I(a) \mid a \in A\}$ is finite: If $a, b \in A_\sigma$, then $I(a) = I(b)$. In order to define a suitable homomorphism $\eta : \mathbb{M}(A, I) \to S$ into a finite monoid S, let us consider the finitely generated trace monoid $\mathbb{M}(\Sigma, I_\Sigma)$. The closure properties of recognizable trace languages (see Section 2) imply that

$$L = \mathbb{M}(\Sigma, I_\Sigma) \setminus \bigcup_{\sigma \in \Sigma} \mathbb{M}(\Sigma, I_\Sigma) \sigma\sigma \mathbb{M}(\Sigma, I_\Sigma)$$

is recognizable. Hence, there exists a homomorphism $h : \mathbb{M}(\Sigma, I_\Sigma) \to S$ into a finite monoid S and a subset $F \subseteq S$ such that $L = h^{-1}(F)$. Now define $g : A \to \Sigma$ by $g(a) = \sigma$ if $a \in A_\sigma$. We can extend g homomorphically to $g : \mathbb{M}(A, I) \to \mathbb{M}(\Sigma, I_\Sigma)$. Let $\eta = g \circ h$. Then $\eta^{-1}(F) = \mathrm{IRR}(R)$. Note that for every $q \in S$, the set $\eta^{-1}(q) \cap A_\sigma$ is either empty or A_σ. Thus, every set $\eta^{-1}(q) \cap A$ is first-order definable in $\mathcal{A}$.

From the previous discussion it follows that the structure

$$\mathcal{B} = (\mathbb{M}(A, I), (\eta^{-1}(q))_{q\in S}, \mathrm{suc}, (\widehat{A_\sigma}/_I)_{\sigma\in\Sigma}, (\widehat{E_a}/_I)_{a\in\Gamma}, (\widehat{a}/_I)_{a\in\Gamma}) \tag{2}$$

(see Section 5.1 for the definition of the operator $\widehat{\ }$) is a factorized unfolding of $\mathcal{A}$.[5] We next present a first-order interpretation of the rooted Cayley-graph $(\mathcal{C}(\mathbb{P}, \Gamma), 1)$ in $\mathcal{B}$.

Lemma 6.4. *$(\mathcal{C}(\mathbb{P}, \Gamma), 1)$ is first-order interpretable in $\mathcal{B}$.*

Proof. By Lemma 6.2 it suffices to show that $(\mathrm{IRR}(R), (F_a)_{a\in\Gamma}, \varepsilon)$ is first-order interpretable in $\mathcal{B}$. First, recall that $\mathrm{IRR}(R) = \eta^{-1}(F)$. Moreover, ε is the only trace t such that there is no s with $(s, t) \in \mathrm{suc}$. Finally, by Lemma 6.3 we have $(s, t) \in F_a$ for $s, t \in \mathrm{IRR}(R)$ and $a \in \Gamma_\sigma$ if and only if in $\mathcal{B}$:

- $(s, t) \in \mathrm{suc}$, $s \notin \widehat{A_\sigma}/_I$, and $t \in \widehat{a}/_I$ (i.e., $t = sa$) or
- $(t, s) \in \mathrm{suc}$, $t \notin \widehat{A_\sigma}/_I$, $s \in \widehat{A_\sigma}/_I$, but there is no u with $(s, u) \in \widehat{E_a}/_I$ (note that if $s \in \widehat{A_\sigma}/_I$, i.e., $s = vb$ with $b \in A_\sigma$, but there is no u with $(s, u) \in \widehat{E_a}/_I$, then $b \circ_\sigma a = 1_\sigma$), or
- $(s, t) \in \widehat{E_a}/_I$.

This proves the lemma. $\square$

Now we can finish the proof of Theorem 4.1. Assume that $\mathcal{M}_\sigma$ is finitely generated by $\Gamma_\sigma \subseteq \mathcal{M}_\sigma \setminus \{1_\sigma\}$. Thus, $\mathbb{P}$ is finitely generated by $\Gamma = \bigcup_{\sigma\in\Sigma} \Gamma_\sigma$.

Let us first prove (1) from Theorem 4.1. If $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}_\sigma, \Gamma_\sigma), 1_\sigma)$ is decidable, then, since every constant $a \in \Gamma_\sigma$ is first-order definable in the rooted Cayley-graph $(\mathcal{C}(\mathcal{M}_\sigma, \Gamma_\sigma), 1_\sigma)$, also the structure $(\mathcal{C}(\mathcal{M}_\sigma, \Gamma_\sigma)\backslash\{1_\sigma\}, (a)_{a\in\Gamma_\sigma})$ has a decidable first-order theory. By the Feferman-Vaught Theorem [13], the same holds for the disjoint union of these structures with the unary predicates $A_\sigma = \mathcal{M}_\sigma \setminus \{1_\sigma\}$ added. But this is precisely the structure $\mathcal{A}$ from (1). We can therefore apply Corollary 5.8 and obtain that $\mathrm{FOTh}(\mathcal{B})$ is decidable. Since $(\mathcal{C}(\mathbb{P}, \Gamma), 1)$ is first-order interpretable in $\mathcal{B}$ (Lemma 6.4), it follows that the first-order theory of $(\mathcal{C}(\mathbb{P}), 1)$ is indeed decidable.

Now assume that $I_\Sigma = \emptyset$, i.e., $\mathbb{M}(A, I) = A^*$. The argumentation is similar to the first-order case: If $\mathrm{MSOTh}(\mathcal{C}(\mathcal{M}_\sigma, \Gamma_\sigma), 1_\sigma)$ is decidable for every $\sigma \in \Sigma$, then also $\mathrm{MSOTh}(\mathcal{A})$ is decidable, [43]. Hence, by Lemma 5.5, also $\mathrm{MSOTh}(\widehat{\mathcal{A}}, (\eta^{-1}(q))_{q\in S})$ is decidable. But the structure $\mathcal{B}$ from (2) (for $I = \emptyset$) is a reduct of this structure. Hence, $\mathrm{MSOTh}(\mathcal{B})$ is decidable, and the result follows again from Lemma 6.4. Note that the cl-predicate from $\widehat{A}$ is actually not needed here. This concludes the proof of Theorem 4.1.

Remark 6.5. Concerning the complexity of $\mathrm{FOTh}(\mathcal{C}(\mathbb{P}), 1)$, note that Theorem 5.6 (more precisely Remark 5.7) allows us to reduce $\mathrm{FOTh}(\mathcal{C}(\mathbb{P}), 1)$ in doubly exponential time to $\mathrm{FOTh}(\mathcal{A})$. Recall that $\mathcal{A}$ is essentially the disjoint union of the Cayley-graphs of the monoids $\mathcal{M}_\sigma$. To the knowledge of the authors, all known proofs for decomposition theorems (in the style of Feferman-Vaught's Theorem) that allow to reduce the theory of a

[5] Here we identify the constant a with the unary relation $\{a\}$, thus $\widehat{a} = A^* a$.

disjoint union (or direct product) to the theories of the factors, lead to a nonelementary blow-up in terms of complexity. Therefore, we are only able to give a nonelementary upper bound for $\mathrm{FOTh}(\mathcal{C}(\mathbb{P}), 1)$ even if all the theories $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$ can be decided in elementary time.

For monadic second-order logic the situation is clear: Note that $\mathcal{C}(\mathbb{Z}/2\mathbb{Z})$ is a graph with two nodes, thus its monadic second-order theory is in PSPACE (it is in fact PSPACE-complete). But in $\mathcal{C}(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$ we can define $\mathbb{Z}$ with the successor relation, which has a nonelementary MSO-theory [33].

7 Proof of Theorem 5.6

We have to translate a first-order sentence φ over the signature of the factorized unfolding $\mathcal{B}$ into an equivalent one over the basic structure $\mathcal{A}$. In a first step, we show how to restrict quantifications in φ to "short traces". This reduction can be found in Section 7.3, it uses a method by Ferrante and Rackoff whose essence is described in Section 7.2. The precise notion of "short trace" uses the concept of Foata's normal form and can be found in Section 7.1. After this reduction to short traces, i.e., after the translation of φ into a local sentence, the local sentence is tranformed into an equivalent one over the signature of $\mathcal{A}$, see Section 7.4 for the details.

For the remainder of this section, let us fix the factorized unfolding

$$\mathcal{B} = (\mathbb{M}(A, I), (\eta^{-1}(q))_{q \in S}, (R_i/_I)_{1 \le i \le \kappa}) \tag{3}$$

of $\mathcal{A}$. Recall that

- $I \subseteq A \times A$ is first-order definable in $\mathcal{A}$,
- $\eta : \mathbb{M}(A, I) \to S$ is a monoid homomorphism into a finite monoid S such that $\eta^{-1}(q) \cap A$ is first-order definable in $\mathcal{A}$ for every $q \in S$, and
- R_i is a k_i-suffix definable relation in $\mathcal{A}$ for $1 \le i \le \kappa$.

Assume moreover that there are only finitely many different sets $I(a)$ for $a \in A$.

7.1 Foata normalforms

An (A, I)-*clique* is a subset $C \subseteq A$ such that $(a, b) \in I$ for all $a, b \in C$ with $a \neq b$. Since $\{I(a) \mid a \in \Sigma\}$ is finite, every (A, I)-clique is finite. Thus, for an (A, I)-clique C, we can define a unique trace $[C] = [a_1 a_2 \cdots a_n]_I$, where $a_1, a_2, \ldots, a_n$ is an arbitrary enumeration of C. Let $\mathcal{F}(A, I)$ denote the set of all (A, I)-cliques. For $t \in \mathbb{M}(A, I)$ let $\max(t) = \{a \in A \mid \exists u \in A^* : t = [ua]_I\}$ and $\min(t) = \{a \in A \mid \exists u \in A^* : t = [au]_I\}$. Note that $\min(t)$ and $\max(t)$ are (A, I)-cliques. The set of all traces that have t as a prefix is denoted by $t\,\mathbb{M}(A, I) = \{tu \mid u \in \mathbb{M}(A, I)\}$. Thus, $t \in [\min(t)]\mathbb{M}(A, I)$.

The *Foata normal form* $\mathrm{FNF}(t)$ of $t \in \mathbb{M}(A, I)$ is a word over the set $\mathcal{F}(A, I)$ of finite (A, I)-cliques. It is defined inductively:

$$\mathrm{FNF}([\varepsilon]_I) = \varepsilon \quad \text{and} \quad \mathrm{FNF}(t) = \min(t)\ \mathrm{FNF}(s)$$

where s is the trace satisfying $t = [\min(t)]s$. Since $\mathbb{M}(A, I)$ is cancellative, s is given uniquely by this requirement. The *height* of t, briefly $\mathrm{height}(t)$, is the length of its Foata normal form $\mathrm{FNF}(t)$. Alternatively, $\mathrm{height}(t)$ can be defined as the number of nodes in a longest directed path in the dependence graph D_t. Thus, $\mathrm{height}(st) \leq \mathrm{height}(s) + \mathrm{height}(t)$. The *reversed Foata normal form* of $t \in \mathbb{M}(A, I)$ is defined as follows:

$$\mathrm{rFNF}([\varepsilon]_I) = \varepsilon \quad \text{and} \quad \mathrm{rFNF}(t) = \mathrm{rFNF}(s)\ \max(t)$$

where s is the trace uniquely given by $t = s[\max(t)]$. Then $\mathrm{height}(t)$ also equals the length of $\mathrm{rFNF}(t)$. If the word $A_1 A_2 \cdots A_n$, where $A_i \in \mathcal{F}(A, I)$, is the (reversed) Foata normal form of t, then we say that the factorization $t = [A_1][A_2] \cdots [A_n]$ is *in (reversed) Foata normal form.*

7.2 The method of Ferrante and Rackhoff

The major tool from mathematical logic that we use for the proof of Theorem 5.6 is a method of Ferrante and Rackhoff. Let $\mathcal{B} = (A, (R_i)_{i \in J})$ be a relational structure, where R_i has arity n_i. The *Gaifman-graph* $G_{\mathcal{B}}$ of the structure $\mathcal{B}$ is the following undirected graph:

$$G_{\mathcal{B}} = (A, \{(a, b) \in A \times A \mid \bigvee_{i \in J} \exists (c_1, \ldots, c_{n_i}) \in R_i\ \exists j, k : c_j = a \neq b = c_k\}).$$

We will mainly be interested in restrictions of the structure $\mathcal{B}$ to certain spheres in this graph. To ease notations, we will also write $S_{\mathcal{B}}(r, \widetilde{a})$ for $\mathcal{B}{\restriction} S_{G_{\mathcal{B}}}(r, \widetilde{a})$, i.e., $S_{\mathcal{B}}(r, \widetilde{a})$ is the substructure of $\mathcal{B}$ induced by the r-sphere around the tuple $\widetilde{a}$ in the Gaifman-graph of $\mathcal{B}$.

A *norm function* on $\mathcal{B}$ is just a function $\lambda : A \to \mathbb{N}$. We write $\mathcal{B} \models \exists x \leq n : \varphi$ in order to express that there exists $a \in A$ such that $\lambda(a) \leq n$ and $\mathcal{B} \models \varphi(a)$, and similarly for $\forall x \leq n : \varphi$. Following Ferrante and Rackoff [15], we define H-bounded structures:

Definition 7.1. *Let λ be a norm function on the structure $\mathcal{B}$. Let furthermore $H : \{(j, d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$ be a function such that the following holds: For any $j \leq d \in \mathbb{N}$, any $\widetilde{a} = (a_1, a_2, \ldots, a_{j-1}) \in A^{j-1}$ with $\lambda(a_i) \leq H(i, d)$, and any $a \in A$, there exists $a_j \in A$ with $\lambda(a_j) \leq H(j, d)$ and*

$$(S_{\mathcal{B}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (S_{\mathcal{B}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j).^{6}$$

Then $\mathcal{B}$ (together with the norm function λ) is called H-bounded.

The original definition by Ferrante and Rackoff [15] differs slightly from this one in two aspects: firstly, their function H has a third argument that describes a uniform bound for $\lambda(a_i)$ with $i < j$; instead, we restrict to the case where the norm of these elements is bounded by appropriate values of the function H. Secondly, Ferrante and Rackoff require the tuples $(\widetilde{a}, a)$ and $(\widetilde{a}, a_j)$ to be indistinguishable by a first-order formula of quantifier

[6] Thus, there is an isomorphism from $S_{\mathcal{B}}(7^{d-j}, \widetilde{a}, a)$ to $S_{\mathcal{B}}(7^{d-j}, \widetilde{a}, a_j)$ that fixes every a_i for $i < j$ and maps a to a_j.

depth $d-j$; instead, we require certain spheres around these tuples to be isomorphic. By Gaifman's theorem, this implies that the given tuples are indistinguishable by any first-order formula, i.e., our requirement is more restrictive, but easier to establish. The following result was shown by Ferrante and Rackoff for their version of H-bounded structures. See [25] for a proof.

Proposition 7.2 (cf. [15]). *Let $\mathcal{B}$ be a relational structure with norm λ and let $H : \{(j,d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$ be a function such that $\mathcal{B}$ is H-bounded. Then for any first-order formula $\varphi \equiv Q_1 x_1\, Q_2 x_2 \cdots Q_d x_d : \psi$ where ψ is quantifier-free and $Q_i \in \{\exists, \forall\}$, we have $\mathcal{B} \models \varphi$ if and only if*

$$\mathcal{B} \models Q_1 x_1 \leq n_1\, Q_2 x_2 \leq n_2 \cdots Q_d x_d \leq n_d : \psi(x_1, \ldots, x_d). \tag{4}$$

7.3 From properties of $\mathcal{B}$ to local properties of $\mathcal{B}$

In this section, we will reduce an arbitrary first-order sentence over the signature of $\mathcal{B}$ to a sentence of the form (4) using Corollary 7.2. First of all, this requires the definition of a norm function on $\mathcal{B}$: Let $k = \max\{k_i \mid 1 \leq i \leq \kappa\}$. On $\mathbb{M}(A,I)$ we define a norm function $\lambda : \mathbb{M}(A,I) \to \mathbb{N}$ by $\lambda(t) = |t|$. According to Section 7.2, $\exists x \leq n : \phi(x)$ is an abbreviation for $\exists x : |x| \leq n \wedge \phi(x)$. Now, in order to make Corollary 7.2 applicable, we next investigate the metric on $\mathbb{M}(A,I)$ that is induced by the structure $\mathcal{B}$ (more precisely, by its Gaifman-graph, cf. Section 2).

Lemma 7.3. *Let $u = s[A_1]\cdots[A_m]y = tv \in \mathbb{M}(A,I)$ with $s[A_1]\cdots[A_m]$ in reversed Foata normal form and $|v| \leq m$. Then $t = s[A_1]\cdots[A_{m-|v|}]w$ for some $w \in \mathbb{M}(A,I)$.*

Proof. The lemma is shown by induction on $|v|$. The case $v = \varepsilon$ is trivial. Thus, let $v = av'$ for some $a \in A$, i.e., $u = s[A_1]\cdots[A_m]y = (ta)v'$. By induction we have $ta = s[A_1]\cdots[A_{m-|v|+1}]w'$ for some $w' \in \mathbb{M}(A,I)$. Then $a \in \max(s[A_1]\cdots[A_{m-|v|+1}]w')$. Since $s[A_1][A_2]\cdots[A_{m-|v|+1}]$ is in reversed Foata normal form, we obtain $a \in \max([A_{m-|v|+1}]w')$. Hence, there is a trace w satisfying $[A_{m-|v|+1}]w' = wa$, i.e., $t = s[A_1]\cdots[A_{m-|v|}]w$. ☐

Recall that for $r \in \mathbb{N}$ and $u \in \mathbb{M}(A,I)$ we denote by $S_{\mathcal{B}}(r,u)$ the substructure of $\mathcal{B}$ induced by the r-sphere around u in the Gaifman-graph $G_{\mathcal{B}}$. The distance function $d_{G_{\mathcal{B}}}$ in the Gaifman-graph $G_{\mathcal{B}}$ will be denoted by d in the following. Recall also that k was chosen such that every relation R_i is k-suffix definable.

Lemma 7.4. *Let $u = s[A_1]\cdots[A_{k\cdot r}]$ be in reversed Foata normal form. Then we have $S_{\mathcal{B}}(r,u) \subseteq s\,\mathbb{M}(A,I)$.*

Proof. Let us take $v \in \mathbb{M}(A,I)$ with $d(u,v) \leq r$. We have to show that $v \in s\,\mathbb{M}(A,I)$. By assumption there exists a path $u_0, u_1, \ldots u_m$ in the Gaifman-graph of $\mathcal{B}$ such that $u_0 = u$, $u_m = v$, and $m \leq r$. Inductively we will show that $u_i = s[A_1]\cdots[A_{kr-ki}]y_i$ for some y_i, thus $v = s[A_1]\cdots[A_{kr-km}]y_m \in s\,\mathbb{M}(A,I)$. The case $i = 0$ is clear. Now assume that $u_i = s[A_1]\cdots[A_{kr-ki}]y_i$ and $i < m$. Since (u_i, u_{i+1}) is an edge in the Gaifman-graph of $\mathcal{B}$ and all nonunary relations of $\mathcal{B}$ result from k-suffix definable relations, we have

$u_i = s[A_1]\cdots[A_{kr-ki}]y_i = zw$ and $u_{i+1} = zw'$ for some $z, w, w' \in \mathbb{M}(A, I)$ with $|w| \leq k$ (and $|w'| \leq k$). Lemma 7.3 implies that $u_{i+1} = s[A_1]\cdots[A_{kr-ki-k}]y'w'$ for some $y' \in \mathbb{M}(A, I)$. Thus, we can set $y_{i+1} = y'w'$. □

Thus, the r-sphere around $u = s[A_1][A_2]\cdots[A_{kr}]$ is contained in $s\,\mathbb{M}(A, I)$. The next lemma will be used to shorten s, i.e., to find v properly shorter than u that is the center of an isomorphic r-sphere. Let $s, t \in \mathbb{M}(A, I)$ be two traces. Since $\mathbb{M}(A, I)$ is cancellative, the mapping $f = f_{s,t} : s\,\mathbb{M}(A, I) \to t\,\mathbb{M}(A, I)$ defined by $f(su) = tu$ is a bijection. We will show that, under some assumptions on s and t, it is an isomorphism from $(S_{\mathcal{B}}(r, u), u)$ to $(S_{\mathcal{B}}(r, v), v)$.

Lemma 7.5. *Let $u = s[A_1]\cdots[A_{kr+k}]$ and $v = t[A_1]\cdots[A_{kr+k}]$ be in reversed Foata normal form and $\eta(s) = \eta(t)$. Then the mapping $f = f_{s,t}$ is an isomorphism from $(S_{\mathcal{B}}(r, u), u)$ to $(S_{\mathcal{B}}(r, v), v)$.*

Proof. Lemma 7.4 implies $S_{\mathcal{B}}(r, u) \subseteq s[A_1]\cdots[A_k]\mathbb{M}(A, I)$, thus f is defined on $S_{\mathcal{B}}(r, u)$. Since $\eta(f(su)) = \eta(tu) = \eta(t) \cdot \eta(u) = \eta(s) \cdot \eta(u) = \eta(su)$, f preserves all unary predicates $\eta^{-1}(q)$ for $q \in S$. Now assume that $(u_1, \ldots, u_n) \in R/_I$, where $u_i \in S_{\mathcal{B}}(r, u)$ and $R/_I$ is a relation of $\mathcal{B}$. Thus, R is k-suffix definable. Hence, there exist $y, w_1, \ldots, w_n \in \mathbb{M}(A, I)$ such that $u_i = yw_i$, $|w_i| \leq k$, and $(y'w_1, \ldots, y'w_n) \in R/_I$ for all $y' \in \mathbb{M}(A, I)$. Since $S_{\mathcal{B}}(r, u) \subseteq s[A_1]\cdots[A_k]\mathbb{M}(A, I)$, we have $yw_i = u_i = s[A_1]\cdots[A_k]v_i$ for some $v_i \in \mathbb{M}(A, I)$. Thus, Lemma 7.3 and $|w_i| \leq k$ implies $y = sy_i$ for some trace y_i. Since $\mathbb{M}(A, I)$ is cancellative, it follows $y_1 = \cdots = y_n =: z$. Thus, $f(u_i) = f(szw_i) = tzw_i$ and $(f(u_1), \ldots, f(u_n)) \in R/_I$. It follows that f maps $S_{\mathcal{B}}(r, u)$ injectively and structure preserving into $S_{\mathcal{B}}(r, v)$. Since we may exchange the roles of s and t, it follows that f maps $S_{\mathcal{B}}(r, u)$ bijectively to $S_{\mathcal{B}}(r, v)$. □

Recall that (A, I) has only finitely many neighborhoods, i.e., that the set $\{I(a) \mid a \in A\}$ is finite. Thus, also $\{D(a) \mid a \in A\}$ is finite, where $D = (A \times A) \setminus I$.

Lemma 7.6. *There exists a homomorphism $h : \mathbb{M}(A, I) \to Q$ into some finite monoid Q such that for all $s, t \in \mathbb{M}(A, I)$, we have:*

if $h(s) = h(t)$ and $a \in \max(s)$, then there exists $b \in \max(t)$ with $D(a) = D(b)$.

Proof. Let $\mathcal{D}$ be the powerset of $\{D(a) \mid a \in A\}$, thus $\mathcal{D}$ is finite. For $s \in \mathbb{M}(A, I)$, define $f(s) = \{D(a) \mid a \in \max(s)\} \in \mathcal{D}$. Let $s, t \in \mathbb{M}(A, I)$ such that $f(s) = f(t)$. We show that $f(sc) = f(tc)$ for all $c \in A$: Clearly, $c \in \max(sc) \cap \max(tc)$. Now let $a \in A \setminus \{c\}$. Then $a \in \max(sc)$ if and only if $(a, c) \in I$ and $a \in \max(s)$. Hence $f(sc) = \{D(c)\} \cup \{D(a) \mid D(a) \in f(s), c \notin D(a)\}$. Thus, indeed, $f(sc) = f(tc)$.

Now consider the image $f(\mathcal{D})$ of $\mathbb{M}(A, I)$ under f. Let Q be the transformation monoid of $f(\mathcal{D})$, i.e., $Q = (f(\mathcal{D})^{f(\mathcal{D})}, \circ)$ and define a mapping $h : A \to Q$ such that $h(a)(f(s)) = f(sa)$ which is well defined by the previous paragraph. Clearly $h(a) \circ h(b) = h(b) \circ h(a)$ for $(a, b) \in I$. Thus, we can extend h to a monoid homomorphism $h : \mathbb{M}(A, I) \to Q$ with $h(t)(f(s)) = f(st)$. Now suppose $h(s) = h(t)$. Then $a \in \max(s)$ implies $D(a) \in f(s) = h(s)(f(\varepsilon)) = f(t)$. Hence, there is $b \in \max(t)$ with $D(a) = D(b)$. □

Lemma 7.7. *Let h be the homomorphism from Lemma 7.6 and let $s, s', t, t' \in \mathbb{M}(A, I)$ with $h(s') = h(t')$, $s = s'[\max(s)]$, and $t = t'[\max(s)]$. Then* $\max(t) = \max(s)$ *and* $\mathrm{height}(t) = \mathrm{height}(t') + 1$.

Proof. Clearly, $\max(s) \subseteq \max(t)$. So let $a \in \max(t) \setminus \max(s)$. Since $t = t'[\max(s)]$, we get $a \in \max(t')$ and $(a, c) \in I$ for every $c \in \max(s)$. Since $h(s') = h(t')$, it follows $D(a) = D(b)$, i.e., $I(a) = I(b)$ for some $b \in \max(s')$. Thus, also $(b, c) \in I$ for every $c \in \max(s)$. But this implies $b \in \max(s)$, i.e., $(b, b) \in I$, a contradiction. Thus, indeed, $\max(t) = \max(s)$. This implies $\mathrm{height}(t) = \mathrm{height}(t'[\max(s)]) = \mathrm{height}(t'[\max(t)]) = \mathrm{height}(t') + 1$. ☐

By Lemma 7.6 we can find a homomorphism η' from $\mathbb{M}(A, I)$ into some finite monoid S' (namely $S \times Q$) such that the following implications hold:

- if $\eta'(s) = \eta'(t)$ and $a \in \max(s)$, then there exists $b \in \max(t)$ with $D(a) = D(b)$.
- If $\eta'(s) = \eta'(t)$, then $\eta(s) = \eta(t)$.

Lemma 7.8. *Let $u \in \mathbb{M}(A, I)$ and $r, \ell \in \mathbb{N}$ such that $\ell \geq k(r+1)$,* $\mathrm{height}(u) > \max\{k(r+1) + |S'| + 1, \ell\}$. *Then there exists $v \in \mathbb{M}(A, I)$ with*

$$\ell < \mathrm{height}(v) \leq \ell + |S'| + 1 \quad \textit{and} \quad (S_{\mathcal{B}}(r, u), u) \cong (S_{\mathcal{B}}(r, v), v).$$

Proof. If $\mathrm{height}(u) \leq \ell+|S'|+1$ we can set $u = v$. Thus, assume that $\mathrm{height}(u) > \ell+|S'|+1$. Then there are (A, I)-cliques $A_i \subseteq A$ and $s \in \mathbb{M}(A, I)$ such that $u = s\,[A_1][A_2]\cdots[A_{k(r+1)}]$ is in reversed Foata normal form and $\mathrm{height}(s) > \ell - k(r + 1) + |S'| + 1$. Let $s' \in \mathbb{M}(A, I)$ with $s = s'\,[\max(s)]$. Then $\mathrm{height}(s') \geq \ell - k(r + 1) + |S'| + 1$, and we can write $s' = s_1 s_2$ with $\mathrm{height}(s_1) = \ell - k(r + 1)$ and $\mathrm{height}(s_2) > |S'|$. A simple pigeon hole argument shows that there exists $s_2' \in \mathbb{M}(A, I)$ such that $\eta'(s_2) = \eta'(s_2')$ and $\mathrm{height}(s_2') \leq |S'|$. Define $t' = s_1 s_2'$ and $t = t'\,[\max(s)]$. Thus, $\eta'(s') = \eta'(t')$. Hence, by Lemma 7.7 we get $\mathrm{height}(t) = \mathrm{height}(t') + 1$ and $\max(t) = \max(s)$. This ensures in particular

$$\mathrm{height}(t) = \mathrm{height}(t') + 1 \leq \mathrm{height}(s_1) + \mathrm{height}(s_2') + 1 \leq \ell - k(r + 1) + |S'| + 1$$

and

$$\mathrm{height}(t) > \mathrm{height}(t') \geq \mathrm{height}(s_1) = \ell - k(r + 1).$$

Now set $v = t[A_1][A_2]\cdots[A_{k(r+1)}]$. A $k(r + 1)$-fold application of Lemma 7.7 implies $\mathrm{height}(v) = \mathrm{height}(t) + k(r + 1)$ and therefore

$$\ell < \mathrm{height}(v) \leq \ell + |S'| + 1.$$

Since $\eta'(s) = \eta'(t)$, we can apply Lemma 7.5, implying $(S_{\mathcal{B}}(r, u), u) \cong (S_{\mathcal{B}}(r, v), v)$, which finishes the proof. ☐

Lemma 7.9. *The factorized unfolding $\mathcal{B}$ from (3) is H-bounded by a function H with $H(j, d) \leq H(d, d) \in 2^{O(d)}$ for $j \leq d$.*

Proof. Recall that the norm of a trace t was defined as its length $|t|$. Since $\{I(a) \mid a \in A\}$ is finite, there is $\alpha \in \mathbb{N}$ such that any (A, I)-clique contains at most α elements. We define $H(i, d)$ inductively: $H(1, d) = \alpha \cdot (k(7^{d-1}+1)+2(|S'|+1))$ and $H(j, d) = \alpha \cdot (H(j-1, d) + 4 \cdot 7^{d-j} \cdot k + |S'| + 1)$ for $1 < j \leq d$. Then $H(j, d) \leq H(d, d)$ is bounded by $2^{O(d)}$ for $j \leq d$.

Let $d \in \mathbb{N}$ and $t \in \mathbb{M}(A, I)$ with $|t| > H(1, d)$. Let $\ell = k(7^{d-1}+1)+|S'|+1 < H(1, d)/\alpha$ and $r = 7^{d-1}$. Then $\ell \geq k(r+1)$ and $\ell = k(r+1)+|S'|+1 < H(1, d)/\alpha < |t|/\alpha \leq \mathrm{height}(t)$. Hence, by Lemma 7.8, there is $t_1 \in \mathbb{M}(A, I)$ with $\mathrm{height}(t_1) \leq \ell + |S'| + 1 \leq H(1, d)$ and $(S_{\mathcal{B}}(7^{d-1}, t), t) \cong (S_{\mathcal{B}}(7^{d-1}, t_1), t_1)$. This proves the base case for the H-boundedness of $\mathcal{B}$.

Next, let $1 < j \leq d \in \mathbb{N}$, $\widetilde{t} = (t_1, t_2, \ldots, t_{j-1}) \in \mathbb{M}(A, I)^{j-1}$ with $|t_i| \leq H(i, d)$ and $t \in \mathbb{M}(A, I)$ with $|t| > H(j, d)$. In order to apply Lemma 7.8, let $\ell = H(j-1, d)+4 \cdot 7^{d-j} \cdot k$ and $r = 7^{d-j}$. Thus, $\ell \geq H(1, d) \geq k(7^{d-1} + 1) \geq k(r + 1)$. Moreover, $|t| > H(j, d) > H(1, d) \geq \alpha \cdot (k(7^{d-j} + 1) + |S'| + 1)$. Hence, $\mathrm{height}(t) \geq |t|/\alpha > k(r + 1) + |S'| + 1$. Furthermore, $|t| > H(j, d) > \alpha \cdot \ell$ implies $\mathrm{height}(t) > \ell$. Thus, by Lemma 7.8, there exists $t_j \in \mathbb{M}(A, I)$ with

$$\ell < \mathrm{height}(t_j) \leq \ell + |S'| + 1 \quad \text{and} \quad (S_{\mathcal{B}}(7^{d-j}, t), t) \cong (S_{\mathcal{B}}(7^{d-j}, t_j), t_j).$$

Thus, $\ell < |t_j| \leq \alpha \cdot (\ell + |S'| + 1)$. In the Gaifman-graph $G_{\mathcal{B}}$, the distance between t_i and t_j is at least $(|t_j| - |t_i|)/k$. Since $|t_i| \leq H(i, d) \leq H(j - 1, d)$ for $1 \leq i < j$, we obtain $d(t_i, t_j) \geq (|t_j| - |t_i|)/k > (\ell - H(j - 1, d))/k = 4 \cdot 7^{d-j}$. Hence the spheres $S_{\mathcal{B}}(7^{d-j}, \widetilde{t})$ and $S_{\mathcal{B}}(7^{d-j}, t_j)$ are disjoint and no edge in $G_{\mathcal{B}}$ connects elements from the former to elements from the latter sphere. The same holds for the spheres $S_{\mathcal{B}}(7^{d-j}, \widetilde{t})$ and $S_{\mathcal{B}}(7^{d-j}, t)$. Thus,

$$(S_{\mathcal{B}}(7^{d-j}, \widetilde{t}, t), \widetilde{t}, t) \cong (S_{\mathcal{B}}(7^{d-j}, \widetilde{t}, t_j), \widetilde{t}, t_j),$$

and the factorized unfolding $\mathcal{B}$ is indeed H-bounded. □

Now let $\varphi \equiv Q_1x_1Q_2x_2 \cdots Q_dx_d : \psi(x_1, \ldots, x_d)$ be a first-order sentence over the signature of $\mathcal{B}$ with d quantifiers $Q_i \in \{\exists, \forall\}$. Since $\mathcal{B}$ is H bounded by the previous lemma, Proposition 7.2 implies that $\mathcal{B} \models \varphi$ if and only if

$$\mathcal{B} \models Q_1x_1 \leq H(1, d)Q_2x_2 \leq H(2, d) \cdots Q_dx_d \leq H(d, d) : \psi(x_1, \ldots, x_d). \tag{5}$$

7.4 From local properties of $\mathcal{B}$ to properties of $\mathcal{A}$

It remains to reduce local sentences of the form (5) to sentences that speak about the structure $\mathcal{A}$. This is achieved by the following proposition.

Proposition 7.10. *Let $\psi(x_1, \ldots, x_d)$ be a Boolean formula over the signature of $\mathcal{B}$. Let $n_1, \ldots, n_d \in \mathbb{N}$, and $Q_1, \ldots, Q_d \in \{\exists, \forall\}$. Then we can effectively construct a sentence θ over the signature of $\mathcal{A}$ such that*

$$\mathcal{B} \models Q_1x_1 \leq H(1, d)\, Q_2x_2 \leq H(2, d) \cdots Q_dx_d \leq H(d, d) : \psi.$$

if and only if $\mathcal{A} \models \theta$. Moreover, θ has quantifier alternation depth $d+O(1)$ and size bounded by $n^d \cdot |\psi| \cdot 2^{O(n)}$ where $n = \max\{n_1, \ldots, n_d\}$.

Proof. We will encode a trace $x \in \mathbb{M}(A, I)$ with $|x| \leq n$ by a sequence $y_1 y_2 \cdots y_m$ of first-order variables $y_i \in A$ of length $m \leq n$, with the meaning that $x = [y_1 y_2 \cdots y_m]_I$. First, for every $m \leq n$, we have to construct a first-order formula in $2m$ free variables over the signature of $\mathcal{A}$, which expresses that $[y_1 y_2 \cdots y_m]_I = [z_1 z_2 \cdots z_m]_I$ in $\mathbb{M}(A, I)$. This can be done inductively as follows: If $m = 0$, then this formula is the truth value true. If $m > 0$, then $[y_1 y_2 \cdots y_m]_I = [z_1 z_2 \cdots z_m]_I$ if and only if

$$\bigvee_{i=1}^{m} \left(y_1 = z_i \ \wedge \ \bigwedge_{j=1}^{i-1} (z_i, z_j) \in I \ \wedge \ [y_2 \cdots y_m]_I = [z_1 \cdots z_{i-1} z_{i+1} \cdots z_m]_I \right)$$

(recall that by assumption the independence relation I can be defined by a fixed first-order formula over the signature of $\mathcal{A}$). The above recursive definition would lead to a formula of exponential size for $[y_1 y_2 \cdots y_m]_I = [z_1 z_2 \cdots z_m]_I$ since an equation of the form $[y_2 \ldots y_m]_I = [u_1 \ldots u_{m-1}]_I$ appears m times. Using a trick from Ferrante [14, Lem. 2] we can be more space economical: The above formula is equivalent to

$$\exists u_1 \cdots \exists u_{m-1} \left\{ \begin{array}{l} [y_2 \cdots y_m]_I = [u_1 \cdots u_{m-1}]_I \ \wedge \\ \displaystyle\bigvee_{i=1}^{m} \left(\begin{array}{l} y_1 = z_i \ \wedge \ \displaystyle\bigwedge_{j=1}^{i-1} (z_j = u_j \ \wedge \ (z_i, u_j) \in I) \ \wedge \\ \displaystyle\bigwedge_{j=i+1}^{m} z_j = u_{j-1} \end{array} \right) \end{array} \right\}.$$

Let s_m be the size of this formula with $2m$ free variables. Then s_m is bounded by $s_{m-1} + O(m^2)$. Thus, $s_n \in O(n^3)$. Moreover the quantifier alternation depth in the above formula is 0, since we only use existential quantifiers.

Now a bounded existential quantification $\exists x_i \leq n_i$ in (4) can be replaced by $\bigvee_{j=0}^{n_i} \exists y_1 \cdots \exists y_j$, where x is represented by the sequence $y_1 \cdots y_j$, and similarly for a universal quantifier. Since there are only d quantifiers in (4), these replacements increase the size of the formula at most by a factor n^d. Furthermore, the quantifier alternation depth is unchanged.

Next, consider an atomic formula $R/_I(x_1, \ldots, x_r)$ in ψ, where R is one of the k-suffix definable relations R_i $(1 \leq i \leq \kappa)$. Since R is k-suffix definable, we can assume that

$$R = \{(uu_1,\, uu_2,\, \ldots, uu_r) \mid u, u_i \in A^*, |u_i| = \ell_i, \mathcal{A} \models \phi(u_1, u_2, \ldots, u_r)\}$$

for some $\ell_i \leq k$, where ϕ is a fixed first-order formula over the signature of $\mathcal{A}$. Assume that the trace $x_i \in \mathbb{M}(A, I)$ is represented by the sequence $y_{i,1} \cdots y_{i,m_i}$ $(m_i \leq n)$. If for some $1 \leq i \leq r$, we have $m_i < \ell_i$, then we can replace $R/_I(x_1, \ldots, x_r)$ by the truth value false. The same can be done if $m_i - \ell_i \neq m_j - \ell_j$ for two different i, j. Thus, assume that $m_i - \ell_i = \ell \geq 0$ for all $1 \leq i \leq r$. Then we can replace $R/_I(x_1, \ldots, x_r)$ by the formula

$$\exists_{\substack{1 \leq i \leq r \\ 1 \leq j \leq \ell_i}} z_{i,j} \, \exists z_1 \cdots \exists z_\ell \left\{ \begin{array}{l} \displaystyle\bigwedge_{1 \leq i \leq r} [y_{i,1} \cdots y_{i,m_i}]_I = [z_1 \cdots z_\ell z_{i,1} \cdots z_{i,\ell_i}]_I \\ \wedge \ \phi(z_{1,1}, \ldots, z_{1,\ell_1}, \ldots, z_{r,1}, \ldots, z_{r,\ell_r}) \end{array} \right\},$$

which has fixed quantifier alternation depth and size bounded by $O(n^3)$. Similarly, an atomic formula of the form $x = y$ can be replaced by a formula of size $s_n \in O(n^3)$ and fixed quantifier alternation depth.

Finally, we want to express $\eta(x) = q$ for some $q \in S$. Assume that $x \in \mathbb{M}(A, I)$ is represented by the sequence $y_1 \cdots y_m$, where $m \leq n$. Then we can replace $\eta(x) = q$ by

$$\bigvee_{\substack{(q_1,\ldots,q_m)\in S^m \\ q_1\cdot q_2\cdots q_m = q}} \bigwedge_{1\leq i\leq m} \eta(y_i) = q_i.$$

Recall that $\eta(y_i) = q_i$ can be expressed by a fixed first-order formula over the signature of $\mathcal{A}$. Thus, the size of the above formula is bounded by $O(|S|^n)$ and its quantifier alternation depth is $O(1)$.

Altogether, any of the atomic subformulas in ψ gets replaced by a formula of quantifier alternation depth $O(1)$ and size bounded by $2^{O(n)}$. Hence, the size of the resulting sentence θ is bounded by $n^d \cdot |\psi| \cdot 2^{O(n)}$, and its quantifier alternation depth is bounded by $d + O(1)$. $\square$

Since the function H satisfies $H(j, d) \leq H(d, d) \in 2^{O(d)}$, the previous proposition implies that the sentence in (5) is equivalent to a first-order sentence over the signature of $\mathcal{A}$ of size $2^{2^{O(|\varphi|)}}$ and quantifier alternation depth $d + O(1)$. This finishes the proof of Theorem 5.6.

8 Some results on factorized unfoldings

This final section contains two auxiliary result on factorized unfoldings. The first one shows that the requirement on $\{I(a) \mid a \in A\}$ in Theorem 5.6 to be finite is necessary for the theorem to hold. The second result of this section gives a much simpler proof of Theorem 5.6 in case the structure $\mathcal{A}$ is finite.

The structure $(\mathbb{N}^*, \widehat{S}, \preceq, \mathrm{cp})$ from Proposition 5.3 has an undecidable first-order theory. Thus, allowing the relation $\preceq/_I$, which is the prefix order on traces, in factorized unfoldings would make Theorem 5.6 fail (already for $I = \emptyset$). In Theorem 5.6, we also assume that there are only finitely many different sets $I(a)$. The reason is again that otherwise the result would fail: Let $V = \{(m, n) \in \mathbb{N}^2 \mid m \leq n\}$ and $E = \{(\ell, m, n) \in \mathbb{N}^3 \mid \ell, m \leq n\}$. On $A = V \cup E$, define the relation R by

$$R = \{((m, n),\ (\ell, m, n)) \mid \ell, m \leq n\} \cup \{((\ell, n),\ (\ell, m, n)) \mid \ell, m \leq n\}.$$

Thus, $\mathrm{dom}(R) = V$ and $\mathrm{ran}(R) = E$. Furthermore, let I be the set of pairs of distinct elements from $V \cup E$ that agree on their last component. Then there are infinitely many sets $I(a)$, but any of these sets is finite. We will consider the structure $\mathcal{A} = (A, R, I)$. The decidability of $\mathrm{FOTh}(\mathbb{N}, \leq)$ implies the decidability of $\mathrm{FOTh}(\mathcal{A})$.

Theorem 8.1. *Let $\mathcal{B} = (\mathbb{M}(A, I), \mathrm{cl}/_I, \mathrm{suc}/_I, \widehat{R}/_I)$, which is a factorized unfolding of $\mathcal{A}$. Then $\mathrm{FOTh}(\mathcal{B})$ is undecidable.*

Proof. We will reduce the first-order theory of all finite directed graphs (which is undecidable by [47]) to the first-order theory of $\mathcal{B}$.

The idea is to represent a finite graph by the finite (A,I)-clique $\max(s)$ of a trace $s \in \mathbb{M}(A,I)$, which by the definition of I is a subset of

$$V \cap (\mathbb{N} \times \{n\}) \;\cup\; E \cap (\mathbb{N}^2 \times \{n\})$$

for some $n \in \mathbb{N}$. Those elements from V (resp. E) in $\max(s)$ represent the nodes (resp. edges) of G. To represent the set $\max(s)$ in $\mathcal{B}$, we use the $\mathrm{cl}/_I$-relation. More precisely, for $s \in \mathbb{M}(A,I)$ let $G(s)$ denote the set of traces t such that $(s,t) \in \mathrm{cl}/_I$ in $\mathcal{B}$. In other words, $G(s)$ is the set of traces sa (with $a \in A$) such that $a \in \max(s)$. By f_s we denote the bijection from $G(s)$ to $\max(s)$ given by $sa \mapsto a$ (this is well-defined, since $sa = sb$ implies $a = b$). Now let $t \in G(s)$. Then $f_s(t) \in V$ if and only if there is $u \in \mathbb{M}(A,I)$ with $(s,u) \in \mathrm{suc}/_I$ and $(t,u) \in \widehat{R}/_I$ (recall that $V = \mathrm{dom}(R)$). Similarly, $f_s(t) \in E$ if and only if there is $u \in \mathbb{M}(A,I)$ with $(s,u) \in \mathrm{suc}$ and $(u,t) \in \widehat{R}/_I$. Now let $v, e \in G(s)$ with $f_s(v) \in V$ and $f_s(e) \in E$. Then $(f_s(v), f_s(e)) \in R$ if and only if $(v,e) \in \widehat{R}/_I$. Thus, we can write a formula $\mathrm{graph}(x)$ with one free variable x such that $\mathcal{B} \models \mathrm{graph}(s)$ if and only if $\max(s)$ is a directed graph, i.e., any element of $\max(s) \cap E$ is adjacent with at least one element from $\max(s) \cap V$.

Now let φ be a sentence over the signature of directed graphs. Using the ideas explained above, we can construct a formula $\varphi'(x)$ with one free variable such that for any trace $s \in \mathbb{M}(A,I)$ satisfying $\mathrm{graph}(s)$, the graph of maximal elements of s satisfies φ if and only if $\mathcal{B} \models \varphi'(s)$. Thus, φ is true in all finite graphs (i.e., belongs to the theory of all finite graphs) if and only if $\mathcal{B} \models \forall x : \mathrm{graph}(x) \Rightarrow \varphi'(x)$. □

In order to prove the undecidability results in Proposition 5.3 and Theorem 8.1 we used infinite structures. Infinity is needed as the next theorem shows. See [23, 3] for the definition of an automatic structure.

Theorem 8.2. *Let $\mathcal{A}$ be a finite relational structure with universe A, and let*

$$\mathcal{B} = (\mathbb{M}(A,I), (\eta^{-1}(q))_{q \in S}, (R_i/_I)_{1 \le i \le \kappa})$$

be any factorized unfolding of $\mathcal{A}$. Then the structure $(\mathcal{B}, \preceq/_I)$ is automatic and has therefore a decidable first-order theory.

Proof. The free monoid $\mathcal{F}(A,I)^*$ generated by the set of (A,I)-cliques maps naturally onto $\mathbb{M}(A,I)$, let h denote the canonical homomorphism defined by $h(C) = [C]$. Let $\mathrm{FNF} \subseteq \mathcal{F}(A,I)^*$ denote the set of Foata normal forms. Then a word $C_1 C_2 \cdots C_n$ over $\mathcal{F}(A,I)$ belongs to FNF if and only if for every $1 \le i < n$ and every $a \in C_{i+1}$, there is $b \in C_i$ with $(a,b) \notin I$. Since A is finite, the set FNF is recognizable. Moreover, h maps FNF bijectively to $\mathbb{M}(A,I)$. Let $\mathrm{suc}_a = \{(t, ta) \mid t \in \mathbb{M}(A,I)\}$ for $a \in A$. In a forthcoming paper, we will show that the structure $(\mathbb{M}(A,I), \preceq/_I, (\mathrm{suc}_a)_{a \in A})$ is automatic with respect to FNF and h. Using the closure of automatic structures under first-order interpretations [23], it suffices to show that

– the set $\{w \in \mathrm{FNF} \mid \eta(h(w)) = q\}$ is regular for every $q \in S$ and
– every relation $R_i/_I$, $1 \leq i \leq \kappa$, is first-order definable in $(\mathbb{M}(A,I), \preceq/_I, (\mathrm{suc}_a)_{a \in A})$.

Using an automaton with state space S, the first point is easy to check. For the second point we can argue as follows: Since A is finite and R_i is k-suffix definable for some k, we can write $R_i/_I$ as $R_i/_I = \bigcup_{(s_1,\ldots,s_n)\in F_i} \{(ts_1, \ldots, ts_n) \mid t \in \mathbb{M}(A,I)\}$, where $F_i \subseteq \mathbb{M}(A,I)^n$ is a *finite* relation. Now, for $s = [a_1 a_2 \cdots a_m]_I \in \mathbb{M}(A,I)$, $a_k \in A$, define

$$\mathrm{suc}_s(x_0, x_m) \;\equiv\; \exists x_1 \cdots \exists x_{m-1} \left\{ \bigwedge_{j=1}^{m} \mathrm{suc}_{a_j}(x_{j-1}, x_j) \right\}.$$

Then $(y_1, \ldots, y_n) \in R_i/_I$ if and only if

$$\bigvee_{(s_1,\ldots,s_n)\in F_i} \exists x \left\{ \bigwedge_{1 \leq j \leq n} \mathrm{suc}_{s_j}(x, y_j) \right\}.$$

Thus, the relation $R_i/_I$ is even first-order definable in $(\mathbb{M}(A,I), (\mathrm{suc}_a)_{a\in A})$ □

Remark 8.3. We can also say something on the complexity of the decision procedure for the first-order theory of $(\mathcal{B}, \preceq/_I)$ in the previous theorem: Suppose that any two distinct letters from A are independent. Then one can reduce $\mathrm{FOTh}(\mathcal{B}, \preceq)$ to Presburger's Arithmetic, which is decidable in $\mathrm{DSPACE}(2^{2^{O(n)}})$ [2] and hence elementary. On the other hand, the theory $\mathrm{FOTh}(\{a,b\}^*, \mathrm{suc}_a, \mathrm{suc}_b, \preceq)$ is not elementarily decidable, see [9, Example 8.3].

Note that Theorem 8.2 implies in particular that the prefix order on finite traces over a finite independence alphabet (A, I) has a decidable first-order theory.

9 Open problems

Many open problems remain for Cayley-graphs of finitely generated monoids. The most ambitious goal would be to obtain a complete (algebraic or combinatorial) characterization of those monoids such that the corresponding Cayley-graph has a decidable first-order theory or MSO-theory, respectively. But due to the missing symmetry in Cayley-graphs of monoids this problem might be too difficult. A promising class for further results are cancellative monoids. Their Cayley-graphs have at least bounded degree. Is there a cancellative monoid with a decidable word problem such that the corresponding Cayley-graph has an undecidable first-order theory? Is there a cancellative monoid such that its Cayley-graph has finite tree-width but an undecidable MSO-theory?

In [25] we have shown that for Cayley-graphs of finitely generated groups the decidability of the full first-order theory is equivalent to the decidability of the Σ_1-theory. The corresponding problem for monoids is again open.

As already mentioned, in statement (1) from Theorem 4.1 it remains open whether the complexity of $\mathrm{FOTh}(\mathcal{C}(\mathbb{P}), 1)$ is bounded elementarily in the complexity of the theories $\mathrm{FOTh}(\mathcal{C}(\mathcal{M}_\sigma), 1_\sigma)$, where the $\mathcal{M}_\sigma$ are the factors of the graph product.

For MSO-theories, statements (2) from Theorem 4.1 and Proposition 4.3 leave a gap. A plausible conjecture is that $\mathrm{MSOTh}(\mathcal{C}(\mathbb{P}), 1)$ is decidable if and only if the four conditions in Proposition 4.3 are satisfied. One might first try to prove this conjecture for graph products of finite monoids. In particular, if the independence relation (Σ, I_Σ) is a chain of four nodes (also called P4) and every node is labeled with a finite monoid, then it is not clear whether the corresponding graph product has a Cayley-graph with a decidable MSO-theory.

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