Axiomatising divergence*

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Abstract. This paper develops sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalence. The axiomatisations can be extended to a considerable fragment of the linear time – branching time spectrum with silent moves, partially solving an open problem posed in [5].

1 Motivation

The study of comparative concurrency semantics is concerned with a uniform classification of process behaviour, and has cumulated in Rob van Glabbeek's seminal papers on the *linear time-branching time spectrum* [4, 5]. The main ('vertical') dimension of the spectrum with silent moves [5] spans between trace equivalence (TE) and branching bisimulation (BB), and identifies different ways to discriminate processes according to their branching structure, where BB induces the finest, and TE the coarsest reasonable semantics. Due to the presence of silent moves, this spectrum is spread in another ('horizontal') dimension, determined by the semantics of *divergence*. In the fragment spanning from weak bisimulation (WB) to BB, seven different criteria to distinguish divergence induce a 'horizontal' lattice, and this lattice appears for all the bisimulation relations.

To illustrate the spectrum, van Glabbeek lists a number of examples and counterexamples showing the differences among the various semantics [5]. *Process algebra* provides a different – and to our opinion more elegant – way to compare semantic issues, by providing distinguishing axioms that capture the essence of an equivalence (or preorder). For the 'vertical' dimension of the spectrum, these distinguishing axioms are well-known (see e.g. [4, 7, 2]). However, the 'horizontal' dimension has resisted an axiomatic treatment so far. We believe that this is mainly due to the fact that divergence only makes sense in the presence of recursion, and that recursion is hard to tackle axiomatically. Isolated points in the 'horizontal' dimension have however been axiomatised, most

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notably Milner's weak bisimulation (WB) congruence [10], and also convergent WB preorder [11], as well as divergence insensitive BB congruence [6] and stable WB congruence [8]. It is worth to mention the works of [3] and [1], which axiomatised divergence sensitive WB congruence and convergent WB preorder, respectively, but without showing completeness in the presence of recursion.

This paper develops complete axiomatisations for the 'horizontal' dimension of weak bisimulation equivalence. A lattice of distinguishing axioms is shown to characterise the distinct semantics of divergence, and to precisely reflect the 'horizontal' lattice structure of the spectrum. We are confident that these axioms form the basis of complete axiomatisation for the bisimulation spectrum spanning from WB to BB.

The paper is organised as follows. [1] Section 2 introduces the necessary notation and definitions, while Section 3 recalls the weak bisimulation equivalences and Section 4 introduces the axiom systems. Section 5 is devoted to soundness of the axioms and sets the ground for the completeness proof. Section 6 is devoted to the main step of the proof, only focusing on closed expressions, while Section 7 covers open expressions. Section 8 concludes the paper.

2 Preliminaries

We assume a set of variables $\mathbb{V}$, and a set of actions $\mathbb{A}$, containing the silent action τ. We consider the set of open finite state agents with silent moves and explicit divergence, given as the set $\mathbb{E}$ of expressions generated by the grammar

$$\mathcal{E} \quad ::= \quad a.\mathcal{E} \quad | \quad \mathcal{E}+\mathcal{E} \quad | \quad recX.\mathcal{E} \quad | \quad X \quad | \quad \Delta(\mathcal{E})$$

where $X \in \mathbb{V}$ and $a \in \mathbb{A}$. $\Delta(E)$ is an expression that adds divergence explicitly to the root of E. The syntactic equality on $\mathbb{E}$ is denoted by $\equiv$. With $\mathbb{V}(E)$ we denote the set of all variables that are free in $E \in \mathbb{E}$, i.e., not bounded by a $recX$-operator. We define $\mathbb{P} = \{E \in \mathbb{E} \mid \mathbb{V}(E) = \emptyset\}$. We use $E, F, G, H, \ldots$ (resp. $P, Q, R, \ldots$) to range over expressions from $\mathbb{E}$ (resp. $\mathbb{P}$). If $\boldsymbol{F} = F_1, \ldots, F_n$ is a sequence of expressions, $\boldsymbol{X} = X_1, \ldots, X_n$ is a sequence of variables, and $E \in \mathbb{E}$ then $E\{\boldsymbol{F}/\boldsymbol{X}\}$ denotes the expression that results from E by simultaneously replacing all free occurrences of X_i in E by F_i ($1 \leq i \leq n$). The variable X is *guarded* in E, if every free occurrence of X in E lies within a subexpression of the form $a.F$ with $a \in \mathbb{A}\backslash\{\tau\}$, otherwise X is called unguarded in E. E is guarded if for every subexpression $recY.F$ of E the variable Y is guarded in F.

The semantics of $\mathbb{E}$ is given as the least transition relation satisfying the following rules:

$$\frac{}{a.E \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{F+E \xrightarrow{a} E'}$$

[1] Note to the reviewers: The appendix contains those relevant proofs that are likely to be omitted in a published version due to space constraints.

$$\frac{E\{recX.E/X\} \xrightarrow{a} E'}{recX.E \xrightarrow{a} E'} \quad \frac{E \xrightarrow{a} E'}{\Delta(E) \xrightarrow{a} E'} \quad \frac{}{\Delta(E) \xrightarrow{\tau} \Delta(E)}$$

The rules are standard, except that, as indicated before, $\Delta(E)$ can diverge, in addition to exhibiting all the behaviour of E.

3 The bisimulations

Since we are working in the context of silent steps, we define a few standard abbreviations: $E \Longrightarrow F$ if $E \xrightarrow{\tau}^* F$; $E \overset{a}{\Longrightarrow} F$ if $E \Longrightarrow \xrightarrow{a} \Longrightarrow F$; $E \overset{\widehat{a}}{\Longrightarrow} F$ if ($E \overset{a}{\Longrightarrow} F$ and $a \neq \tau$) or ($E \Longrightarrow F$ and $a = \tau$). We write $E \xrightarrow{a}$ (resp. $E \longrightarrow$) if $E \xrightarrow{a} F$ for some $F \in \mathbb{E}$ (resp. $E \xrightarrow{a} F$ for some $a \in \mathbb{A}$, $F \in \mathbb{E}$). With $E \overset{a}{\not\longrightarrow}$ and $E \not\longrightarrow$ we denote the corresponding negated conditions. We let $E \Uparrow$ denote $E \xrightarrow{\tau}^\omega$, i.e., E has the possibility to diverge. Finally, $E \Uparrow\!\!\!|$ denotes that there is some F such that $E \Longrightarrow F$ and either $F \Uparrow$ or $F \not\longrightarrow$ (or equivalently, $E \Uparrow$ or $E \Longrightarrow F \not\longrightarrow$ for some F), i.e., E may either diverge, or silently decide to terminate. For a relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ define the following conditions (in all conditions $P, Q, P' \in \mathbb{P}$ and $a \in \mathbb{A}$ are implicitly $\forall$-quantified):

(WB) if $(P,Q) \in \mathcal{R} \wedge P \xrightarrow{a} P'$ then $Q \overset{\widehat{a}}{\Longrightarrow} Q' \wedge (P',Q') \in \mathcal{R}$ for some Q',
(S) if $(P,Q) \in \mathcal{R} \wedge P \overset{\tau}{\not\longrightarrow}$ then $Q \Longrightarrow Q' \overset{\tau}{\not\longrightarrow}$ for some Q',
(0) if $(P,Q) \in \mathcal{R} \wedge P \not\longrightarrow$ then $Q \Longrightarrow Q' \not\longrightarrow$ for some Q',
(Δ) if $(P,Q) \in \mathcal{R} \wedge P \Uparrow$ then $Q \Uparrow$,
(λ) if $(P,Q) \in \mathcal{R} \wedge P \Uparrow\!\!\!|$ then $Q \Uparrow\!\!\!|$.

Let $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation. We say that $\mathcal{R}$ is a

- *weak bisimulation* (WB^ϵ or simply WB) if $\mathcal{R}$ satisfies (WB).
- *stable weak bisimulation* (WB^S) if $\mathcal{R}$ satisfies (WB) and (S).
- *completed weak bisimulation* (WB^0) if $\mathcal{R}$ satisfies (WB) and (0).
- *divergent weak bisimulation* (WB^λ) if $\mathcal{R}$ satisfies (WB) and (λ).
- *divergent stable weak bisimulation* (WB^Δ) if $\mathcal{R}$ satisfies (WB) and (Δ).

Note that for the definition of a WB^λ, instead of condition (λ) we could also require that if $(P,Q) \in \mathcal{R}$ and ($P \Uparrow$ or $P \not\longrightarrow$) then $Q \Uparrow\!\!\!|$. In the sequel, we let $*$ range over the set $\{\Delta, \lambda, S, 0, \epsilon\}$. The relation $\sim^* \subseteq \mathbb{P} \times \mathbb{P}$ is defined as the union of all WB^*, it is easily seen to be itself a WB^* as well as an equivalence relation.

Theorem 1. [5] *The above equivalences are ordered by inclusion according to the lattice in Figure 1. The upper relation contains the lower if and only if both are connected by a line.*

$\sim^*$ is not a congruence with respect to '+' (which is a well known deficiency), and for $* \in \{\Delta, \lambda, S, 0\}$ also not a congruence with respect to $\Delta(.)$. For instance $\tau.0 \sim^\Delta 0$, but $\Delta(\tau.0) \not\sim^\Delta \Delta(0)$. To obtain the coarsest congruences in $\sim^*$ on $\mathbb{P}$, we define each $\simeq^*$ to be the relation that contains exactly the pairs $(P,Q) \in \mathbb{P} \times \mathbb{P}$ that satisfy the following *root conditions*:

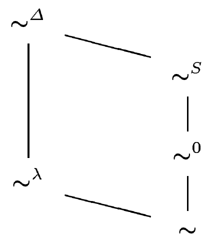

Fig. 1. Inclusions between the relations $\sim^*$

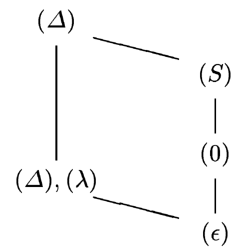

Fig. 2. Implications between the distinguishing axioms

- if $P \xrightarrow{a} P'$ then $Q \overset{a}{\Longrightarrow} Q'$ and $P' \sim^* Q'$ for some Q'
- if $Q \xrightarrow{a} Q'$ then $P \overset{a}{\Longrightarrow} P'$ and $P' \sim^* Q'$ for some P'

We lift the relations defined on $\mathbb{P}$ to $\mathbb{E}$ as usual: Let $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ and $E, F \in \mathbb{E}$. Let $\boldsymbol{X} = X_1, \ldots, X_n$ be a sequence of variables that contains all variables from $\mathbb{V}(E) \cup \mathbb{V}(F)$. Then $(E, F) \in \mathcal{R}$ if for all $\boldsymbol{P} = P_1, \ldots, P_n$ with $P_i \in \mathbb{P}$ we have $(E\{\boldsymbol{P}/\boldsymbol{X}\}, F\{\boldsymbol{P}/\boldsymbol{X}\}) \in \mathcal{R}$..

Theorem 2 (see Appendix B)**.** *The relation $\simeq^*$ is the coarsest congruence contained in $\sim^*$ w.r.t. the operators of $\mathbb{E}$. Furthermore the inclusions listed in Theorem 1 carry over from $\sim^*$ to $\simeq^*$.*

4 Axioms

This section introduces a lattice of axioms characterising the above weak bisimulations. For $* \in \{\Delta, S, 0, \epsilon\}$, the axioms for $\simeq^*$ are given in Table 1, plus the axiom $(*)$ from Table 2. The axioms for $\simeq^\lambda$ are given in Table 1, plus the axioms (Δ) and (λ) from Table 2. We write $E =^* F$ if $E = F$ can be derived by application of the axioms for $\simeq^*$.

The axioms $(S1) - (S4)$, $(\tau 1) - (\tau 3)$, and $(rec1) - (rec4)$ are standard [10]. The axiom $(rec5)$ makes divergence explicit if introduced due to silent recursion; it defines the nature of the Δ-operator. $(rec6)$ states the redundancy of recursion on an unguarded variable in the context of divergence.

We discuss the distinguishing axioms in reverse order relative to how they are listed in Table 2. Axiom (λ) characterises the property of *WB*$^\lambda$ that divergence cannot be distinguished when terminating. Axiom (ϵ) represents Milner's 'fair' setting, where divergence is never distinguished. The remaining three axioms state that divergence cannot be distinguished if the process can still perform an action to escape the divergence (0), that it cannot be distinguished if the process can perform a silent step to escape divergence (S), and that two consecutive divergences cannot be properly distinguished (Δ). It is a simple exercise to verify the implications between the distinguishing axioms as summarized in the lattice in Figure 2. It nicely reflects the inclusions between the respective congruences. The upper axioms turn into derivable laws given the lower ones (plus the core axioms from Table 1) as axioms.

$(S1)\ E + F = F + E$	$(\tau 1)\ a.\tau.E = a.E$
$(S2)\ E + (F + G) = (E + F) + G$	$(\tau 2)\ \tau.E + E = \tau.E$
$(S3)\ E + E = E$	$(\tau 3)\ a.(E + \tau.F) = a.(E + \tau.F) + a.F$
$(S4)\ E + 0 = E$	

$(rec1)$ if Y is not free in $recX.E$ then $recX.E = recY.(E\{Y/X\})$
$(rec2)$ $recX.E = E\{recX.E/X\}$
$(rec3)$ if X is guarded in E and $F = E\{F/X\}$ then $F = recX.E$
$(rec4)$ $recX.(X + E) = recX.E$
$(rec5)$ $recX.(\tau.(X + E) + F) = recX.\Delta(E + F)$
$(rec6)$ $recX.(\Delta(X + E) + F) = recX.\Delta(E + F)$

Table 1. Core axioms

(Δ)	$\Delta(\Delta(E) + F) = \tau.(\Delta(E) + F)$
(S)	$\Delta(\tau.E + F) = \tau.(\tau.E + F)$
(0)	$\Delta(a.E + F) = \tau.(a.E + F)$
(ϵ)	$\Delta(E) = \tau.E$
(λ)	$\Delta(0) = \tau.0$

Table 2. Distinguishing axioms

A few laws, derivable with the axioms for $\simeq^{\Delta}$ (and thus for all $\simeq^{*}$) give further insight, and will be useful for the further discussion.

Lemma 1 (see Appendix G). *The following laws can be derived:*

$$(\Delta') \quad \Delta(E) =^{\Delta} \Delta(E) + E$$
$$(\tau\Delta) \quad \Delta(E) =^{\Delta} \tau.\Delta(E) + E$$
$$(\tau\Delta') \quad \Delta(E) =^{\Delta} \tau.\Delta(E)$$
$$(rec7) \quad recX.(\tau.(X + E) + F) =^{\Delta} recX.(\tau.X + E + F)$$

5 Soundness and completeness

This section is devoted to the soundness of the axioms for $\simeq^*$ and the preparation of the proof of their completeness. For the latter we follow the lines of the proof of [10], and work as much as possible in the setting of WB^{Δ}, the finest setting.

Theorem 3 (soundness). *If $E, F \in \mathbb{E}$ and $E =^* F$ then $E \simeq^* F$.*

Proof. Unique solution of guarded equations, which entails soundness of ($rec3$) is shown in Appendix D. The recursion axioms ($rec5$) and ($rec6$) are treated in Appendix E. See Appendix F for the other axioms. □

In order to show completeness, i.e., that $E \simeq^* F$ implies $E =^* F$, we proceed along the lines of [10], except for the treatment of expressions from $\mathbb{E}\backslash\mathbb{P}$. As in [10] the first step consists in transforming every expression into a guarded one:

Theorem 4 (see Appendix G). *Let $E \in \mathbb{E}$. There exists a guarded F with $E =^{\Delta} F$ (and thus $\mathbb{V}(E) = \mathbb{V}(F)$).*

We do not consider $* = \epsilon$ in the sequel because by using axiom (ϵ), for every $E \in \mathbb{E}$ we find an E' such that E' does not contain the Δ-operator and $E =^{\epsilon} E'$. This allows to apply Milner's result [10] that in the absence of the Δ-operator the axioms from Table 1 without ($\tau\Delta$) together with ($rec7$) and Milner's law $recX(\tau.X + E) = recX(\tau.E)$ are complete for $\simeq^{\epsilon}$. The latter law follows immediately from ($rec5$) and (ϵ).

The basic ingredient of the completeness proof are equations systems. Let $V \subseteq \mathbb{V}$ be a set of variables and let $\boldsymbol{X} = X_1, \ldots, X_n$ be an ordered sequence of variables, where $X_i \notin V$. An *equation system over the free variables V and the formal variables $\boldsymbol{X}$* is a set of equations $\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\}$ such that $E_i \in \mathbb{E}$ and $\mathbb{V}(E_i) \subseteq \{X_1, \ldots, X_n\} \cup V$ for $1 \leq i \leq n$. Let $\boldsymbol{F} = F_1, \ldots, F_n$ be an ordered sequence of expressions. Then $\boldsymbol{F}$ $*$-provably satisfies the equation system $\mathcal{E}$ if $F_i =^* E_i\{\boldsymbol{F}/\boldsymbol{X}\}$ for all $1 \leq i \leq n$. An expression F $*$-provably satisfies $\mathcal{E}$ if there exists a sequence of expressions $F_1, \ldots, F_n$, which $*$-provably satisfies $\mathcal{E}$ and such that $F \equiv F_1$. We say that $\mathcal{E}$ is *guarded* if there exists a linear order $\prec$ on the variables $\{X_1, \ldots, X_n\}$ such that whenever the variable X_j is unguarded in the expression E_i then $X_j \prec X_i$.

For the next definition we take for each formal variable X_i ($1 \leq i \leq n$) a corresponding formal variable X_i^{Δ} such that $X_i^{\Delta} \notin \{X_1, \ldots, X_n\} \cup V$. The symbols $\alpha, \beta, \gamma, \ldots$ denote either Δ or $\_$. If e.g. $\alpha = \_$ then $X_i^{\alpha} \equiv X_i$ and $\alpha(E) \equiv E$. A *standard equation system (SES)* $\mathcal{E}$ over the free variables V and the formal variables $X_1, X_1^{\Delta}, \ldots, X_n, X_n^{\Delta}$ is an equation system of the form

$$\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\} \cup \{X_i^{\Delta} = \Delta(X_i) \mid 1 \leq i \leq n\}$$

where E_i is a sum of expressions $a.X_j$ ($a \in \mathbb{A}$, $1 \leq j \leq n$), $\tau.X_j^{\Delta}$ ($1 \leq j \leq n$), and variables $Y \in V$. We also say briefly that $\mathcal{E}$ is an SES over the free variables V and the formal variables $\boldsymbol{X} = X_1, \ldots, X_n$. If the sequence $F_1, \Delta(F_1), \ldots, F_n, \Delta(F_n)$ $*$-provably satisfies the SES $\mathcal{E}$ then we say briefly that $\boldsymbol{F} = F_1, \ldots, F_n$ $*$-provably satisfies $\mathcal{E}$. Furthermore $E_i\{\boldsymbol{F}/\boldsymbol{X}\}$ denotes the expression that results from substituting in E_i the variable X_i^α by $\alpha(F_i)$, where $1 \leq i \leq n$ and $\alpha \in \{\_, \Delta\}$. We write $X_i^\alpha \xrightarrow{a}_{\mathcal{E}} X_j^\beta$ if E_i contains the summand $a.X_j^\beta$. Note that $X_i \xrightarrow{a}_{\mathcal{E}} X_j^\beta$ if and only if $X_i^\Delta \xrightarrow{a}_{\mathcal{E}} X_j^\beta$. The notions $X_i^\alpha \Longrightarrow_{\mathcal{E}} X_j^\beta$, $X_i^\alpha \overset{a}{\Longrightarrow}_{\mathcal{E}} X_j^\beta$, $X_i \not\rightarrow_{\mathcal{E}}, \dots$ are derived from the relations $\xrightarrow{a}_{\mathcal{E}}$ analogously to the corresponding notions for $\mathbb{E}$. Note that $X_i \not\rightarrow_{\mathcal{E}}$ if and only if E_i is a sum of free variables. If the SES $\mathcal{E}$ is clear from the context then we will omit the subscript $\mathcal{E}$ in these relations. Note that $\mathcal{E}$ is guarded if and only if the relation $\xrightarrow{\tau}_{\mathcal{E}}$ is acyclic. Finally we say that the SES $\mathcal{E}$ is *saturated* if for all $1 \leq i, j \leq n$, α, β, and $Y \in V$ we have:

1. If $X_i \overset{a}{\Longrightarrow} X_j^\alpha$ then also $X_i \xrightarrow{a} X_j^\alpha$.
2. If $X_i \Longrightarrow X_j^\alpha$ and Y is a summand of E_j then Y is also a summand of E_i.

The introduction of the new variables X_i^Δ and the special form of an SES is crucial in order to carry over Milner's saturation property in the presence of the Δ-operator:

Theorem 5 (see Appendix H). *Every guarded expression E $*$-provably satisfies a guarded and saturated SES over the free variables $\mathbb{V}(E)$.*

Using axiom (*rec*3), the following theorem can be shown analogously to [10].

Theorem 6. *Let $E, F \in \mathbb{E}$ and let $\mathcal{E}$ be a guarded equation system such that both E and F $*$-provably satisfy $\mathcal{E}$. Then $E =^* F$.*

6 Joining two equation systems

In this section we will restrict to expressions from $\mathbb{P}$. The main technical result of this section is

Theorem 7. *Let $P, Q \in \mathbb{P}$ such that $P \simeq^* Q$. Furthermore P (resp. Q) $*$-provably satisfies the guarded and saturated SES $\mathcal{E}_1 = \{X_i = E_i \mid 1 \leq i \leq m\}$ (resp. $\mathcal{E}_2 = \{Y_j = F_j \mid 1 \leq j \leq n\}$). Then there exists a guarded equation system $\mathcal{E}$ such that both P and Q $*$-provably satisfy $\mathcal{E}$.*

Let us postpone the proof of Theorem 7 for a moment and first see how completeness for $\mathbb{P}$ can be deduced:

Theorem 8 (completeness for $\mathbb{P}$). *If $P, Q \in \mathbb{P}$ and $P \simeq^* Q$ then $P =^* Q$.*

Proof. By Theorem 4, both P and Q can be turned into guarded expression $P', Q' \in \mathbb{P}$ via the axioms for $\simeq^\Delta$. Due to soundness, we have that $P' \simeq^* Q'$, and by Theorem 5 P' (resp. Q') $*$-provably satisfies a guarded and saturated SES $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) without free variables. By Theorem 7 there is some guarded equation system $\mathcal{E}$ which is $*$-provably satisfied by P' and Q'. Theorem 6 gives $P' =^* Q'$, and hence $P =^* Q$, concluding the proof. □

In order to prove Theorem 7, we need the following two lemmas.

Lemma 2. *Let $\mathcal{E}$ be a guarded SES over the formal variables $X_1, \ldots, X_n$, and let X_i be such that there do not exist k, α, and $a \in \mathbb{A}\backslash\{\tau\}$ with $X_i \overset{a}{\Longrightarrow} X_k^\alpha$. Then there exist j, β with $X_i \Longrightarrow X_j^\beta \not\rightarrow$.*

Proof. Induction along the $\Longrightarrow_\mathcal{E}$, which is a partial order for a guarded SES. □

For the further consideration it is useful to define a macro $\mathcal{M}^*(P)$ for $P \in \mathbb{P}$ by

$$\mathcal{M}^*(P) = \begin{cases} P \Uparrow & \text{if } * = \Delta, \\ P \overset{\tau}{\longrightarrow} & \text{if } * = S, \\ P \longrightarrow & \text{if } * = 0, \\ P \Lsh\!\!\Uparrow & \text{if } * = \lambda. \end{cases}$$

Lemma 3 (see Appendix B)**.** *If $\Delta(P) \sim^* \Delta(Q)$ then one of the following three cases holds:*

1. *$\mathcal{M}^*(P)$ and $P \sim^* \Delta(Q)$*
2. *$\mathcal{M}^*(Q)$ and $\Delta(P) \sim^* Q$*
3. *Neither $\mathcal{M}^*(P)$ nor $\mathcal{M}^*(Q)$, and $P \sim^* Q$*

Now we are able to prove Theorem 7.

Proof (Theorem 7). Assume that $\mathcal{E}_1$ is $*$-provably satisfied by the expressions $P_1, \ldots, P_m \in \mathbb{P}$, where $P \equiv P_1$, and that $\mathcal{E}_2$ is $*$-provably satisfied by the expressions $Q_1, \ldots, Q_n \in \mathbb{P}$, where $Q \equiv Q_1$. Thus $P_i =^* E_i\{\boldsymbol{P}/\boldsymbol{X}\}$ and $Q_j =^* F_j\{\boldsymbol{Q}/\boldsymbol{Y}\}$, and hence also $P_i \simeq^* E_i\{\boldsymbol{P}/\boldsymbol{X}\}$ and $Q_j \simeq^* F_j\{\boldsymbol{Q}/\boldsymbol{Y}\}$. Since $P, Q \in \mathbb{P}$, both $\mathcal{E}_1$ and $\mathcal{E}_2$ do not have free variables. The proof of the following two claims can be found in Appendix H.

Claim 1 *If $\alpha(P_i) \sim^* \beta(Q_j)$ then the following implications hold:*

1. *If $X_i \overset{a}{\longrightarrow} X_k^\gamma$ then either ($a = \tau$ and $\gamma(P_k) \sim^* \beta(Q_j)$) or there exist ℓ, δ such that $Y_j \overset{a}{\longrightarrow} Y_\ell^\delta$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
2. *If $Y_j \overset{a}{\longrightarrow} Y_\ell^\delta$ then either ($a = \tau$ and $\alpha(P_i) \sim^* \delta(Q_\ell)$) or there exist k, γ such that $X_i^\alpha \overset{a}{\longrightarrow} X_k^\gamma$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
3. *Let $* = \Delta$. If $\alpha = \Delta$ then either $\beta = \Delta$ or $Y_j \overset{\tau}{\longrightarrow} Y_\ell^\Delta$ for some ℓ.*
4. *Let $* = \Delta$. If $\beta = \Delta$ then either $\alpha = \Delta$ or $X_i \overset{\tau}{\longrightarrow} X_k^\Delta$ for some k.*
5. *Let $* = \lambda$. If $\alpha = \Delta$ or ($\alpha = \_$ and $X_i \not\rightarrow$) then either $\beta = \Delta$, or ($\beta = \_$ and $Y_j \not\rightarrow$), or $Y_j \overset{\tau}{\longrightarrow} Y_\ell^\Delta$ for some ℓ, or $Y_j \overset{\tau}{\longrightarrow} Y_\ell \not\rightarrow$ for some ℓ.*
6. *Let $* = \lambda$. If $\beta = \Delta$ or ($\beta = \_$ and $Y_j \not\rightarrow$) then either $\alpha = \Delta$, or ($\alpha = \_$ and $X_i \not\rightarrow$), or $X_i \overset{\tau}{\longrightarrow} X_k^\Delta$ for some k, or $X_i \overset{\tau}{\longrightarrow} X_k \not\rightarrow$ for some k.*

Claim 2 *If $P_i \simeq^* Q_j$ then the following implications hold:*

1. *If $X_i \overset{a}{\longrightarrow} X_k^\alpha$ then there exist ℓ, β such that $Y_j \overset{a}{\longrightarrow} Y_\ell^\beta$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*
2. *If $Y_j \overset{a}{\longrightarrow} Y_\ell^\beta$ then there exist k, α such that $X_i \overset{a}{\longrightarrow} X_k^\alpha$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*

Now take for all $1 \leq i \leq m$, $1 \leq j \leq n$, and α, β with $\alpha(P_i) \sim^* \beta(Q_j)$ a variable $Z_{i,j}^{\alpha,\beta}$, and let $\boldsymbol{Z} = Z_{1,1}^{-,-}, \ldots$ be a sequence consisting of these variables. If furthermore $\alpha = \_$ or $\beta = \_$ then we define $G_{i,j}^{\alpha,\beta}$ as the sum, which contains the summand

$$\begin{array}{ll} a.Z_{k,\ell}^{\gamma,\delta} & \text{if } X_i \xrightarrow{a} X_k^\gamma,\ Y_j \xrightarrow{a} Y_\ell^\delta, \text{ and } \gamma(P_k) \sim^* \delta(Q_\ell), \\ \tau.Z_{k,j}^{\gamma,\beta} & \text{if } X_i \xrightarrow{\tau} X_k^\gamma \text{ but } \neg\exists \ell, \delta : Y_j \xrightarrow{\tau} Y_\ell^\delta \ \wedge\ \gamma(P_k) \sim^* \delta(Q_\ell) \\ & \text{(this implies by Claim 1(1) that } \gamma(P_k) \sim^* \beta(Q_j)), \\ \tau.Z_{i,\ell}^{\alpha,\delta} & \text{if } Y_j \xrightarrow{\tau} Y_\ell^\delta \text{ but } \neg\exists k, \gamma : X_i \xrightarrow{\tau} X_k^\gamma \ \wedge\ \gamma(P_k) \sim^* \delta(Q_\ell) \\ & \text{(this implies by Claim 1(2) that } \alpha(P_i) \sim^* \delta(Q_\ell)). \end{array}$$

Furthermore $G_{i,j}^{\alpha,\beta}$ does not contain any other summands. Now the equation system $\mathcal{E}$ over the formal variables $\boldsymbol{Z}$ contains for each variable in $\boldsymbol{Z}$ the corresponding equation below, where for equation (E1) by Lemma 3 one of the three cases listed in (E1) holds (if the first and the second case hold, then we choose arbitrarily one of the two corresponding equations for (E1).

$$\begin{array}{lll} \text{(E1)} & Z_{i,j}^{\Delta,\Delta} = & \begin{cases} Z_{i,j}^{-,\Delta} & \text{if } \mathcal{M}^*(P_i) \text{ and } P_i \sim^* \Delta(Q_j) \\ Z_{i,j}^{\Delta,-} & \text{if } \mathcal{M}^*(Q_j) \text{ and } \Delta(P_i) \sim^* Q_j \\ \Delta(Z_{i,j}^{-,-}) & \text{if neither } \mathcal{M}^*(P_i) \text{ nor } \mathcal{M}^*(Q_j), \text{ and } P_i \sim^* Q_j \end{cases} \\ \text{(E2)} & Z_{i,j}^{\alpha,\beta} = & \tau.G_{i,j}^{\alpha,\beta} \quad \text{if } \alpha = \Delta \neq \beta \text{ or } \alpha \neq \Delta = \beta \\ \text{(E3)} & Z_{i,j}^{-,-} = & G_{i,j}^{-,-} \end{array}$$

Note that $\mathcal{E}$ is not an SES. From the guardedness of $\mathcal{E}_1$ and $\mathcal{E}_2$ it follows easily that also $\mathcal{E}$ is guarded. We will show that P $*$-provably satisfies $\mathcal{E}$, that also Q $*$-provably satisfies $\mathcal{E}$ can be shown analogously. For this we define for each variable $Z_{i,j}^{\alpha,\beta}$ in $\boldsymbol{Z}$ the corresponding expression $R_{i,j}^{\alpha,\beta}$ by

$$R_{i,j}^{\Delta,\Delta} \equiv R_{i,j}^{\Delta,-} \equiv \Delta(P_i),$$

$$R_{i,j}^{-,\Delta} \equiv \tau.P_i,$$

$$R_{i,j}^{-,-} \equiv \begin{cases} P_i & \text{if } \forall \ell, \delta \left\{ Y_j \xrightarrow{a} Y_\ell^\delta \ \Rightarrow\ \exists k, \gamma \left\{ \begin{array}{l} X_i \xrightarrow{a} X_k^\gamma \ \wedge \\ \gamma(P_k) \sim^* \delta(Q_\ell) \end{array} \right\} \right\} \\ \tau.P_i & \text{if } \exists \ell, \delta \left\{ Y_j \xrightarrow{\tau} Y_\ell^\delta \ \wedge\ \neg\exists k, \gamma \left\{ \begin{array}{l} X_i \xrightarrow{\tau} X_k^\gamma \ \wedge \\ \gamma(P_k) \sim^* \delta(Q_\ell) \end{array} \right\} \right\} \end{cases}$$

and let $\boldsymbol{R} = R_{1,1}^{-,-}, \ldots$ be the sequence corresponding to the sequence $\boldsymbol{Z}$. First note that $R_{1,1}^{-,-} \equiv P_1 \equiv P$ by $P_1 \simeq^* Q_1$ and Claim 2(2). It remains to check that all equations are $*$-provably satisfied when every variable $Z_{i,j}^{\alpha,\beta}$ is replaced by $R_{i,j}^{\alpha,\beta}$. We start with equation (E1) defining $Z_{i,j}^{\Delta,\Delta}$. Since this equation belongs to $\mathcal{E}$ only if $\Delta(P_i) \sim^* \Delta(Q_j)$, we may assume this for the following cases 1-3.

Case 1. The equation $Z_{i,j}^{\Delta,\Delta} = Z_{i,j}^{\Delta,-}$ belongs to $\mathcal{E}$.

This case is trivial since $R_{i,j}^{\Delta,\Delta} \equiv R_{i,j}^{\Delta,-} \equiv \Delta(P_i)$.

Case 2. The equation $Z_{i,j}^{\Delta,\Delta} = Z_{i,j}^{-,\Delta}$ belongs to $\mathcal{E}$.

Thus $\mathcal{M}^*(P_i)$ and $P_i \sim^* \Delta(Q_j)$. Since $R_{i,j}^{-,\Delta} \equiv \tau.P_i$ and $R_{i,j}^{\Delta,\Delta} \equiv \Delta(P_i)$ we have to prove that $\tau.P_i =^* \Delta(P_i)$. We distinguish on the value of $*$.

Case 2.1. $* = \Delta$

Then $P_i \sim^\Delta \Delta(Q_j)$ and hence $X_i \xrightarrow{\tau} X_k^\Delta$ for some k by Claim 1(4). Thus there exists an expression R with (we use the derived law $(\tau\Delta')$ from Lemma 1)

$$\tau.P_i =^\Delta \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R + \tau.\Delta(P_k)) =^\Delta$$
$$\tau.(R + \Delta(P_k)) =^\Delta \Delta(R + \Delta(P_k)) =^\Delta \Delta(P_i).$$

Case 2.2. $* = S$

Then $P_i \sim^S \Delta(Q_j)$ and $\mathcal{M}^S(P_i)$, i.e., $P_i \xrightarrow{\tau}$. Since $P_i \simeq^S E_i\{\boldsymbol{P}/\boldsymbol{X}\}$, also $E_i\{\boldsymbol{P}/\boldsymbol{X}\} \xrightarrow{\tau}$, i.e, $X_i \xrightarrow{\tau}$, and there exist expressions R, P_k with

$$\tau.P_i =^S \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R + \tau.P_k) =^S \Delta(R + \tau.P_k) =^S \Delta(P_i).$$

Case 2.3. $* = 0$

Then $P_i \sim^0 \Delta(Q_j)$ and $\mathcal{M}^0(P_i)$, i.e., $P_i \longrightarrow$. Since $P_i \simeq^0 E_i\{\boldsymbol{P}/\boldsymbol{X}\}$, also $E_i\{\boldsymbol{P}/\boldsymbol{X}\} \longrightarrow$, i.e., $X_i \longrightarrow$, and there exist expressions R, P_k with

$$\tau.P_i =^0 \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\Delta \tau.(R + a.P_k) =^0 \Delta(R + a.P_k) =^0 \Delta(P_i).$$

Case 2.4. $* = \lambda$

Then $P_i \sim^\lambda \Delta(Q_j)$ and hence $X_i \not\rightarrow$, or $X_i \xrightarrow{\tau} X_k^\Delta$, or $X_i \xrightarrow{\tau} X_k \not\rightarrow$ for some k by Claim 1(6). Thus we can distinguish the following three cases:

Case 2.4.1. $X_i \not\rightarrow$, i.e., $E_i \equiv 0$ (note that if we would deal with free variables, then we could only conclude here that E_i must be a sum of free variables)

We obtain $\tau.P_i =^\lambda \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} \equiv \tau.0 =^\lambda \Delta(0) =^\lambda \Delta(P_i)$.

Case 2.4.2. $X_i \xrightarrow{\tau} X_k^\Delta$. We can conclude as in case 2.1.

Case 2.4.3. $X_i \xrightarrow{\tau} X_k \not\rightarrow$, thus there exists R with

$$\tau.P_i =^\lambda \tau.E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^\lambda \tau.(R + \tau.0) =^\lambda$$
$$\tau.(R + \Delta(0)) =^\Delta \Delta(R + \Delta(0)) =^\lambda \Delta(P_i).$$

Case 3. The equation $Z_{i,j}^{\Delta,\Delta} = \Delta(Z_{i,j}^{-,-})$ belongs to $\mathcal{E}$.

Thus $P_i \sim^* Q_j$ and neither $\mathcal{M}^*(P_i)$ nor $\mathcal{M}^*(Q_j)$ holds. We have either $R_{i,j}^{-,-} \equiv P_i$ or $R_{i,j}^{-,-} \equiv \tau.P_i$. The case that $R_{i,j}^{-,-} \equiv P_i$ is trivial, thus let us assume that $R_{i,j}^{-,-} \equiv \tau.P_i$. Then there exist ℓ, δ such that $Y_j \xrightarrow{\tau} Y_\ell^\delta$ but there do not exist k, γ with $X_i \xrightarrow{\tau} X_k^\gamma$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$. Since $\Delta(P_i) \sim^* \Delta(Q_j)$, it follows $\Delta(P_i) \sim^* \delta(Q_\ell)$ by Claim 1(2). By a case distinction on the value of $*$ we will deduce a contradiction to $\neg\mathcal{M}^*(Q_j)$.

Case 3.1. $* = \Delta$

By Claim 1(3) and $\Delta(P_i) \sim^{\Delta} \delta(Q_\ell)$ either $\delta = \Delta$ or $Y_\ell \stackrel{\tau}{\longrightarrow} Y_p^{\Delta}$ for some p. By saturation of $\mathcal{E}_2$ and $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^{\delta}$ we have $Y_j \stackrel{\tau}{\longrightarrow} Y_p^{\Delta}$ for some p. Thus $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \Uparrow$ and $Q_j \Uparrow$, a contradiction to $\neg\mathcal{M}^{\Delta}(Q_j)$.

Case 3.2. $* = S$

We have $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^{\delta}$, i.e., $Y_j \stackrel{\tau}{\longrightarrow}$, i.e., $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \stackrel{\tau}{\longrightarrow}$. But since $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \simeq^S Q_j$, we obtain $Q_j \stackrel{\tau}{\longrightarrow}$, a contradiction to $\neg\mathcal{M}^S(Q_j)$.

Case 3.3. $* = 0$

We can conclude analogously to case 3.2. with $\stackrel{\tau}{\longrightarrow}$ replaced by $\longrightarrow$.

Case 3.4. $* = \lambda$

By Claim 1(5) and $\Delta(P_i) \sim^{\lambda} \delta(Q_\ell)$ either $\delta = \Delta$, or ($\delta = \_$ and $Y_\ell \not\longrightarrow$), or $Y_\ell \stackrel{\tau}{\longrightarrow} Y_k^{\Delta}$ for some k, or $Y_\ell \stackrel{\tau}{\longrightarrow} Y_k \not\longrightarrow$ for some k. Since $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^{\delta}$, saturation of $\mathcal{E}_2$ implies that either $Y_j \stackrel{\tau}{\longrightarrow} Y_k^{\Delta}$ or $Y_j \stackrel{\tau}{\longrightarrow} Y_k \not\longrightarrow$ for some k. Hence $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \Uparrow\!\!\!\Uparrow$. Since $Q_j \simeq^{\lambda} F_j\{\boldsymbol{Q}/\boldsymbol{Y}\}$, this is a contradiction to $\neg\mathcal{M}^{\lambda}(Q_j)$.

It remains to check the equations (E2) and (E3). Fix α, β such that $\alpha(P_i) \sim^* \beta(Q_j)$ and either $\alpha = \_$ or $\beta = \_$. We will distinguish two main cases:

Case 4. It holds

$$\forall \ell, \delta \ (Y_j \stackrel{a}{\longrightarrow} Y_\ell^{\delta} \ \Rightarrow \ \exists k, \gamma : X_i \stackrel{a}{\longrightarrow} X_k^{\gamma} \ \wedge \ \gamma(P_k) \sim^* \delta(Q_\ell)). \tag{1}$$

With axiom (τ1) and (S1)-(S3) we obtain $G_{i,j}^{\alpha,\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^{\Delta} E_i\{\boldsymbol{P}/\boldsymbol{X}\} =^* P_i$ (this step is analogous to [10]).

Case 4.1. $\alpha = \beta = \_$

We have $R_{i,j}^{\_,\_} \equiv P_i =^* G_{i,j}^{\_,\_}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E3) is satisfied.

Case 4.2. $\alpha = \_$, $\beta = \Delta$

We obtain $R_{i,j}^{\_,\Delta} \equiv \tau.P_i =^* \tau.G_{i,j}^{\_,\Delta}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E2) is satisfied.

Case 4.3. $\alpha = \Delta$, $\beta = \_$

Thus $\Delta(P_i) \sim^* Q_j$. By inspecting the equation (E2) and using the fact that $R_{i,j}^{\Delta,\_} \equiv \Delta(P_i)$ and $G_{i,j}^{\alpha,\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^* P_i$, we see that it remains to show $\Delta(P_i) =^* \tau.P_i$. We distinguish on the value of $*$.

Case 4.3.1. $* = \Delta$

Thus $\Delta(P_i) \sim^{\Delta} Q_j$ and $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^{\Delta}$ for some ℓ by Claim 1(3). Hence by (1) there exist k, γ with $X_i \stackrel{\tau}{\longrightarrow} X_k^{\gamma}$ and $\gamma(P_k) \sim^{\Delta} \Delta(Q_\ell)$. By Claim 1(4) either $\gamma = \Delta$ or $X_k \stackrel{\tau}{\longrightarrow} X_p^{\Delta}$ for some p. By saturation of $\mathcal{E}_1$ we obtain in both cases $X_i \stackrel{\tau}{\longrightarrow} X_p^{\Delta}$ for some p. Now we can use the deduction form case 2.1.

Case 4.3.2. $* = S$

We have $\Delta(P_i) \sim^S Q_j$. If $Q_j \stackrel{\tau}{\not\longrightarrow}$ then $P_i \stackrel{\tau}{\longrightarrow}$. With $P_i \simeq^S E_i\{\boldsymbol{P}/\boldsymbol{X}\}$ we obtain $E_i\{\boldsymbol{P}/\boldsymbol{X}\} \stackrel{\tau}{\longrightarrow}$, i.e, $X_i \stackrel{\tau}{\longrightarrow}$, and we can use the deduction from case 2.2. On the other hand if $Q_j \stackrel{\tau}{\longrightarrow}$ then $F_j\{\boldsymbol{Q}/\boldsymbol{Y}\} \stackrel{\tau}{\longrightarrow}$, i.e, $Y_j \stackrel{\tau}{\longrightarrow}$. Thus $X_i \stackrel{\tau}{\longrightarrow}$ by (1), and we can argue in the same way as above.

Case 4.3.3. $* = 0$. Analogous to case 4.3.2.

Case 4.3.4. $* = \lambda$

Since $\Delta(P_i) \sim^{\lambda} Q_j$, Claim 1(5) implies either $Y_j \not\longrightarrow$, or $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^{\Delta}$, or $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell \not\longrightarrow$ for some ℓ. Hence we can distinguish three different cases.

Case 4.3.4.1. $Y_j \not\longrightarrow$

By Claim 1(1) there cannot exist an $a \in \mathbb{A} \backslash \{\tau\}$ with $X_i \stackrel{a}{\Longrightarrow}$. Lemma 2 and the saturation of $\mathcal{E}_1$ implies that either $X_i \not\rightarrow$ (see case 2.4.1.), or $X_i \stackrel{\tau}{\longrightarrow} X_k^\Delta$ for some k (see case 2.4.2.) or $X_i \stackrel{\tau}{\longrightarrow} X_k \not\rightarrow$ for some k (see case 2.4.3.).

Case 4.3.4.2. $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^\Delta$ for some ℓ

By (1) there exist k, γ with $X_i \stackrel{\tau}{\longrightarrow} X_k^\gamma$ and $\gamma(P_k) \sim^\lambda \Delta(Q_\ell)$. By Claim 1(6) either $\gamma = \Delta$, or ($\gamma = \_$ and $X_k \not\rightarrow$), or $X_k \stackrel{\tau}{\longrightarrow} X_p^\Delta$ for some p, or $X_k \stackrel{\tau}{\longrightarrow} X_p \not\rightarrow$ for some p. By saturation we obtain either $X_i \stackrel{\tau}{\longrightarrow} X_p^\Delta$ for some p (see case 2.4.2.), or $X_i \stackrel{\tau}{\longrightarrow} X_p \stackrel{\tau}{\not\rightarrow}$ for some p (see case 2.4.3.).

Case 4.3.4.3. $Y_j \stackrel{\tau}{\longrightarrow} Y_\ell \not\rightarrow$ for some ℓ

By (1) there exist k, γ with $X_i \stackrel{\tau}{\longrightarrow} X_k^\gamma$ and $\gamma(P_k) \sim^\lambda Q_\ell$. Using Claim 1(6) we can argue in the same way as in case 4.3.4.2.

Case 5. $\exists \ell, \delta \ (Y_j \stackrel{\tau}{\longrightarrow} Y_\ell^\delta \ \wedge \ \neg \exists k, \gamma : X_i \stackrel{\tau}{\longrightarrow} X_k^\gamma \ \wedge \ \gamma(P_k) \sim^* \delta(Q_\ell))$

Analogously to case 4, axiom (τ1) and (S1)-(S3) yields

$$G_{i,j}^{\alpha;\beta}\{\boldsymbol{R}/\boldsymbol{Z}\} =^\Delta E_i\{\boldsymbol{P}/\boldsymbol{X}\} + \tau.\alpha(P_i) =^* P_i + \tau.\alpha(P_i).$$

Case 5.1. $\alpha = \beta = \_$

We have $R_{i,j}^{-;-} \equiv \tau.P_i =^\Delta P_i + \tau.P_i =^* G_{i,j}^{-;-}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E3) is satisfied.

Case 5.2. $\alpha = \_$, $\beta = \Delta$

We obtain $R_{i,j}^{-;\Delta} \equiv \tau.P_i =^\Delta \tau.\tau.P_i =^\Delta \tau.(P_i + \tau.P_i) =^* \tau.G_{i,j}^{-;\Delta}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E2) is satisfied.

Case 5.3. $\alpha = \Delta$, $\beta = \_$

With ($\tau\Delta'$) we get $R_{i,j}^{\Delta;-} \equiv \Delta(P_i) =^\Delta \tau.\Delta(P_i) =^\Delta \tau.(P_i + \tau.\Delta(P_i)) =^* \tau.G_{i,j}^{-;\Delta}\{\boldsymbol{R}/\boldsymbol{Z}\}$, thus (E2) is again satisfied. This concludes the proof of Theorem 7 and hence of Theorem 8. □

7 Completeness for open expressions

In order to prove completeness for the whole set $\mathbb{E}$ we will argue in a purely syntactical way by investigating our axioms. The following observation is crucial:

Lemma 4 (see Appendix I). *Let $* \neq 0$ and $E, F \in \mathbb{E}$. If $a \in \mathbb{A} \backslash \{\tau\}$ does neither occur in E nor in F then $E\{a.0/X\} =^* F\{a.0/X\}$ implies $E =^* F$.*

Note that Lemma 4 is false for $* = 0$. We have $\tau.a.0 =^0 \Delta(a.0)$ but $\tau.X \neq^0 \Delta(X)$ (since $\tau.0 \not\simeq^o \Delta(0)$). Hence, in the following theorem we have to exclude $* = 0$.

Theorem 9. *Let $* \neq 0$ and $E, F \in \mathbb{E}$. If $E \simeq^* F$ then $E =^* F$.*

Proof. Let $E \simeq^* F$. We prove by induction on $|\mathbb{V}(E) \cup \mathbb{V}(F)|$ that $E =^* F$. If $\mathbb{V}(E) \cup \mathbb{V}(F) = \emptyset$ then in fact $E, F \in \mathbb{P}$ and $E =^* F$ by Theorem 8. Thus let $X \in \mathbb{V}(E) \cup \mathbb{V}(F)$. Since $E \simeq^* F$, we have $E\{a.0/X\} \simeq^* F\{a.0/X\}$. Thus by induction $E\{a.0/X\} =^* F\{a.0/X\}$ and hence $E =^* F$ by Lemma 4. □

In order to obtain a complete axiomatisation of $\simeq^0$ for open expressions, we have to introduce the following additional axiom ($\mathbb{E}$).

(𝔼) If $E\{0/X\} = F\{0/X\}$ and $E\{a.0/X\} = F\{a.0/X\}$ where $a \in \mathbb{A} \backslash \{\tau\}$ does neither occur in E nor in F then $E = F$.

The proof that this axiom is sound for $\simeq^0$ can be found at the end of Appendix I. Furthermore if we add this axiom to the standard axioms for $\simeq^0$ then we can prove completeness in the same way as in the proof of Theorem 9.

Theorem 10. *Let $E, F \in \mathbb{E}$. If $E \simeq^0 F$ then $E =^0 F$ can be derived by the standard axioms for $\simeq^0$ plus the axiom* (𝔼).

8 Conclusion

This paper has developed sound and complete axiomatisations for the divergence sensitive spectrum of weak bisimulation equivalences. We have not covered the weak bisimulation preorders $WB^{\downarrow}$ and $WB^{\downarrow\downarrow}$ considered in [5]. We claim however that adding the axiom $\Delta(E) \leq E + F$ to the axioms of WB^{λ} (respectively WB^{Δ}) is enough to obtain completeness of $WB^{\downarrow\downarrow}$ ($WB^{\downarrow}$). Note that $WB^{\downarrow}$ is axiomatised in [11], so only $WB^{\downarrow\downarrow}$ needs further work.

We are confident that our axiomatisation form the basis of a complete equational characterisation of the bisimulation fragment of the linear time – branching time spectrum with silent moves. On the technical side, we are currently investigating whether the somewhat unsatisfactory auxiliary axiom (𝔼) is indeed necessary for achieving completeness of open expressions for $\simeq^0$.

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Appendix

A Properties of the transition relations $\xrightarrow{a}$

In this section we will prove several lemmas that we will use quite extensively (later sometimes also without explicit reference) in the further discussion. In all lemmas let $E, F, G, H \in \mathbb{E}$ and $a \in \mathbb{A}$. First we define analogously to the notion of guardedness the following notions: The variable X is *weakly guarded* in E, if every free occurrence of X in E lies within a subexpression of the form $a.F$, otherwise X is called *totally unguarded* in E. Thus for instance X is unguarded but weakly guarded in $\tau.X$. Furthermore X is totally unguarded in $\Delta(X)$.

Lemma 5. *If $G \xrightarrow{a} H$ then $G\{E/X\} \xrightarrow{a} H\{E/X\}$.*

Proof. Induction on the height of the derivation tree for the transition $G \xrightarrow{a} H$. Most cases are trivial. Let us just consider the case $G \equiv \mathit{rec}Y.G'$, where w.l.o.g. $X \not\equiv Y$ and $Y \notin \mathbb{V}(E)$. [2] We have $G'\{\mathit{rec}Y.G'/Y\} \xrightarrow{a} H$. The induction hypothesis implies

$$(G'\{E/X\})\{\mathit{rec}Y.G'\{E/X\}/Y\} \equiv (G'\{\mathit{rec}Y.G'/Y\})\{E/X\} \xrightarrow{a} H\{E/X\}.$$

Thus $\mathit{rec}Y.G'\{E/X\} \xrightarrow{a} H\{E/X\}$. □

Lemma 6. *If $E \xrightarrow{a} F$ and X is totally unguarded in G then $G\{E/X\} \xrightarrow{a} F$.*

Proof. First let us claim the following implication:

If $E \xrightarrow{a} F$ and $Y \in \mathbb{V}(F)$ then $Y \in \mathbb{V}(E)$.

This implication can be easily shown by an induction on the height of the derivation tree for the transition $E \xrightarrow{a} F$. Now we can prove the lemma by induction on the structure of the expression G: The cases $G \equiv X$, $G \equiv \Delta(H)$, and $G \equiv G_1 + G_2$ are clear. Finally assume that $G \equiv \mathit{rec}Y.H$ and that $X \not\equiv Y$, $Y \notin \mathbb{V}(E)$. Thus by the implication stated above also $Y \notin \mathbb{V}(F)$. By induction we have $H\{E/X\} \xrightarrow{a} F$, thus by Lemma 5 $(H\{E/X\})\{\mathit{rec}Y.H\{E/X\}/Y\} \xrightarrow{a} F\{\mathit{rec}Y.H\{E/X\}/Y\} \equiv F$. Thus $\mathit{rec}Y.H\{E/X\} \xrightarrow{a} F$. □

Lemma 7. *Let $G\{E/X\} \xrightarrow{a} F$ be derivable by a derivation tree of height n. Then one of the following two cases holds:*

1. *X is totally unguarded in G and $E \xrightarrow{a} F$, which can be derived by a derivation tree of height $\leq n$.*
2. *$G \xrightarrow{a} H$ and $F \equiv H\{E/X\}$. Furthermore if X is guarded in G and $a = \tau$ then X is also guarded in H.*

[2] The latter can be assumed since during a substitution bounded variables are renamed in order to avoid new variable bindings.

Proof. Induction on n: The case that G has the form $X \in \mathbb{V}$, $a.G'$, or $G_1 + G_2$ is clear. If $G \equiv \Delta(G')$ and $\Delta(G'\{E/X\}) \stackrel{a}{\longrightarrow} F$ then either $F \equiv \Delta(G'\{E/X\})$ and $a = \tau$, and we obtain the second case, or $G'\{E/X\} \stackrel{a}{\longrightarrow} F$, which can be derived by a derivation tree of height $n-1$. By induction, either

- X is totally unguarded in G' (and hence totally unguarded in G) and $E \stackrel{a}{\longrightarrow} F$ by a derivation tree of height $\leq n-1$ or
- $G' \stackrel{a}{\longrightarrow} H$ (and thus $G \stackrel{a}{\longrightarrow} H$), $F \equiv H\{E/X\}$, and if X is guarded in G' (i.e., guarded in G) and $a = \tau$ then X is also guarded in H.

Finally assume that $G \equiv recY.G'$ and w.l.o.g. that $X \not\equiv Y$ and $Y \notin \mathbb{V}(E)$. Thus

$$(G'\{recY.G'/Y\})\{E/X\} \equiv (G'\{E/X\})\{recY.G'\{E/X\}/Y\} \stackrel{a}{\longrightarrow} F$$

and the induction hypothesis implies that either

1. X is totally unguarded in $G'\{recY.G'/Y\}$ (i.e., totally unguarded in G) and $E \stackrel{a}{\longrightarrow} F$, which can be derived by a derivation tree of height $\leq n-1$, or
2. $G'\{recY.G'/Y\} \stackrel{a}{\longrightarrow} H$ (i.e., $G \stackrel{a}{\longrightarrow} H$) and $F \equiv H\{E/X\}$. Furthermore if X is guarded in $G'\{recY.G'/Y\}$ (i.e., guarded in G) and $a = \tau$ then X is also guarded in H. □

The preceding three lemmas easily imply the next two lemmas:

Lemma 8. *$G\{E/X\} \stackrel{\tau}{\not\longrightarrow}$ (resp. $G\{E/X\} \not\longrightarrow$) if and only if $G \stackrel{\tau}{\not\longrightarrow}$ (resp. $G \not\longrightarrow$) and (X is weakly guarded in G or $E \stackrel{\tau}{\not\longrightarrow}$ (resp. $E \not\longrightarrow$)).*

Lemma 9. *$G\{E/X\} \Uparrow$ if and only if $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $E \Uparrow$).*

Lemma 10. *$recX.G \stackrel{a}{\longrightarrow} E$ if and only if $G \stackrel{a}{\longrightarrow} H$ and $E \equiv H\{recX.G/X\}$ for some H.*

Proof. If $G \stackrel{a}{\longrightarrow} H$ and $E \equiv H\{recX.G/X\}$ then $G\{recX.G/X\} \stackrel{a}{\longrightarrow} E$, i.e, $recX.G \stackrel{a}{\longrightarrow} E$ by Lemma 5. For the other direction assume that $recX.G \stackrel{a}{\longrightarrow} E$ can be derived by a derivation tree of height n but there does not exist a derivation tree for this transition of height $< n$. Then $G\{recX.G/X\} \stackrel{a}{\longrightarrow} E$ can be derived by a derivation tree of height $n-1$. By Lemma 7 either $G \stackrel{a}{\longrightarrow} H$ and $E \equiv H\{recX.G/X\}$ for some H, or X is totally unguarded in G and $recX.G \stackrel{a}{\longrightarrow} E$ can be derived by a derivation tree of height $\leq n-1$. But the second alternative gives a contradiction. □

Lemma 11. *$recX.G \Uparrow$ if and only if $G \Uparrow$ or ($G \stackrel{\tau}{\Longrightarrow} H$ and X is totally unguarded in H).*

Proof. If $G \Uparrow$ then by Lemma 9 also $G\{recX.G/X\} \Uparrow$, i.e., $recX.G \Uparrow$. If $G \stackrel{\tau}{\Longrightarrow} H$ and X is totally unguarded in H then $G\{recX.G/X\} \stackrel{\tau}{\Longrightarrow} H\{recX.G/X\}$, i.e, $recX.G \stackrel{\tau}{\longrightarrow} E \Longrightarrow H\{recX.G/X\}$ for some E. Since X is totally unguarded in

H, Lemma 6 implies $recX.G \xrightarrow{\tau} E \Longrightarrow H\{recX.G/X\} \xrightarrow{\tau} E$, thus $recX.G \Uparrow$. Finally assume that $recX.G \Uparrow$. Then $recX.G \xrightarrow{\tau} E$ and $E \Uparrow$. By Lemma 10 we have $G \xrightarrow{\tau} G'$ and $E \equiv G'\{recX.G/X\} \Uparrow$ for some G'. By Lemma 9 either $G' \Uparrow$ (and thus $G \Uparrow$), or $G' \Longrightarrow H$ (and thus $G \overset{\tau}{\Longrightarrow} H$) for some H such that X is totally unguarded in H. □

B Properties of $\sim^*$ and $\simeq^*$

Lemma 12. *Let $P, Q \in \mathbb{P}$. If $P + R \sim^* Q + R$ for all $R \in \mathbb{P}$ then $P \simeq^* Q$.*

Proof. Assume that $P + R \sim^* Q + R$ for all $R \in \mathbb{P}$ and $P \not\simeq^* Q$. By the definition of $\simeq^*$, there is some $a \in \mathbb{A}$ such that, w.l.o.g. $P \xrightarrow{a} P'$ but whenever $Q \overset{a}{\Longrightarrow} Q'$ then $P' \not\sim^* Q'$. Choose $R \equiv b.0$ where $b \in \mathbb{A}$ does neither occur in P nor in Q. Clearly, $P + R \xrightarrow{a} P'$, and since $P + R \sim^* Q + R$ there is some Q' such that $Q + R \overset{\widehat{a}}{\Longrightarrow} Q'$ and $P' \sim^* Q'$. However, if $a = \tau$ and $Q' \equiv Q + R$, then $P' \not\sim^* Q'$ since Q' may do a b-transition whereas P' does not have this possibility; otherwise $Q \overset{a}{\Longrightarrow} Q'$ (since $R \overset{a}{\Longrightarrow} P'$ is impossible because $b \neq a$), so again $P' \not\sim^* Q'$ and we conclude in either case by contradiction. □

Theorem 11 (restated Theorem 2). *The relation $\simeq^*$ is the coarsest congruence contained in $\sim^*$ w.r.t. the operators of $\mathbb{E}$. Furthermore the inclusions listed in Theorem 1 carry over from $\sim^*$ to $\simeq^*$.*

Proof. That the inclusions from Figure 1 also hold for the relations $\simeq^*$ is easy to check. The inclusion $\simeq^* \subseteq \sim^*$ can be verified by proving that the relation $\simeq^* \cup \sim^*$ is a *WB**. The condition (*WB*) is trivial due to the root condition for $\simeq^*$. Also the verification of the remaining condition $(*)$, where $* \in \{S, 0, \Delta, \lambda\}$, is straight-forward, let us only consider the condition (λ). Thus assume that $P \Uparrow\!\!\!\Uparrow$ and $P \simeq^* Q$ (the case $P \sim^* Q$ is clear). Then either $P \not\rightarrow$ or $P \xrightarrow{\tau} P' \Uparrow\!\!\!\Uparrow$ for some P'. In the first case, $P \simeq^\lambda Q$ implies $Q \not\rightarrow$. In the second case, $P \simeq^\lambda Q$ implies $Q \overset{\tau}{\Longrightarrow} Q'$ and $P' \sim^\lambda Q'$ for some Q'. Since $P' \Uparrow\!\!\!\Uparrow$ we obtain $Q' \Uparrow\!\!\!\Uparrow$ and thus $Q \Uparrow\!\!\!\Uparrow$.

It remains to show that $\simeq^*$ is the coarsest congruence w.r.t. to the operators of $\mathbb{E}$. Congruence with respect to '+' and action prefix is clear (for the congruence w.r.t. action prefix we have to use the inclusion $\simeq^* \subseteq \sim^*$). Congruence w.r.t. Δ can be see as follows. Assume that $P \simeq^* Q$. First note that $\{(\Delta(P), \Delta(Q)), (\Delta(Q), \Delta(P))\} \cup \sim^*$ is a *WB**. Thus we have $\Delta(P) \sim^* \Delta(Q)$. From this we deduce easily that the pair $(\Delta(P), \Delta(Q))$ satisfies also the root condition, i.e., $\Delta(P) \simeq^* \Delta(Q)$. The congruence proof w.r.t. the recursion operator (the only hard part) is shifted to Appendix C.

It remains to argue that $\simeq^*$ is in fact the coarsest congruence contained in $\sim^*$. Assume that $\mathcal{R} \subseteq \sim^*$ is a congruence with respect to the operators of $\mathbb{E}$. Let $(P, Q) \in \mathcal{R}$. Thus for all $R \in \mathbb{P}$ we have $(P+R, Q+R) \in \mathcal{R}$, i.e., $P+R \sim^* Q+R$. By Lemma 12 we have $P \simeq^* Q$. Thus $\mathcal{R} \subseteq \simeq^*$. □

Lemma 13 (restated Lemma 3). *If $\Delta(P) \sim^* \Delta(Q)$ then one of the following three cases holds:*

1. $\mathcal{M}^*(P)$ *and* $P \sim^* \Delta(Q)$
2. $\mathcal{M}^*(Q)$ *and* $\Delta(P) \sim^* Q$
3. Neither $\mathcal{M}^*(P)$ *nor* $\mathcal{M}^*(Q)$, *and* $P \sim^* Q$

Proof. Assume that $\Delta(P) \sim^* \Delta(Q)$. First we claim that

$$\text{if } \mathcal{M}^*(P) \text{ then } P \sim^* \Delta(P).$$

In order to see this, note that if $\mathcal{M}^*(P)$ then $\mathrm{Id}_{\mathbb{P}} \cup \{(P, \Delta(P)), (\Delta(P), P)\}$ is a *WB**. Thus if $\mathcal{M}^*(P)$ then $P \sim^* \Delta(P) \sim^* \Delta(Q)$. If $\mathcal{M}^*(Q)$ we can argue analogously. Thus assume that neither $\mathcal{M}^*(P)$ nor $\mathcal{M}^*(Q)$. We have to prove that $P \sim^* Q$. For this we show that $\mathcal{R} = \{(P,Q),(Q,P)\} \cup \sim^*$ is a *WB**. Since neither $\mathcal{M}^*(P)$ nor $\mathcal{M}^*(Q)$, it is easy to see that $\mathcal{R}$ satisfies the condition $(*)$ from the definition of a *WB**. Hence it remains to check condition (*WB*). For pairs in $\sim^*$ this is trivial. Thus, let us consider w.l.o.g. the pair (P,Q), where $P \xrightarrow{a} P'$. Then also $\Delta(P) \xrightarrow{a} P'$. Since $\Delta(P) \sim^* \Delta(Q)$ we obtain $\Delta(Q) \stackrel{\widehat{a}}{\Longrightarrow} Q'$ for some Q' with $P' \sim^* Q'$. If in fact $\Delta(Q) \stackrel{a}{\Longrightarrow} Q'$ then we also have $Q \stackrel{a}{\Longrightarrow} Q'$. On the other hand if $Q' \equiv \Delta(Q)$ and $a = \tau$ then $P \xrightarrow{\tau} P' \sim^* \Delta(Q)$. But for all choices of $* \in \{\Delta, S, 0, \lambda\}$ this gives a contradiction to $\neg\mathcal{M}^*(P)$. □

C Congruence with respect to *rec*

This section is devoted to the congruence property of $\simeq^*$ w.r.t. recursion. Let $\mathcal{S} \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation. We say that $\mathcal{S}$ is an *observational congruence up to* $\sim^*$ if $(P,Q) \in \mathcal{S}$ implies for all $a \in \mathbb{A}$ and $P' \in \mathbb{P}$ the following:

(*WB'*) if $P \xrightarrow{a} P'$ then $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \;\mathcal{S}\; R \sim^* Q'$ for some $R, Q' \in \mathbb{P}$
(Δ') if $* = \Delta$ and $P \Uparrow$ then also $Q \Uparrow$
(λ') if $* = \lambda$ and $P \Uparrow$ then $Q \Uparrow$ or $Q \stackrel{\tau}{\Longrightarrow} Q' \not\rightarrow$ for some Q'

Note that if $\mathcal{S}$ is an observational congruence up to $\sim^*$ then $(P,Q) \in \mathcal{S}$ and $P \stackrel{\tau}{\not\rightarrow}$ (resp. $P \not\rightarrow$) implies by (*WB'*) that also $Q \stackrel{\tau}{\not\rightarrow}$ (resp. $Q \not\rightarrow$). We are aiming to show that $P \simeq^* Q$ holds already if there is an observational congruence up to $\sim^*$ between P and Q. This will be expressed in Lemma 16. We need the following lemmas beforehand.

Lemma 14. *If* $\mathcal{S}$ *is an observational congruence up to* $\sim^*$ *then* $(P,Q) \in \mathcal{S}$ *and* $P \stackrel{\widehat{a}}{\Longrightarrow} P'$ *imply* $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ *and* $P' \;\mathcal{S}\; R \sim^* Q'$ *for some* $Q', R \in \mathbb{P}$.

Proof. It is sufficient to show that

$$\text{if } P \;\mathcal{S}\; R \sim^* Q \text{ and } P \xrightarrow{a} P' \text{ then } Q \stackrel{\widehat{a}}{\Longrightarrow} Q' \text{ and} \\ P' \;\mathcal{S}\; R' \sim^* Q' \text{ for some } Q', R' \in \mathbb{P},$$

because this implies that we can trace the chain $P \stackrel{\widehat{a}}{\Longrightarrow} P'$ transition by transition choosing $R \equiv Q$ for the first step. So, let $P \;\mathcal{S}\; R \sim^* Q$ and $P \xrightarrow{a} P'$. We derive using property (*WB'*) $R \stackrel{a}{\Longrightarrow} R''$ and $P' \;\mathcal{S}\; R' \sim^* R''$ for some $R', R'' \in \mathbb{P}$. Now, $R \stackrel{a}{\Longrightarrow} R''$ and $R \sim^* Q$ imply $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $R'' \sim^* Q'$ for some $Q' \in \mathbb{P}$. As a whole, we obtain $P' \mathcal{S} R' \sim^* R'' \sim^* Q'$, i.e., $P' \mathcal{S} R' \sim^* Q'$. □

Lemma 15. *If $\mathcal{S}$ is an observational congruence up to $\sim^*$ then $\sim^* \mathcal{S} \sim^*$ is a WB*.*

Proof. Note that since $\mathcal{S}$ is symmetric, also $\sim^* \mathcal{S} \sim^*$ is symmetric. Assume that $P \sim^* R_1 \, \mathcal{S} \, R_2 \sim^* Q$.

(*WB*): Assume that $P \xrightarrow{a} P'$. We have to show that $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $P' \sim^* \mathcal{S} \sim^* Q'$ for some Q'. Since $P \sim^* R_1$ there exists an R_1' such that $R_1 \stackrel{\widehat{a}}{\Longrightarrow} R_1'$ and $P' \sim^* R_1'$. By Lemma 14 and $(R_1, R_2) \in \mathcal{S}$ there exists an R_2' with $R_2 \stackrel{\widehat{a}}{\Longrightarrow} R_2'$ and $R_1' \, \mathcal{S} \sim^* R_2'$. Finally $R_2 \sim^* Q$ implies $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $R_2' \sim^* Q'$ for some Q'. Altogether we have $P' \sim^* R_1' \, \mathcal{S} \, \sim^* R_2' \sim^* Q'$.

(Δ): Let $* = \Delta$ and $P \Uparrow$. Then $Q \Uparrow$ follows immediately.

(S): Let $* = S$ and $P \stackrel{\tau}{\not\longrightarrow}$. Since $P \sim^S R_1$ we have $R_1 \Longrightarrow R_1' \stackrel{\tau}{\not\longrightarrow}$ for some R_1'. Since $(R_1, R_2) \in \mathcal{S}$ there exists by Lemma 14 an R_2' with $R_2 \Longrightarrow R_2'$ and $R_1' \; \mathcal{S} \; R' \sim^S R_2'$. Since $R_1' \stackrel{\tau}{\not\longrightarrow}$ also $R' \stackrel{\tau}{\not\longrightarrow}$. Next $R' \sim^S R_2'$ and $R' \stackrel{\tau}{\not\longrightarrow}$ implies $R_2' \Longrightarrow R_2'' \stackrel{\tau}{\not\longrightarrow}$ for some R_2''. Now $R_2 \Longrightarrow R_2' \Longrightarrow R_2''$ and $R_2 \sim^S Q$ implies $Q \Longrightarrow Q''$ and $R_2'' \sim^S Q''$ for some Q''. Finally since $R_2'' \stackrel{\tau}{\not\longrightarrow}$ there exists a Q' with $Q'' \Longrightarrow Q' \stackrel{\tau}{\not\longrightarrow}$.

(0): Analogously to (S) with $\xrightarrow{\tau}$ replaced by $\longrightarrow$.

(λ): Let $* = \lambda$ and assume that $P \Uparrow$. Since $P \sim^\lambda R_1$, either $R_1 \Uparrow$ or $R_1 \Longrightarrow R_1' \not\longrightarrow$ for some R_1'. If $R_1 \Uparrow$ then by $(R_1, R_2) \in \mathcal{S}$ we have $R_2 \Uparrow\!\!\Uparrow$. Since $R_2 \sim^\lambda Q$ this implies $Q \Uparrow\!\!\Uparrow$. On the other hand if $R_1 \Longrightarrow R_1' \not\longrightarrow$ then by $(R_1, R_2) \in \mathcal{S}$ and Lemma 14 there exist R, R_2' with $R_2 \Longrightarrow R_2'$ and $R_1' \, \mathcal{S} \, R \sim^\lambda R_2'$. Since $R_1' \not\longrightarrow$ also $R \not\longrightarrow$. Since $R \sim^\lambda R_2'$ we have $R_2' \Uparrow\!\!\Uparrow$. Thus $R_2 \Uparrow\!\!\Uparrow$ and we can conclude as above for $R_1 \Uparrow$. □

Lemma 16. *If $\mathcal{S}$ is an observational congruence up to $\sim^*$ then $\mathcal{S} \subseteq \simeq^*$.*

Proof. Assume that $(P, Q) \in \mathcal{S}$, where $\mathcal{S}$ is an observational congruence up to $\sim^*$. By Lemma 15 the relation $\sim^* \mathcal{S} \sim^*$ is a WB*. Since $\mathrm{Id}_{\mathbb{P}} \subseteq \sim^*$, we have $\mathcal{S} \sim^* \subseteq \sim^*$. Therefore if $P \xrightarrow{a} P'$ then (*WB*$'$) implies $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \sim^* Q'$ for some $Q' \in \mathbb{P}$. Furthermore since $\mathcal{S}$ and $\sim^*$ are symmetric, also the symmetric root condition holds. Thus $P \simeq^* Q$. □

In order to prove $recX.E \simeq^* recX.F$ if $E \sim^* F$, it is sufficient to construct an observational congruence up to $\sim^*$ containing the pair $(recX.E, recX.F)$. This will be done in Lemma 18. We will need the following statement.

Lemma 17. *Let $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \simeq^* F$, i.e, $E\{P/X\} \simeq^* F\{P/X\}$ for all $P \in \mathbb{P}$. Then the following holds:*

1. *If $E \stackrel{\tau}{\Longrightarrow} E'$, where X is totally unguarded in E', then also $F \stackrel{\tau}{\Longrightarrow} F'$ for some F' such that X is totally unguarded in F'.*
2. *If $* = \Delta$ and $E \Uparrow$ then also $F \Uparrow$.*
3. *If $* = \lambda$ and $E \Uparrow$ then either $F \Uparrow$ or $F \stackrel{\tau}{\Longrightarrow} F' \not\longrightarrow$ for some F' such that X is weakly guarded in F'.*

Proof. We will only prove the last statement, the other statements can be shown similarly. Thus assume that $* = \lambda$ and $E \Uparrow$. Let $a \in \mathbb{A}\backslash\{\tau\}$. Since $E \Uparrow$, also $E\{a.0/X\} \Uparrow$ by Lemma 9. Then $E\{a.0/X\} \simeq^{\lambda} F\{a.0/X\}$ implies that either $F\{a.0/X\} \Uparrow$ or $F\{a.0/X\} \stackrel{\tau}{\Longrightarrow} Q \not\rightarrow$ for some Q. If $F\{a.0/X\} \Uparrow$ then Lemma 9 implies $F \Uparrow$. If $F\{a.0/X\} \stackrel{\tau}{\Longrightarrow} Q \not\rightarrow$ then $a \neq \tau$ and Lemma 7 implies $Q \equiv G\{a.0/X\}$ and $F \stackrel{\tau}{\Longrightarrow} G$ for some G. Finally, since $G\{a.0/X\} \not\rightarrow$, X must be weakly guarded in G by Lemma 8. □

Lemma 18. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \simeq^* F$. Furthermore let*

$$\mathcal{R} = \{\langle G\{recX.E/X\}, G\{recX.F/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\}.$$

Then $\mathcal{S} = (\mathcal{R} \cup \mathcal{R}^{-1})$ is an observational congruence up to $\sim^*$.

Proof. By symmetry it suffices to consider a pair $\langle G\{recX.E/X\}, G\{recX.F/X\}\rangle$.

(*WB′*): We proceed by an induction on the height of the derivation tree for the transition $G\{recX.E/X\} \stackrel{a}{\longrightarrow} P$.

Case 1. $G \equiv 0$ or $G \equiv a.H$ for some $a \in \mathbb{A}$: trivial

Case 2. $G \equiv X$: Assume $recX.E \stackrel{a}{\longrightarrow} P$. Thus $E\{recX.E/X\} \stackrel{a}{\longrightarrow} P$, which can be derived by a smaller derivation tree. Thus, the induction hypothesis implies $E\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q'$ and $P\,\mathcal{S}\,R \sim^* Q'$ for some R, Q'. Since $E \simeq^* F$, we have $E\{recX.F/X\} \simeq^* F\{recX.F/X\}$. This implies $F\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ and $Q' \sim^* Q$ for some Q and thus finally $recX.F \stackrel{a}{\Longrightarrow} Q$ and $P\,\mathcal{S}\,R \sim^* Q' \sim^* Q$.

Case 3. $G \equiv \Delta(H)$: Assume $\Delta(H\{recX.E/X\}) \stackrel{a}{\longrightarrow} P$. The case $a = \tau$ and $P \equiv \Delta(H\{recX.E/X\})$ is trivial. On the other hand if $H\{recX.E/X\} \stackrel{a}{\longrightarrow} P$ then by induction $H\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \sim^* Q$. Thus also $\Delta(H\{recX.F/X\}) \stackrel{a}{\longrightarrow} Q$.

Case 4. $G \equiv H_1 + H_2$: Assume $H_1\{recX.E/X\} + H_2\{recX.E/X\} \stackrel{a}{\longrightarrow} P$. W.l.o.g. it holds $H_1\{recX.E/X\} \stackrel{a}{\longrightarrow} P$. By induction $H_1\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \sim^* Q$. Thus $H_1\{recX.F/X\} + H_2\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$.

Case 5. $G \equiv recY.H$: By renaming bounded variables we may assume that $Y \not\equiv X$ and $Y \notin \mathbb{V}(recX.E) \cup \mathbb{V}(recX.F)$ which implies $(recY.H)\{recX.E/X\} \equiv recY.H\{recX.E/X\}$ and $(recY.H)\{recX.F/X\} \equiv recY.H\{recX.F/X\}$. Assume $recY.H\{recX.E/X\} \stackrel{a}{\longrightarrow} P$, hence

$$\begin{aligned}&(H\{recX.E/X\})\{recY.H\{recX.E/X\}/Y\} \equiv \\ &(H\{recY.H/Y\})\{recX.E/X\} \stackrel{a}{\longrightarrow} P\end{aligned}$$

by a smaller derivation tree. By induction $(H\{recY.H/Y\})\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$ for some $Q \in \mathbb{P}$ with $P\,\mathcal{S}\,R \sim^* Q$, which implies $recY.H\{recX.F/X\} \stackrel{a}{\Longrightarrow} Q$.

(Δ'): Assume that $* = \Delta$ and $G\{recX.E/X\} \Uparrow$. We have to show that also $G\{recX.F/X\} \Uparrow$. First consider the case $G \equiv X$, i.e., let $recX.E \Uparrow$. By Lemma 11 either $E \Uparrow$ or $E \stackrel{\tau}{\Longrightarrow} E'$ for some E' such that X is totally unguarded in E'. If $E \Uparrow$ then $E \simeq^{\Delta} F$ and Lemma 17(2) imply $F \Uparrow$, thus also $recX.F \Uparrow$ by Lemma 11. Similarly, if $E \stackrel{\tau}{\Longrightarrow} E'$, where X is totally unguarded in E', then

$E \simeq^{\Delta} F$ and Lemma 17(1) imply $F \stackrel{\tau}{\Longrightarrow} F'$ for some F' such that X is totally unguarded in F'. Thus $recX.F \Uparrow$ by Lemma 11.

Now assume that G is arbitrary and that $G\{recX.E/X\} \Uparrow$. By Lemma 9 either $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$). If $G \Uparrow$ then also $G\{recX.F/X\} \Uparrow$. On the other hand assume that $G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$. Since $recX.E \Uparrow$, from the previous paragraph we obtain $recX.F \Uparrow$. Finally $G \Longrightarrow H$ and X totally unguarded in H imply $G\{recX.F/X\} \Uparrow$ by Lemma 9.

(λ'): Assume that $* = \lambda$ and $G\{recX.E/X\} \Uparrow$. Again we first consider the case $G \equiv X$, i.e., $recX.E \Uparrow$. As for $* = \Delta$, we have either $E \Uparrow$ or $E \stackrel{\tau}{\Longrightarrow} E'$ for some E' such that X is totally unguarded in E'. If $E \stackrel{\tau}{\Longrightarrow} E'$ for some E' such that X is totally unguarded in E' then analogously to (Δ') we obtain $recX.F \Uparrow$. Thus assume $E \Uparrow$. Then $E \simeq^{\lambda} F$ and Lemma 17(3) imply either $F \Uparrow$ or $F \stackrel{\tau}{\Longrightarrow} F' \not\rightarrow$ for some F' such that X is weakly guarded in F'. If $F \Uparrow$ then also $recX.F \Uparrow$ by Lemma 11. If $F \stackrel{\tau}{\Longrightarrow} F' \not\rightarrow$ for some F' such that X is weakly guarded in F' then $F\{recX.F/X\} \stackrel{\tau}{\Longrightarrow} F'\{recX.F/X\}$, i.e, $recX.F \stackrel{\tau}{\Longrightarrow} F'\{recX.F/X\}$, by Lemma 5. Furthermore, since $F' \not\rightarrow$ and X is weakly guarded in F', we have $F'\{recX.F/X\} \not\rightarrow$ by Lemma 8.

If G is arbitrary and $G\{recX.E/X\} \Uparrow$ then as for (Δ') either $G \Uparrow$ or ($G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$). If $G \Uparrow$ then also $G\{recX.F/X\} \Uparrow$. Thus assume that $G \Longrightarrow H$, X is totally unguarded in H, and $recX.E \Uparrow$. From the previous paragraph we obtain either $recX.F \Uparrow$ or $recX.F \stackrel{\tau}{\Longrightarrow} Q \not\rightarrow$ for some Q. If $recX.F \Uparrow$ then $G\{recX.F/X\} \Uparrow$ by Lemma 9. On the other hand if $recX.F \stackrel{\tau}{\Longrightarrow} Q \not\rightarrow$, then, since X is totally unguarded in H, $G\{recX.F/X\} \Longrightarrow H\{recX.F/X\} \stackrel{\tau}{\Longrightarrow} Q \not\rightarrow$ by Lemma 5 and Lemma 6. □

Eventually, we have all the means to derive that $\simeq^*$ is a congruence with respect to *rec*.

Corollary 1. If $E, F \in \mathbb{E}$ then $E \simeq^* F$ implies $recX.E \simeq^* recX.F$.

Proof. Due to the definition of $\simeq^*$ for expressions with free variables, it suffices to consider only those $E, F \in \mathbb{E}$ where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Assume that $E \simeq^* F$ holds. Then the relation $\mathcal{S}$ appearing in Lemma 18 is an observational congruence up to $\sim^*$. Choosing $G \equiv X$ implies $\langle recX.E, recX.F\rangle \in \mathcal{S}$ and thus $recX.E \simeq^* recX.F$ by Lemma 16. □

D Unique solution of guarded equations

Abusing notation relative to Appendix C, we shall for this section redefine the notion of an observational congruence up to $\sim^*$ as follows: A symmetric relation $\mathcal{S} \subseteq \mathbb{P} \times \mathbb{P}$ is an observational congruence up to $\sim^*$ if $(P, Q) \in \mathcal{S}$ implies for all $a \in \mathbb{A}$ and $P' \in \mathbb{P}$ the following:

(*WB'*) if $P \stackrel{a}{\Longrightarrow} P'$ then $Q \stackrel{a}{\Longrightarrow} Q'$ and $P' \sim^* R_1 \; \mathcal{S} \; R_2 \sim^* Q'$ for some $Q', R_1, R_2 \in \mathbb{P}$

(Δ') if $* = \Delta$ and $P \Uparrow$ then $Q \Uparrow$
(S') if $* = S$ and $P \Longrightarrow P' \stackrel{\tau}{\not\longrightarrow}$ then $Q \Longrightarrow Q' \stackrel{\tau}{\not\longrightarrow}$ for some $Q' \in \mathbb{P}$
($0'$) if $* = 0$ and $P \Longrightarrow P' \not\longrightarrow$ then $Q \Longrightarrow Q' \not\longrightarrow$ for some $Q' \in \mathbb{P}$
(λ') if $* = \lambda$ and $P \Uparrow\!\!\Uparrow$ then $Q \Uparrow\!\!\Uparrow$

Lemma 19. *If $\mathcal{S}$ is an observational congruence up to $\sim^*$ then $\mathcal{S} \subseteq \simeq^*$.*

Proof. First we prove that $\sim^* \mathcal{S} \sim^*$ is a *WB**, the rest of the proof is analogous to the proof of Lemma 16. Assume that $P \sim^* R_1 \mathcal{S} R_2 \sim^* Q$.

(*WB*): Assume that $P \stackrel{a}{\longrightarrow} P'$. Then there exists an R_1' with $R_1 \stackrel{\widehat{a}}{\Longrightarrow} R_1'$ and $P' \sim^* R_1'$. The case $R_1 \equiv R_1'$ is clear. Thus let us assume that $R_1 \stackrel{a}{\Longrightarrow} R_1'$. Then (*WB*$'$) implies $R_2 \stackrel{a}{\Longrightarrow} R_2'$ and $R_1' \sim^* \mathcal{S} \sim^* R_2'$ for some R_2'. Finally $R_2 \stackrel{a}{\Longrightarrow} R_2'$ and $R_2 \sim^* Q$ implies $Q \stackrel{\widehat{a}}{\Longrightarrow} Q'$ and $R_2' \sim^* Q'$ for some Q'.

(Δ): straight-forward

(S): Assume that $* = S$ and $P \stackrel{\tau}{\not\longrightarrow}$. Then $P \sim^S R_1$ implies $R_1 \Longrightarrow R_1' \stackrel{\tau}{\not\longrightarrow}$ for some R_1'. With $(R_1, R_2) \in \mathcal{S}$ and (S') we get $R_2 \Longrightarrow R_2' \stackrel{\tau}{\not\longrightarrow}$, which finally with $R_2 \sim^S Q$ implies $Q \Longrightarrow Q' \stackrel{\tau}{\not\longrightarrow}$ for some Q'.

(0): Analogously to (S) with $\stackrel{\tau}{\longrightarrow}$ replaced by $\longrightarrow$.

(λ): straight-forward ☐

Lemma 20. *Assume that*

- *$E \in \mathbb{E}$, $P, Q \in \mathbb{P}$,*
- *X is guarded in E, $\mathbb{V}(E) \subseteq \{X\}$,*
- *$P \simeq^* E\{P/X\}$ and $Q \simeq^* E\{Q/X\}$.*

Then $\mathcal{S} = \{\langle G\{P/X\}, G\{Q/X\}\rangle, \langle G\{Q/X\}, G\{P/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\}$ is an observational congruence up to $\sim^*$.

Proof. By symmetry it suffices to consider a pair $\langle G\{P/X\}, G\{Q/X\}\rangle$, where $\mathbb{V}(G) \subseteq \{X\}$.
(*WB*$'$): Before we prove (*WB*$'$) we first prove the following weaker condition (wb$'$):

$$\text{If } G\{P/X\} \Longrightarrow P' \text{ then } G\{Q/X\} \Longrightarrow Q' \text{ and}$$
$$P' \sim^* R_1 \ \mathcal{S} \ R_2 \sim^* Q' \text{ for some } R_1, R_2, Q' \in \mathbb{P}.$$

The case $G\{P/X\} \equiv P'$ is clear. Thus assume that $G\{P/X\} \stackrel{\tau}{\Longrightarrow} P'$. The congruence property of $\simeq^*$ and $P \simeq^* E\{P/X\}$ implies

$$G\{P/X\} \simeq^* G\{E\{P/X\}/X\} \equiv G\{E/X\}\{P/X\}.$$

Since $G\{P/X\} \stackrel{\tau}{\Longrightarrow} P'$ there exists an $R \in \mathbb{P}$ with

$$G\{E/X\}\{P/X\} \stackrel{\tau}{\Longrightarrow} R \text{ and } P' \sim^* R.$$

Note that X is guarded in $G\{E/X\}$ and furthermore $\mathbb{V}(G\{E/X\}) \subseteq \{X\}$. Thus we can apply Lemma 5 and 7 to each $\stackrel{\tau}{\longrightarrow}$-transition in the sequence $G\{E/X\}\{P/X\} \stackrel{\tau}{\Longrightarrow} R$. We obtain an H with

– $R \equiv H\{P/X\}$,
– $G\{E/X\}\{Q/X\} \stackrel{\tau}{\Longrightarrow} H\{Q/X\}$, and
– X is guarded in H and $\mathbb{V}(H) \subseteq \{X\}$.

Finally $G\{E/X\}\{Q/X\} \equiv G\{E\{Q/X\}/X\} \simeq^* G\{Q/X\}$ implies

$$G\{Q/X\} \stackrel{\tau}{\Longrightarrow} Q' \text{ and } H\{Q/X\} \sim^* Q'.$$

Thus we have $P' \sim^* R \equiv H\{P/X\} \, \mathcal{S} \, H\{Q/X\} \sim^* Q'$, which proves (wb′).

In order to prove (*WB*′), assume that $G\{P/X\} \stackrel{a}{\Longrightarrow} P'$. As above it follows that there exists an R with

$$G\{E/X\}\{P/X\} \stackrel{a}{\Longrightarrow} R \text{ and } P' \sim^* R,$$

i.e., $G\{E/X\}\{P/X\} \Longrightarrow \stackrel{a}{\longrightarrow} \Longrightarrow R$. Now again we apply Lemma 5 and 7 to the first sequence of $\stackrel{\tau}{\longrightarrow}$-transitions and the subsequent $\stackrel{a}{\longrightarrow}$-transition. We obtain an $H \in \mathbb{E}$ with $\mathbb{V}(H) \subseteq \{X\}$ and

$$G\{E/X\}\{P/X\} \Longrightarrow \stackrel{a}{\longrightarrow} H\{P/X\} \Longrightarrow R,$$
$$G\{E/X\}\{Q/X\} \Longrightarrow \stackrel{a}{\longrightarrow} H\{Q/X\}.$$

The first point together with (wb′) (with H instead of G) implies

$$H\{Q/X\} \Longrightarrow R' \text{ and } R \sim^* R_1 \, \mathcal{S} \, R_2 \sim^* R',$$

i.e., $G\{E/X\}\{Q/X\} \stackrel{a}{\Longrightarrow} R'$. Finally, since $G\{E/X\}\{Q/X\} \simeq^* G\{Q/X\}$, there exists a Q' with $G\{Q/X\} \stackrel{a}{\Longrightarrow} Q'$ and $R' \sim^* Q'$. Thus

$$P' \sim^* R \sim^* R_1 \, \mathcal{S} \, R_2 \sim^* R' \sim^* Q'.$$

(Δ'): Let $* = \Delta$ and $G\{P/X\} \Uparrow$. Since $G\{P/X\} \simeq^{\Delta} G\{E/X\}\{P/X\}$, also $G\{E/X\}\{P/X\} \Uparrow$. Since X is guarded in $G\{E/X\}$, Lemma 9 implies $G\{E/X\} \Uparrow$. Thus $G\{E/X\}\{Q/X\} \Uparrow$ and $G\{Q/X\} \Uparrow$.

(S'): Let $* = S$ and $G\{P/X\} \Longrightarrow P' \stackrel{\tau}{\not\longrightarrow}$. Then there exists an R with $G\{E/X\}\{P/X\} \Longrightarrow R \stackrel{\tau}{\not\longrightarrow}$. Since X is guarded in $G\{E/X\}$, we can trace by Lemma 5 and Lemma 7 the transition sequence $G\{E/X\}\{P/X\} \Longrightarrow R$ and obtain an H with $R \equiv H\{P/X\}$, $G\{E/X\}\{Q/X\} \Longrightarrow H\{Q/X\}$, and X guarded in H. Since $H\{P/X\} \stackrel{\tau}{\not\longrightarrow}$ and X is guarded in H this implies with Lemma 8 $H\{Q/X\} \stackrel{\tau}{\not\longrightarrow}$. Finally $G\{E/X\}\{Q/X\} \Longrightarrow H\{Q/X\} \stackrel{\tau}{\not\longrightarrow}$ implies $G\{Q/X\} \Longrightarrow Q' \stackrel{\tau}{\not\longrightarrow}$ for some Q'.

($0'$): Can be shown analogously to (S') with $\stackrel{\tau}{\longrightarrow}$ replaced by $\longrightarrow$.

(λ'): Let $* = \lambda$ and $G\{P/X\} \Uparrow\!\!\Uparrow$. Since $G\{P/X\} \simeq^{\lambda} G\{E/X\}\{P/X\}$, also $G\{E/X\}\{P/X\} \Uparrow\!\!\Uparrow$. By using the arguments from (Δ') and ($0'$) above we obtain $G\{E/X\}\{Q/X\} \Uparrow\!\!\Uparrow$, which finally gives us $G\{Q/X\} \Uparrow\!\!\Uparrow$. □

Theorem 12. *Assume that*

– $E \in \mathbb{E}$, $P, Q \in \mathbb{P}$,
– X *is guarded in* E, $\mathbb{V}(E) \subseteq \{X\}$,
– $P \simeq^* E\{P/X\}$, *and* $Q \simeq^* E\{Q/X\}$.

Then $P \simeq^* Q$.

Proof. We obtain the conclusion by choosing $G \equiv X$ in the relation $\mathcal{R}$ defined in Lemma 20. □

Theorem 13. *Assume that*

– $E \in \mathbb{E}$, $P \in \mathbb{P}$,
– X *is guarded in* E, $\mathbb{V}(E) \subseteq \{X\}$, *and*
– $P \simeq^* E\{P/X\}$.

Then $P \simeq^* recX.E$.

Proof. We have $recX.E \simeq^* E\{recX.E/X\}$, thus we can apply Theorem 12 with $Q \equiv recX.E$. □

Using the definition of $\simeq^*$ for expressions with free variables, Theorem 12 and Theorem 13 hold for arbitrary expressions from $\mathbb{E}$.

E Soundness of (*rec*5) and (*rec*6)

For (*rec*5) we have to prove that

$$P \equiv recX.(\tau.(X + E) + F) \simeq^{\Delta} recX.(\Delta(E + F)) \equiv Q,$$

where w.l.o.g. $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Define the following relations:

$$\begin{aligned}\mathcal{R}_0 &= \{\langle G\{P/X\}, G\{Q/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\}\\ \mathcal{R}_1 &= \{\langle P + E\{P/X\},\ \Delta(E\{Q/X\} + F\{Q/X\})\rangle\}\\ \mathcal{R} &= \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}\end{aligned}$$

Thus $\mathcal{R}$ is symmetric. We will prove the following statements:

– If $(R_1, R_2) \in \mathcal{R} \land R_1 \xrightarrow{a} R_1'$ then $R_2 \stackrel{a}{\Longrightarrow} R_2' \land (R_1', R_2') \in \mathcal{R}$ for some R_2'.
– If $(R_1, R_2) \in \mathcal{R}$ and $R_1 \Uparrow$ then $R_2 \Uparrow$.

This implies that $\mathcal{R}$ is a WB^{Δ}, i.e., $\mathcal{R} \subseteq \sim^*$. Then by the first point all pairs in $\mathcal{R}$ also satisfy the root condition, thus $\mathcal{R} \subseteq \simeq^{\Delta}$. If we choose $G \equiv X$ in $\mathcal{R}_0$ this implies $P \simeq^{\Delta} Q$.

The second point above, i.e. $(R_1, R_2) \in \mathcal{R}$ and $R_1 \Uparrow$ imply $R_2 \Uparrow$, follows immediately from Lemma 9 and the fact that $P \Uparrow$ and $Q \Uparrow$. In order to prove the first point, i.e, $(R_1, R_2) \in \mathcal{R}$ and $R_1 \xrightarrow{a} R_1'$ imply $R_2 \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2', we first consider the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. We treat this case by an induction on the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$ using a case distinction on the expression G in $\mathcal{R}_0$. Most cases are straight-forward, we only consider the two cases resulting from $G \equiv X$:

Case 1. $R_1 \equiv P \xrightarrow{a} R_1'$ and $R_2 \equiv Q$: Thus $recX.(\tau.(X+E)+F) \xrightarrow{a} R_1'$, i.e, $\tau.(P+E\{P/X\})+F\{P/X\} \xrightarrow{a} R_1'$ by a smaller derivation tree. There are two cases:

Case 1.1. $a = \tau$ and $R_1' \equiv P+E\{P/X\}$: We have $Q \equiv recX.(\Delta(E+F)) \xrightarrow{\tau} \Delta(E\{Q/X\}+F\{Q/X\})$ and $\langle P+E\{P/X\},\ \Delta(E\{Q/X\}+F\{Q/X\})\rangle \in \mathcal{R}$.

Case 1.2. $F\{P/X\} \xrightarrow{a} R_1'$. By induction we obtain $F\{Q/X\} \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $Q \xrightarrow{\tau} \Delta(E\{Q/X\}+F\{Q/X\}) \stackrel{a}{\Longrightarrow} R_2'$.

Case 2. $R_1 \equiv Q \xrightarrow{a} R_1'$ and $R_2 \equiv P$: Thus $recX.(\Delta(E+F)) \xrightarrow{a} R_1'$, i.e, $\Delta(E\{Q/X\}+F\{Q/X\}) \xrightarrow{a} R_1'$ by a smaller derivation tree. There are three cases:

Case 2.1. $a = \tau$ and $R_1' \equiv \Delta(E\{Q/X\}+F\{Q/X\})$. We have $P \xrightarrow{\tau} P + E\{P/X\}$ and $\langle \Delta(E\{Q/X\}+F\{Q/X\}), P+E\{P/X\}\rangle \in \mathcal{R}$.

Case 2.2. $E\{Q/X\} \xrightarrow{a} R_1'$. By induction we obtain $E\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P \xrightarrow{\tau} P+E\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$.

Case 2.2. $F\{Q/X\} \xrightarrow{a} R_1'$. By induction we obtain $F\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus also $P \stackrel{a}{\Longrightarrow} R_2'$. This concludes the consideration of the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

It remains to consider the case $(R_1, R_2) \in \mathcal{R}_1 \cup \mathcal{R}_1^{-1}$. For this we will make use of the case $(R_1, R_2) \in \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. There are two cases:

Case 1. $R_1 \equiv P + E\{P/X\}$ and $R_2 \equiv \Delta(E\{Q/X\}+F\{Q/X\})$. Thus we have $P+E\{P/X\} \xrightarrow{a} R_1'$, and we can distinguish the following two cases:

Case 1.1. $P \xrightarrow{a} R_1'$. Since $\langle P, Q\rangle \in \mathcal{R}_0$ we have $Q \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $\Delta(E\{Q/X\}+F\{Q/X\}) \stackrel{a}{\Longrightarrow} R_2'$.

Case 1.2. $E\{P/X\} \xrightarrow{a} R_1'$. Since $\langle E\{P/X\}, E\{Q/X\}\rangle \in \mathcal{R}_0$, we obtain $E\{Q/X\} \stackrel{a}{\Longrightarrow} R_2'$ and thus $\Delta(E\{Q/X\}+F\{Q/X\}) \stackrel{a}{\Longrightarrow} R_2'$ for some R_2' with $(R_1', R_2') \in \mathcal{R}$.

Case 2. $R_1 \equiv \Delta(E\{Q/X\}+F\{Q/X\})$ and $R_2 \equiv P+E\{P/X\}$. We can distinguish the following three cases:

Case 2.1. $a = \tau$ and $R_1' \equiv \Delta(E\{Q/X\}+F\{Q/X\})$. Since $P \xrightarrow{\tau} P + E\{P/X\}$, we have $P+E\{P/X\} \xrightarrow{\tau} P+E\{P/X\}$.

Case 2.2. $E\{Q/X\} \xrightarrow{a} R_1'$. Since $\langle E\{Q/X\}, E\{P/X\}\rangle \in \mathcal{R}_0$, we obtain $E\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P+E\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$.

Case 2.3. $F\{Q/X\} \xrightarrow{a} R_1'$. Since $\langle F\{Q/X\}, F\{P/X\}\rangle \in \mathcal{R}_0$, we obtain $F\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$ and $(R_1', R_2') \in \mathcal{R}$ for some R_2'. Thus $P \stackrel{a}{\Longrightarrow} R_2'$ and hence $P+E\{P/X\} \stackrel{a}{\Longrightarrow} R_2'$. This concludes the correctness proof of (*rec*5).

For (*rec*6) we have to prove that

$$P \equiv recX.(\Delta(X+E)+F) \simeq^{\Delta} recX.(\Delta(E+F)) \equiv Q,$$

where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. We proceed analogously to (*rec*5). Define

$$\mathcal{R}_0 = \{\langle G\{P/X\}, G\{Q/X\}\rangle \mid \mathbb{V}(G) \subseteq \{X\}\},$$
$$\mathcal{R}_1 = \{\langle \Delta(P + E\{P/X\}), \Delta(E\{Q/X\} + F\{Q/X\})\rangle\}, \text{ and}$$
$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}.$$

Then the following statements can be shown:

- If $(R_1, R_2) \in \mathcal{R} \wedge R_1 \xrightarrow{a} R_1'$ then $R_2 \xRightarrow{a} R_2' \wedge (R_1', R_2') \in \mathcal{R}$ for some R_2'.
- If $(R_1, R_2) \in \mathcal{R}$ and $R_1 \Uparrow$ then $R_2 \Uparrow$.

The proofs are analogous to those for (*rec*5) and left to the reader.

F Soundness

Recall that a strong bisimulation is a relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for all $(P, Q) \in \mathcal{R}$ the following two conditions hold (see also [9]):

- If $P \xrightarrow{a} P'$ then $Q \xrightarrow{a} Q'$ and $(P', Q') \in \mathcal{R}$ for some Q'.
- If $Q \xrightarrow{a} Q'$ then $P \xrightarrow{a} P'$ and $(P', Q') \in \mathcal{R}$ for some Q'.

We write $P \cong Q$ if there exists some strong bisimulation $\mathcal{R}$ containing the pair (P, Q). The following lemma is easy to see.

Lemma 21. *It holds* $\cong \subseteq \simeq^{\Delta}$.

Theorem 14 (restated Theorem 3). *Let* $E, F \in \mathbb{E}$. *If* $E =^* F$ *then* $E \simeq^* F$.

Proof. Due to the definition of $\simeq^*$ for expressions with free variables, it suffices to check the soundness of the axioms only for $\mathbb{P}$. First we check the core axioms from Table 1, which have to be verified for our finest congruence $\simeq^{\Delta}$:

- $(S1)$, $(S2)$, $(S3)$, $(S4)$, (*rec*1), (*rec*2), and (*rec*4) are sound for $\cong$ [9].
- (*rec*3): see Theorem 13 from Appendix D. [3]
- $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$: Soundness for $\simeq^{\Delta}$ can be shown analogously to the soundness for $\simeq$, see e.g. [10].
- (*rec*5) and (*rec*6): see Appendix E.

We continue with the distinguishing axioms from Table 2.

- (Δ): We need to prove that $\Delta(\Delta(P) + Q) \simeq^{\Delta} \tau.(\Delta(P) + Q)$. The symmetric closure of the relation

  $$\{\langle \Delta(\Delta(P) + Q), \tau.(\Delta(P) + Q)\rangle, \langle \Delta(\Delta(P) + Q), \Delta(P) + Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

  can be shown to be a $W\!B^{\Delta}$. Furthermore, $\langle \Delta(\Delta(P) + Q), \tau.(\Delta(P) + Q)\rangle$ satisfies the root condition, thus $\Delta(\Delta(P) + Q) \simeq^{\Delta} \tau.(\Delta(P) + Q)$.

[3] Note that (*rec*3) is also sound for $\cong$ [9], but since (*rec*3) has the form of an implication, this does not imply the soundness with respect to $\simeq^{\Delta}$.

– (λ): We need to prove that $\Delta(0) \simeq^{\lambda} \tau.0$. It is not difficult to prove that

$$\{\langle \Delta(0), \tau.0\rangle, \langle \Delta(0), 0\rangle\}$$

is a WB^{λ} and $\langle \Delta(0), \tau.0\rangle$ satisfies the root condition.

– (S): We have to prove that $\Delta(\tau.P + Q) \simeq^{0} \tau.(\tau.P + Q)$. It is not difficult to prove that the symmetric closure of

$$\{\langle \Delta(\tau.P + Q), \tau.(\tau.P + Q)\rangle, \langle \Delta(\tau.P + Q), \tau.P + Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB^{S} and that $\langle \Delta(\tau.P + Q), \tau.(\tau.P + Q)\rangle$ satisfies the root condition.

– (0): We have to prove that $\Delta(a.P + Q) \simeq^{0} \tau.(a.P + Q)$. The symmetric closure of

$$\{\langle \Delta(a.P + Q), \tau.(a.P + Q)\rangle, \langle \Delta(a.P + Q), a.P + Q\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB^{0} and the pair $\langle \Delta(a.P + Q), \tau.(a.P + Q)\rangle$ satisfies the root condition.

– (ϵ): We have to show that $\Delta(P) \simeq \tau.P$. The symmetric closure of

$$\{\langle \Delta(P), \tau.P\rangle, \langle \Delta(P), P\rangle\} \cup \mathrm{Id}_{\mathbb{P}}$$

is a WB and $\langle \Delta(P), \tau.P\rangle$ satisfies the root condition. □

G Derivable laws for $=^{\Delta}$

Lemma 22 (restated Lemma 1). *The following laws can be derived:*

$$\begin{array}{lr}
(\Delta') & \Delta(E) =^{\Delta} \Delta(E) + E \\
(\tau\Delta) & \Delta(E) =^{\Delta} \tau.\Delta(E) + E \\
(\tau\Delta') & \Delta(E) =^{\Delta} \tau.\Delta(E) \\
(rec7) & recX.(\tau.(X + E) + F) =^{\Delta} recX.(\tau.X + E + F)
\end{array}$$

Proof. First we derive ($\tau\Delta$) as follows, where $X \in \mathbb{V} \setminus \mathbb{V}(E)$:

$$\begin{aligned}
\Delta(E) &=^{\Delta} recX.\Delta(E) && (rec2) \\
&=^{\Delta} recX.\Delta(0 + E) && (\mathrm{S4}) \\
&=^{\Delta} recX.(\tau.(X + 0) + E) && (rec5) \\
&=^{\Delta} \tau.(recX.(\tau.(X + 0) + E) + 0) + E && (rec2) \\
&=^{\Delta} \tau.(recX.\Delta(0 + E) + 0) + E && (rec5) \\
&=^{\Delta} \tau.(recX.\Delta(E)) + E && (\mathrm{S4}) \\
&=^{\Delta} \tau.\Delta(E) + E && (rec2)
\end{aligned}$$

Now $(\tau\Delta')$ can be deduced as follows:

$$\begin{aligned}
\tau.\Delta(E) &=^{\Delta} \tau.\Delta(E) + \Delta(E) && (\tau 2)\\
&=^{\Delta} \tau.\Delta(E) + \tau.\Delta(E) + E && (\tau\Delta)\\
&=^{\Delta} \tau.\Delta(E) + E && (\mathrm{S3})\\
&=^{\Delta} \Delta(E) && (\tau\Delta)
\end{aligned}$$

Law (Δ') is an immediate consequence of $(\tau\Delta)$ and $(\tau\Delta')$. Finally for $(rec7)$ note that by $(rec5)$ both expressions can be transformed into $recX.\Delta(E+F)$. □

Lemma 23. *If X is unguarded in $E \in \mathbb{E}$ then $E =^{\Delta} E + X$.*

Proof. We prove the lemma by an induction on the structure of E. Since X is unguarded in E we only have to consider the following cases.
Case 1. $E \equiv X$: By axiom (S3) we have $X =^{\Delta} X + X$.
Case 2. $E \equiv \tau.E'$: We have

$$\begin{aligned}
E \equiv \tau.E' &=^{\Delta} \tau.E' + E' && (\tau 2)\\
&=^{\Delta} \tau.E' + E' + X && (\text{induction hypothesis})\\
&=^{\Delta} E + X && (\tau 2)
\end{aligned}$$

Case 3. $E \equiv \Delta(E')$. With the derived law (Δ') from Lemma 1, we can conclude analogously to case 2.
Case 4. $E \equiv E_1 + E_2$: W.l.o.g. assume that X is unguarded in E_2. The induction hypothesis implies $E_2 =^{\Delta} E_2 + X$. Thus $E \equiv E_1 + E_2 =^{\Delta} E_1 + E_2 + X \equiv E + X$.
Case 5. $E \equiv recY.E'$: Since X must be free in E we have $X \not\equiv Y$. The induction hypothesis implies $E' =^{\Delta} E' + X$. Thus $E'\{recY.E'/Y\} =^{\Delta} E'\{recY.E'/Y\} + X$. Axiom $(rec2)$ implies $recY.E' =^{\Delta} recY.E' + X$. □

Theorem 15 (restated Theorem 4). *Let $E \in \mathbb{E}$. There exists a guarded F with $E =^{\Delta} F$ (and thus $\mathbb{V}(E) = \mathbb{V}(F)$).*

Proof. The proof follows [10]. We prove the theorem by an induction on the structure of the expression E. Only the case $E \equiv recX.E'$ is interesting. For this case we prove the following stronger statement (†).

Let $E \in \mathbb{E}$. Then there exists a guarded F such that
- X is guarded in F,
- there does not exist a free unguarded occurrence of an arbitrary variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F, [4] and
- $recX.E =^{\Delta} recX.F$.

[4] A specific free occurrence of Y in F is called unguarded if this occurrence does not lie within a subexpression $a.F'$ with $a \neq \tau$.

We prove (†) by an induction on the nesting depth $d(E)$ of recursions in E. We have for instance $d(recX.(a.recY.(a.X + b.Y) + a.(recX.(recX.(0))))) = 3$. First we consider the following case (‡):

> There does not exist an unguarded occurrence of an arbitrary variable $Y \in \mathbb{V}(E)$ which lies within a subexpression $recZ.G$ of E.

This case also covers the induction base $d(E) = 0$. So assume that E satisfies (‡). It remains to remove all unguarded occurrences of X in E. Since E satisfies (‡) we know that no unguarded occurrence of X in E lies within a recursion. If X is guarded in E we are ready. So assume that X occurs unguarded in E. We now list several reduction steps which when iteratedly applied to E terminate with an expression that satisfies (†). During this reduction process we either eliminate an unguarded occurrence of X or we reduce the number of τ-guards and Δ-operators that proceed an unguarded occurrence of X. Since E satisfies (‡), one of the following four cases must apply.
Case 1. $E \equiv \tau.(X + E') + F'$: With the derivable law ($rec7$), we get

$$recX.E \equiv recX(\tau.(X + E') + F') =^{\Delta} recX.(\tau.X + E' + F').$$

We continue with the expression $\tau.X + E' + F'$.
Case 2. $E \equiv \tau.E' + F'$, where X is unguarded in E', but X is weakly guarded in E': Lemma 23 implies $E' =^{\Delta} X + E'$. Thus $E =^{\Delta} \tau.(X + E') + F'$. By case 1 we can continue with the expression $\tau.X + E' + F'$.
Case 3. $E \equiv \Delta(X + E') + F'$: With ($rec5$), ($rec6$), and ($rec7$) we get

$$\begin{aligned} recX.(\Delta(X + E') + F') &=^{\Delta} recX.(\Delta(E' + F')) \\ &=^{\Delta} recX.(\tau.(X + E') + F') =^{\Delta} recX.(\tau.X + E' + F'). \end{aligned}$$

We continue with the expression $\tau.X + E' + F'$.
Case 4. $E \equiv \Delta(E') + F'$, where X is unguarded in E', but X is weakly guarded in E': Again by Lemma 23 we have $E' =^{\Delta} X + E'$. Thus $E =^{\Delta} \Delta(X + E') + F'$. An application of case 3 gives the expression $\tau.X + E' + F'$.

By iterating these four reduction steps we finally arrive at an expression, where all unguarded occurrences of X in E occur in the form $E \equiv X + \cdots$ or $E \equiv \tau.X + \ldots$. Furthermore by axiom (S3) and (τ2) we may assume that there exists at most one occurrence of this form. Thus it remains to consider the following two cases:
Case 5. $E \equiv X + E'$: By axiom ($rec4$) we have

$$recX.E \equiv recX.(X + E') =^{\Delta} recX.E'.$$

Case 6. $E \equiv \tau.X + E'$: By axiom ($rec5$) we have

$$recX.E \equiv recX.(\tau.X + E') =^{\Delta} recX.(\Delta(E')).$$

Note that X is guarded in $\Delta(E')$ if X is guarded in E'. This concludes the consideration of case (‡).

It remains to consider the cases that are not covered by (‡). For this let us choose a subexpression $recX'.E'$ of E such that this subexpression does not lie within another recursion, thus $recX'.E'$ is an outermost recursion. Since $d(E') < d(E)$, the induction hypothesis implies that there exists an expression F with the following properties:

- X' is guarded in F.
- There does not exist an unguarded occurrence of an arbitrary variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F.
- $recX'.E' =^{\Delta} recX'.F$

It follows that in the expression $F\{recX'.F/X'\}$ there does not exist an unguarded occurrence of any variable which lies within a recursion. Axiom (rec2) allows us to replace $recX'.E'$ within E by $F\{recX'.F/X'\}$. If we do this step for every outermost recursion of E, we obtain an expression that satisfies (‡). This concludes the proof. □

H Properties of equation systems

Theorem 16 (restated Theorem 5). *Every guarded expression E $*$-provably satisfies a guarded and saturated SES over the free variables $\mathbb{V}(E)$.*

Proof. First we prove by induction on the structure of the expression E that E $*$-provably satisfies a guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(E)$ and the formal variables $X_1, \ldots, X_m$. Furthermore for the inductive proof we need the following property (§):

> If $Y \in \mathbb{V}(E)$ is guarded in E then there do not exist i, α such that $X_1 \Longrightarrow X_i^{\alpha}$ and Y occurs as a summand in the expression E_i, where $X_i = E_i$ is an equation of $\mathcal{E}$.

Case 1. $E \equiv 0$ or $E \in \mathbb{V}$: trivial
Case 2. $E \equiv a.F$:

By induction F $*$-provably satisfies a guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(F)$ and the formal variables $X_1, \ldots, X_m$. Then $a.F$ $*$-provably satisfies the guarded SES $\{X_0 = a.X_1, X_0^{\Delta} = \Delta(X_0)\} \cup \mathcal{E}$ over the formal variables $X_0, \ldots, X_m$. Furthermore this new SES satisfies (§) if $\mathcal{E}$ satisfies (§).
Case 3. $E \equiv \Delta(F)$:

Again let $\mathcal{E}$ be a guarded SES over the free variables $\mathbb{V}(F)$ and the formal variables $X_1, \ldots, X_m$ that is $*$-provably satisfied by F. Then $\Delta(F)$ $*$-provably satisfies the guarded SES $\{X_0 = \tau.X_1^{\Delta}, X_0^{\Delta} = \Delta(X_0)\} \cup \mathcal{E}$. To see this one can use the derived law $(\tau\Delta')$ from Lemma 1. Furthermore this new SES satisfies (§) if $\mathcal{E}$ satisfies (§).
Case 4. $E \equiv F + G$:

Assume that F (resp. G) $*$-provably satisfy the guarded SES $\mathcal{E}$ (resp. $\mathcal{F}$) over the free variables $\mathbb{V}(F)$ (resp. $\mathbb{V}(G)$) and the formal variables $X_1, \ldots, X_m$ (resp. $Y_1, \ldots, Y_n$), where w.l.o.g. $\{X_1, \ldots, X_m\} \cap \{Y_1, \ldots, Y_n\} = \emptyset$. Let the equation $X_1 = F_1$ (resp. $Y_1 = G_1$) belong to $\mathcal{E}$ (resp. $\mathcal{F}$). Then $F + G$ $*$-provably satisfies the guarded SES $\{Z = F_1 + G_1, Z^\Delta = \Delta(Z)\} \cup \mathcal{E} \cup \mathcal{F}$ over the formal variables $Z, X_1, \ldots, X_m, Y_1, \ldots, Y_n$. Furthermore this new SES satisfies (§) if $\mathcal{E}$ and $\mathcal{F}$ satisfy (§).

Case 4. $E \equiv recX_0.F$, where X_0 is guarded in F:

The case $X_0 \notin \mathbb{V}(F)$ is trivial, thus assume that $X_0 \in \mathbb{V}(F)$. Let F $*$-provably satisfy the guarded SES $\mathcal{E}$ over the free variables $\mathbb{V}(F)$ and the formal variables $X_1, \ldots, X_m$. Assume that $\mathcal{E}$ satisfies (§) and assume that the equation $X_1 = E_1$ belongs to $\mathcal{E}$. Let the SES $\mathcal{F}$ result from $\mathcal{E}$ by replacing each right hand side E_i of an equation of $\mathcal{E}$ by $E_i\{E_1/X_0\}$. Note that due to (§), the free variable X_0 does not appear as a summand in E_1, hence X_0 does not occur in the SES $\mathcal{F}$. Then $recX_0.F$ $*$-provably satisfies the SES $\{X_0 = E_1, X_0^\Delta = \Delta(X_0)\} \cup \mathcal{F}$ over the formal variables $X_0, \ldots, X_m$ and the free variables $\mathbb{V}(E) = \mathbb{V}(F)\backslash\{X_0\}$. Furthermore since $\mathcal{E}$ satisfies (§), this new SES is guarded and satisfies again (§).

It remains to transform a guarded SES $\mathcal{E}$, which is $*$-provably satisfied by an expression E, into a guarded and saturated SES, which is also $*$-provably satisfied by E. We only show, how the first condition of the definition of a saturated SES can be enforced by induction on the length of the transition sequence $X_i \stackrel{a}{\Longrightarrow} X_j^\alpha$, the second condition on free variables can be enforced similarly. First assume that for $\mathcal{E}$ it holds $X_i \stackrel{\tau}{\longrightarrow} X_k^\beta \Longrightarrow\stackrel{a}{\longrightarrow}\Longrightarrow X_j^\alpha$ for some β, k. By induction we may assume that already $X_i \stackrel{\tau}{\longrightarrow} X_k^\beta \stackrel{a}{\longrightarrow} X_j^\alpha$. Assume that the equations $X_i = \tau.X_k^\beta + E_i$ and $X_k = E_k$ belong to $\mathcal{E}$. If $\beta = \_$ then by axiom (τ2) we can replace the equation $X_i = \tau.X_k + E_i$ by $X_i = \tau.X_k + E_i + E_k$. On the other hand if $\beta = \Delta$ then the same replacement is justified by axiom (τ2) and the law (Δ') from Lemma 1. The resulting SES is still $*$-provably satisfied by E, it is guarded, and it satisfies $X_i \stackrel{a}{\longrightarrow} X_j^\alpha$.

It remains to consider the case that $\mathcal{E}$ satisfies $X_i \stackrel{a}{\longrightarrow}\Longrightarrow X_k^\beta \stackrel{\tau}{\longrightarrow} X_j^\alpha$. By induction we may assume that already $X_i \stackrel{a}{\longrightarrow} X_k^\beta \stackrel{\tau}{\longrightarrow} X_j^\alpha$. Assume that the equations $X_i = a.X_k^\beta + E_i$ and $X_k = \tau.X_j^\alpha + E_k$ belong to $\mathcal{E}$. If $\beta = \_$, i.e., $X_i = a.X_k + E_i$, then we can by axiom (τ3) replace the equation $X_i = a.X_k + E_i$ by $X_i = a.X_k + a.X_j^\alpha + E_i$. If $\beta = \Delta$, i.e., $X_i = a.X_k^\Delta + E_i$, then we must have $a = \tau$, since an SES does not contain summands of the form $a.X^\Delta$ for $a \neq \tau$. Thus $X_i = \tau.X_k^\Delta + E_i$ and by axiom (τ2) and (Δ') we can replace this equation by $X_i = \tau.X_k^\Delta + \tau.X_j^\alpha + E_k + E_i$. In both cases the resulting SES is still $*$-provably satisfied by E, it is guarded, and it satisfies $X_i \stackrel{a}{\longrightarrow} X_j^\alpha$. □

Lemma 24. *Let $\mathcal{E} = \{X_i = E_i \mid 1 \leq i \leq n\} \cup \{X_i^\Delta = \Delta(X_i) \mid 1 \leq i \leq n\}$ be a guarded SES without free variables, which is $*$-provably satisfied by the expressions $P_1, \ldots, P_n \in \mathbb{P}$.*

1. *If $\alpha(P_i) \sim^* P \stackrel{\widehat{a}}{\Longrightarrow} Q$ then $X_i^\alpha \stackrel{\widehat{a}}{\Longrightarrow} X_k^\gamma$ and $Q \sim^* \gamma(P_k)$ for some k, γ.*
2. *If $* = \Delta$ and $\alpha(P_i) \Uparrow$ then $X_i^\alpha \Longrightarrow X_k^\Delta$ for some k.*
3. *If $* = \lambda$ and $\alpha(P_i) \Uparrow\!\!\Uparrow$ then there exists k such that either $X_i^\alpha \Longrightarrow X_k^\Delta$ or $X_i^\alpha \Longrightarrow X_k \not\longrightarrow$.*

Proof. Since $\mathcal{E}$ is guarded there exists a linear order $\prec$ on the formal variables $\{X_1, X_1^\Delta, \ldots, X_n, X_n^\Delta\}$ such that $X_i^\alpha \xrightarrow{\tau} X_k^\beta$ implies $X_k^\beta \prec X_i^\alpha$. We prove all statements of the lemma by an induction along the order $\prec$.

1. Let us first consider the case $a = \tau$, i.e, $P \Longrightarrow Q$. The case $P \equiv Q$ is trivial. Thus assume that $P \xrightarrow{\tau} R \Longrightarrow Q$. Then $P \sim^* \alpha(P_i) \simeq^* \alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\})$ implies $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \Longrightarrow R'$ and $R \sim^* R'$ for some R'. If $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \equiv R'$ then we have $R \sim^* R' \simeq^* \alpha(P_i)$. Since the transition sequence $R \Longrightarrow Q$ is shorter than the original sequence $P \stackrel{\tau}{\Longrightarrow} Q$, we can conclude by an induction on the length of the transition sequence. Thus assume that $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \xrightarrow{\tau} P' \Longrightarrow R'$, where $P' \not\equiv \alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\})$. We obtain $E_i\{\boldsymbol{P}/\boldsymbol{X}\} \xrightarrow{\tau} P'$ and thus $P' \equiv \beta(P_j)$ and $X_i^\alpha \xrightarrow{\tau} X_j^\beta$ for some β, j. Since $X_j^\beta \prec X_i^\alpha$ and $\beta(P_j) \Longrightarrow R'$ we obtain inductively $X_j^\beta \Longrightarrow X_\ell^\delta$ and $R' \sim^* \delta(P_\ell)$ for some δ, ℓ. Finally $R \Longrightarrow Q$, $R \sim^* R' \sim^* \delta(P_\ell)$ and $X_\ell^\delta \prec X_i^\alpha$ implies inductively $X_\ell^\delta \Longrightarrow X_k^\gamma$ and $Q \sim^* \gamma(P_k)$ for some k, γ.
   Now assume that $a \neq \tau$, i.e, $P \stackrel{a}{\Longrightarrow} Q$. Since $P \sim^* \alpha(P_i) \simeq^* \alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\})$ we get $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \stackrel{a}{\Longrightarrow} R$ and $Q \sim^* R$ for some R. If $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \xrightarrow{a} P_j \Longrightarrow R$ for some j, then $X_i^\alpha \xrightarrow{a} X_j \Longrightarrow X_k^\gamma$ and $\gamma(P_k) \sim^* R \sim^* Q$ for some k, γ by the previous paragraph. On the other hand if $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \xrightarrow{\tau} \beta(P_j) \stackrel{a}{\Longrightarrow} R$, where $X_i^\alpha \xrightarrow{\tau} X_j^\beta$, then $X_j^\beta \prec X_i^\alpha$ and by induction we get $X_j^\beta \stackrel{a}{\Longrightarrow} X_k^\gamma$ and $\gamma(P_k) \sim^* R \sim^* Q$ for some k, γ.
2. Assume that $* = \Delta$ and $\alpha(P_i) \Uparrow$. Since $\alpha(P_i) \simeq^\Delta \alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\})$, also $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}) \Uparrow$. Thus either $\alpha = \Delta$, or there exists j, β with $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}) \xrightarrow{\tau} \beta(P_j) \Uparrow$. Hence $X_i^\alpha \xrightarrow{\tau} X_j^\beta$ and we can conclude inductively.
3. Assume that $* = \lambda$ and $\alpha(P_i) \Uparrow\!\!\!\Uparrow$. Since $\alpha(P_i) \simeq^\lambda \alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\})$, we have $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \Uparrow\!\!\!\Uparrow$. Thus, either $\alpha = \Delta$ (and we are ready), or $\alpha = \_$ and $X_i \not\rightarrow$ (and we are ready), or there exist j, β such that $X_i^\alpha \xrightarrow{\tau} X_j^\beta$ and $\beta(P_j) \Uparrow\!\!\!\Uparrow$. In the latter case we can conclude by induction. □

Claim 3 (restated Claim 1 from the proof of Theorem 7) *Assume the notation from the proof of Theorem 7. If $\alpha(P_i) \sim^* \beta(Q_j)$ then the following implications hold:*

1. *If $X_i \xrightarrow{a} X_k^\gamma$ then either ($a = \tau$ and $\gamma(P_k) \sim^* \beta(Q_j)$) or there exist ℓ, δ such that $Y_j \xrightarrow{a} Y_\ell^\delta$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
2. *If $Y_j \xrightarrow{a} Y_\ell^\delta$ then either ($a = \tau$ and $\alpha(P_i) \sim^* \delta(Q_\ell)$) or there exist k, γ such that $X_i \xrightarrow{a} X_k^\gamma$ and $\gamma(P_k) \sim^* \delta(Q_\ell)$.*
3. *Let $* = \Delta$. If $\alpha = \Delta$ then either $\beta = \Delta$ or $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ.*
4. *Let $* = \Delta$. If $\beta = \Delta$ then either $\alpha = \Delta$ or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k.*
5. *Let $* = \lambda$. If $\alpha = \Delta$ or ($\alpha = \_$ and $X_i \not\rightarrow$) then either $\beta = \Delta$, or ($\beta = \_$ and $Y_j \not\rightarrow$), or $Y_j \xrightarrow{\tau} Y_\ell^\Delta$ for some ℓ, or $Y_j \xrightarrow{\tau} Y_\ell \not\rightarrow$ for some ℓ.*
6. *Let $* = \lambda$. If $\beta = \Delta$ or ($\beta = \_$ and $Y_j \not\rightarrow$) then either $\alpha = \Delta$, or ($\alpha = \_$ and $X_i \not\rightarrow$), or $X_i \xrightarrow{\tau} X_k^\Delta$ for some k, or $X_i \xrightarrow{\tau} X_k \not\rightarrow$ for some k.*

Proof. By symmetry it suffices to show (1), (3), and (5). For (1) assume that $X_i \xrightarrow{a} X_k^\gamma$. Thus $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \xrightarrow{a} \gamma(P_k)$. Since $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \simeq^* \alpha(P_i) \sim^* \beta(Q_j)$, we have $\beta(Q_j) \overset{\widehat{a}}{\Longrightarrow} R$ for some R with $\gamma(P_k) \sim^* R$. By Lemma 24(1) we obtain $Y_j^\beta \overset{\widehat{a}}{\Longrightarrow} Y_\ell^\delta$ and $\gamma(P_k) \sim^* R \sim^* \delta(Q_\ell)$ for some ℓ, δ. Since $\mathcal{E}_2$ is saturated we obtain the conclusion of (1).

For (3) assume that $* = \Delta$ and $\Delta(P_i) \sim^\Delta \beta(Q_j)$. Thus $\beta(Q_j) \Uparrow$. Lemma 24(2) implies $Y_j^\beta \Longrightarrow Y_\ell^\Delta$ for some ℓ. Saturation of $\mathcal{E}_2$ yields the conclusion of (3).

Finally let $* = \lambda$. If $\alpha = \Delta$ or ($\alpha = \_$ and $X_i \not\rightarrow$) then $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \Uparrow\!\!\Uparrow$. Since $\alpha(E_i\{\boldsymbol{P}/\boldsymbol{X}\}) \simeq^\lambda \alpha(P_i) \sim^\lambda \beta(Q_j)$, we obtain $\beta(Q_j) \Uparrow\!\!\Uparrow$. Lemma 24(3) implies that there exists ℓ such that either $Y_j^\beta \Longrightarrow Y_\ell^\Delta$ or $Y_j^\beta \Longrightarrow Y_\ell \not\rightarrow_{\mathcal{E}}$. Saturation of $\mathcal{E}_2$ yields the conclusion of (5). □

Claim 4 (restated Claim 2 from the proof of Theorem 7) *Assume the notation from the proof of Theorem 7. If $P_i \simeq^* Q_j$ then the following holds:*

1. *If $X_i \xrightarrow{a} X_k^\alpha$ then there exist ℓ, β such that $Y_j \xrightarrow{a} Y_\ell^\beta$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*
2. *If $Y_j \xrightarrow{a} Y_\ell^\beta$ then there exist k, α such that $X_i \xrightarrow{a} X_k^\alpha$ and $\alpha(P_k) \sim^* \beta(Q_\ell)$.*

Proof. Follows analogously to Claim 3. □

I Completeness for open expressions

For this section let us fix a variable $X \in \mathbb{V}$ and an action $a \in \mathbb{A}\setminus\{\tau\}$. In the following we have to deal with substitutions that are allowed to replace different occurrences of the variable X by different expressions. [5] Let $O(E)$ be the set of free occurrences of the variable X in the expression $E \in \mathbb{E}$. Now if $A \subseteq \mathbb{E}$ and $\sigma : O(E) \to A$, i.e., σ is a function mapping every free occurrence of X in E to an expression from A, then we denote by E^σ the expression that results from replacing every free occurrence $o \in O(E)$ of the variable X in E by the expression $\sigma(o) \in A$. For instance, if we denote the two free occurrences of X in $E \equiv X + a.(X + \Delta(Y) + recX.(\tau.X))$ by o_1 and o_2 and define $\sigma(o_1) \equiv b.0$ and $\sigma(o_2) \equiv 0$ then $E^\sigma \equiv b.0 + a.(0 + \Delta(Y) + recX.(\tau.X))$. Finally let $\mathbb{P}_0 = \{P \in \mathbb{P} \mid P \sim^* 0\}$ and $a.\mathbb{P}_0 = \{a.P \in \mathbb{P} \mid P \in \mathbb{P}_0\}$. Since the sets $O(E)$ and $O(a.E)$ (resp. $O(\Delta(E))$, $O(recY.E)$ where $X \neq Y$) are in a natural 1-1-correspondence, we will identify them in the sequel. Furthermore if $E \equiv E_1 + E_2$ then $O(E)$ can be identified with the disjoint union of $O(E_1)$ and $O(E_2)$ and if $\sigma : O(E_1 + E_2) \to \mathbb{E}$ then there are unique $\sigma_i : O(E_i) \to \mathbb{E}$ with $E^\sigma \equiv E_1^{\sigma_1} + E_2^{\sigma_2}$.

Lemma 25. *Assume that $a \in \mathbb{A}$ does not occur in $E \in \mathbb{E}$ and let $\sigma : O(E) \to a.\mathbb{P}_0$. If $E^\sigma =^* G$ then $G \equiv F^\rho$ for some $\rho : O(F) \to a.\mathbb{P}_0$.*

[5] Formally an occurrence of a variable X in E can be defined as the unique path from the root to the occurrence in the expression tree corresponding to E.

Proof. Note that $X \notin \mathbb{V}(E^\sigma)$ and hence also $X \notin \mathbb{V}(G)$. We can write G uniquely as $G \equiv F^\rho$ for some $F \in \mathbb{E}$ and $\rho : O(F) \to a.\mathbb{E}$ such that a does not occur in F. Note that $X \notin \mathbb{V}(H)$ for all $H \in \mathsf{im}(\rho)$. We claim that $\mathsf{im}(\rho) \subseteq a.\mathbb{P}_0$.

First let us reduce the problem to closed expressions. Choose an action $b \in \mathbb{A}\backslash\{\tau, a\}$ and denote for an expression $H \in \mathbb{E}$ by $\widehat{H}$ the expression that results from H by replacing every free variable $Y \in \mathbb{V}(H)\backslash\{X\}$ by $b.0$. Thus $\mathbb{V}(H) \subseteq \{X\}$. Similarly for $\pi : O(H) \to \mathbb{E}$ denote by $\widehat{\pi} : O(H) \to \mathbb{E}$ the function defined by $o \mapsto \widehat{\pi(o)}$. With this definition we have $\widehat{E^\sigma} =^* \widehat{F^\rho}$. We may identify the sets $O(E)$ and $O(\widehat{E})$ and similarly for F. By doing this identification and noting that $\mathsf{im}(\sigma) \subseteq \mathbb{P}$ we obtain $\widehat{E^\sigma} \equiv \widehat{E}^\sigma$. Similarly we have $\widehat{F^\rho} \equiv \widehat{F}^{\widehat{\rho}}$.

We claim that $\mathsf{im}(\widehat{\rho}) \subseteq a.\mathbb{P}_0$. Before proving this, let us first explain how this implies $\mathsf{im}(\rho) \subseteq a.\mathbb{P}_0$. Take some $H \in \mathsf{im}(\rho)$. Then $H \equiv a.H'$ for some $H' \in \mathbb{E}$. If H' would contain a free variable Y, then the expression $\widehat{H} \equiv a.\widehat{H'} \in \mathsf{im}(\widehat{\rho})$ would not be contained in $a.\mathbb{P}_0$ (the expression $\widehat{H'}$ would contain a subexpression $b.0$, which implies $\widehat{H'} \not\sim^* 0$). But this would contradict $\mathsf{im}(\widehat{\rho}) \subseteq a.\mathbb{P}_0$. Thus $H \in \mathbb{P}$ and hence $\rho = \widehat{\rho}$.

It remains to prove $\mathsf{im}(\widehat{\rho}) \subseteq a.\mathbb{P}_0$. By definition we have $\mathsf{im}(\widehat{\rho}) \subseteq a.\mathbb{P}$. Assume that there is a free occurrence o of X in $\widehat{F}$ such that $P_1 \equiv \widehat{\rho}(o) \notin a.\mathbb{P}_0$, i.e., $P_1 \equiv a.P_2$ and $P_2 \not\sim^* 0$ Then we can reach from $\widehat{F}$ by some sequence of transitions an expression where this occurrence is totally unguarded (this can be proved by a simple induction). Furthermore since $a \in \mathbb{A}$ does not occur in $\widehat{F}$, this sequence of transitions does not involve a. Thus $\widehat{F}^{\widehat{\rho}} \xrightarrow{b_1} \cdots \xrightarrow{b_m} \xrightarrow{a} P_2 \not\sim^* 0$ for some $b_i \in \mathbb{A}\backslash\{a\}$. Since $\widehat{E}^\sigma =^* \widehat{F}^{\widehat{\rho}}$, i.e., $\widehat{E}^\sigma \simeq^* \widehat{F}^{\widehat{\rho}}$ and both expressions belong to $\mathbb{P}$, we have $\widehat{E}^\sigma \xrightarrow{c_1} \cdots \xrightarrow{c_n} Q_1 \xrightarrow{a} Q_2 \Longrightarrow Q_3$ for some $c_i \in \mathbb{A}\backslash\{a\}$ and $Q_1, Q_2, Q_3 \in \mathbb{P}$ such that $P_2 \sim^* Q_3$, and thus $Q_3 \not\sim^* 0$. Since a does not occur in $\widehat{E}$, the expressions Q_1, Q_2 must satisfy $Q_1 \in \mathsf{im}(\sigma)$ and $Q_1 \equiv a.Q_2$. Thus $Q_2 \sim^* 0$. With $Q_2 \Longrightarrow Q_3$ this implies $Q_3 \sim^* 0$, a contradiction. □

Lemma 26. *Assume that $a \in \mathbb{A}\backslash\{\tau\}$ does neither occur in E nor in F. If $E^\sigma \equiv F^\rho$ for $\sigma : O(E) \to a.\mathbb{P}_0$ and $\rho : O(F) \to a.\mathbb{P}_0$ then $E \equiv F$.*

Proof. The lemma can be shown by induction on the structure of $E \in \mathbb{E}$. □

Lemma 27 (restated Lemma 4). *Let $* \neq 0$ and $G, H \in \mathbb{E}$. If $a \in \mathbb{A}\backslash\{\tau\}$ does neither occur in G nor in H then $G\{a.0/X\} =^* H\{a.0/X\}$ implies $G =^* H$.*

Proof. Let $* \neq 0$ and $G, H \in \mathbb{E}$. We will prove the following more general statement. Let $\sigma : O(G) \to a.\mathbb{P}_0$ and $\rho : O(H) \to a.\mathbb{P}_0$. We claim that $G^\sigma =^* H^\rho$ implies $G =^* H$. Assume that $G^\sigma =^* H^\rho$. We will prove $G =^* H$ by an induction on the structure of the proof for $G^\sigma =^* H^\rho$. We have to consider the following cases:
Case 1. $G^\sigma \equiv H^\rho$. This case is covered by Lemma 26.
Case 2. $G^\sigma =^* H^\rho$ is an instance of one of the axioms for $\simeq^*$ except of the conditional axiom (*rec*3). We show that already $G =^* H$ is an instance of an axiom for $\simeq^*$. Most of the axioms to not involve action prefixes, and are therefore quite easy to treat. Let us exemplary consider the cases (*rec*2) and (τ3).

($rec2$): We have $G^\sigma \equiv recY.E$ and $H^\rho \equiv E\{recY.E/Y\}$ and w.l.o.g. $X \not\equiv Y$. Thus $G \equiv recY.F$ and $E \equiv F^\sigma$ for some F such that a does not occur in F. Thus $H^\rho \equiv E\{recY.E/Y\} \equiv F^\sigma\{recY.F^\sigma/Y\} \equiv (F\{recY.F/Y\})^\pi$ for some $\pi : O(F\{recY.F/Y\}) \to a.\mathbb{P}_0$. Lemma 26 implies $H \equiv F\{recY.F/Y\}$. Thus $G =^* H$ is an instance of axiom ($rec2$).

($\tau 3$): Assume that $G^\sigma \equiv b.(E+\tau.F)$ and $H^\rho \equiv b.(E+\tau.F)+b.F$ for $E, F \in \mathbb{E}$ and $b \in \mathbb{A}$. From $H^\rho \equiv b.(E+\tau.F)+b.F$ we deduce $H \equiv H_1 + H_2$, $H_1^{\rho_1} \equiv b.(E+\tau.F)$ and $H_2^{\rho_2} \equiv b.F$. We can distinguish the following two cases:
Case 2.1. $a = b$: Thus $G^\sigma \equiv a.(E+\tau.F)$, $H_1^{\rho_1} \equiv a.(E+\tau.F)$ and $H_2^{\rho_2} \equiv a.F$. Since a does neither occur in G nor in H_i, we have $G \equiv H_1 \equiv H_2 \equiv X$. Thus $H \equiv X + X$ and $G =^* H$ is an instance of axiom ($S3$).
Case 2.2. $a \neq b$: It follows $G \equiv b.(G_1 + \tau.G_2)$, $G_1^{\sigma_1} \equiv E$, $G_2^{\sigma_2} \equiv F$, $H_1 \equiv b.(H_{1,1} + \tau.H_{1,2})$, $H_{1,1}^{\rho_{1,1}} \equiv E$, $H_{1,2}^{\rho_{1,2}} \equiv F$, and finally also $H_2 \equiv b.H_3$, $H_3^{\rho_2} \equiv F$. Thus $G_1^{\sigma_1} \equiv H_{1,1}^{\rho_{1,1}}$ and $G_2^{\sigma_2} \equiv H_{1,2}^{\rho_{1,2}} \equiv H_3^{\rho_2}$. Lemma 26 implies $G_1 \equiv H_{1,1}$ and $G_2 \equiv H_{1,2} \equiv H_3$. It follows $H \equiv b.(G_1+\tau.G_2)+b.G_2$, and $G =^* H$ is an instance of axiom ($\tau 3$).

Finally note that case 2. cannot deal with axiom (0): If $G \equiv \Delta(X+0)$ and $H \equiv \tau(X+0)$ then $G = H$ is not an instance of an axiom for $\simeq^0$ but $G\{a.0/X\} = H\{a.0/X\}$ is an instance of axiom (0). This is the reason for excluding the case $* = 0$ in Lemma 4.
Case 3. $G^\sigma =^* H^\rho$ is derived by axiom ($rec3$). Thus there is $E \in \mathbb{E}$ such that $H^\rho \equiv recY.E$ and $G^\sigma =^* E\{G^\sigma/Y\}$ can be derived by a smaller proof. Furthermore Y must be guarded in E and w.l.o.g $X \not\equiv Y$. Since $H^\rho \equiv recY.E$, we have $H \equiv recY.F$ and $E \equiv F^\rho$ for some F. Thus $G^\sigma =^* E\{G^\sigma/Y\} \equiv (F^\rho)\{G^\sigma/Y\} \equiv (F\{G/Y\})^\pi$ for some π by a smaller proof. Hence by induction we get $G =^* F\{G/Y\}$. Furthermore since Y must be guarded in $E \equiv F^\rho$ and $Y \not\equiv X$, Y must be also guarded in F. We obtain $G =^* recY.F$, i.e., $G =^* H$.
Case 4. $G^\sigma =^* H^\rho$ is derived from $G^\sigma =^* F$ and $F =^* H^\rho$, which both have smaller proofs. With Lemma 25 we get $F \equiv E^\pi$ for some E and $\pi : O(E) \to a.\mathbb{P}_0$. Thus $G^\sigma =^* E^\pi$ and $E^\pi =^* H^\rho$ by smaller proofs, which implies by induction $G =^* E$ and $E =^* H$, i.e, $G =^* H$.
Case 5. $G^\sigma \equiv b.E$, $H^\rho \equiv b.F$, and $E =^* F$ by a smaller proof. [6] We can distinguish two cases:
Case 5.1. $a = b$: Since a does neither occur in G nor in H we have $G \equiv X \equiv H$.
Case 5.2. $a \neq b$: It follows $G \equiv b.G'$, $H \equiv b.H'$, $E \equiv G'^\sigma$, and $F \equiv H'^\rho$. Thus $G'^\sigma =^* H'^\rho$ and by induction we have $G' =^* H'$, i.e, $G =^* H$. □

Theorem 17 (soundness of ($\mathbb{E}$) for $\simeq^0$). *Assume that $E\{0/X\} \simeq^0 F\{0/X\}$ and $E\{a.0/X\} \simeq^0 F\{a.0/X\}$, where $a \in \mathbb{A}\backslash\{\tau\}$ does neither occur in E nor in F. Then $E \simeq^0 F$.*

Proof. Let $E\{0/X\} \simeq^0 F\{0/X\}$ and $E\{a.0/X\} \simeq^0 F\{a.0/X\}$ where $a \in \mathbb{A}\backslash\{\tau\}$ does neither occur in E nor in F. We have to show that $E \simeq^0 F$. Due to the definition of $\simeq^0$ for expressions with free variables, it suffices to consider the case

[6] All other cases (like for instance $G^\sigma \equiv E_1 + E_2$, $H^\rho \equiv F_1 + F_2$ and $E_i =^* F_i$ by a smaller proof) can be treated similarly.

that $\mathbb{V}(E) \cup \mathbb{V}(F) = \{X\}$. Thus we have to show that for all $P \in \mathbb{P}$ it holds $E\{P/X\} \simeq^0 F\{P/X\}$. Fix an arbitrary $P \in \mathbb{P}$. We distinguish the following two cases.

Case 1. $P \longrightarrow$. We first claim that the symmetric closure of

$$\mathcal{R} = \{\langle G\{P/X\}, H\{P/X\}\rangle \mid a \text{ does neither occur in } G \text{ nor in } H \text{ and } G\{a.0/X\} \sim^0 H\{a.0/X\}\}$$

is a *WB*^{0}. Consider a pair $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$.

(*WB*): Assume that $G\{P/X\} \stackrel{b}{\longrightarrow} P'$. By Lemma 7 we can distinguish the following two cases:

Case i. $G \stackrel{b}{\longrightarrow} G'$ and $P' \equiv G'\{P/X\}$. Thus $G\{a.0/X\} \stackrel{b}{\longrightarrow} G'\{a.0/X\}$. Since $G\{a.0/X\} \sim^0 H\{a.0/X\}$, we have $H\{a.0/X\} \stackrel{\widehat{b}}{\Longrightarrow} Q$ and $G'\{a.0/X\} \sim^0 Q$ for some Q. Note that $G \stackrel{b}{\longrightarrow} G'$ implies $b \neq a \neq \tau$. Thus by Lemma 7 we have $H \stackrel{\widehat{b}}{\Longrightarrow} H'$ and $G'\{a.0/X\} \sim^0 Q \equiv H'\{a.0/X\}$. Thus $H\{P/X\} \stackrel{\widehat{b}}{\Longrightarrow} H'\{P/X\}$ and $\langle G'\{P/X\}, H'\{P/X\}\rangle \in \mathcal{R}$ (note also that since a does not occur in G, $G \stackrel{b}{\longrightarrow} G'$ implies that a does not occur in G' as well, and similarly for H, H').

Case ii. $P \stackrel{b}{\longrightarrow} P'$ and X is totally unguarded in G. Thus $G\{a.0/X\} \stackrel{a}{\longrightarrow} 0$. Since $G\{a.0/X\} \sim^0 H\{a.0/X\}$, we obtain $H\{a.0/X\} \stackrel{a}{\Longrightarrow}$. Using Lemma 7 and the fact that $a \in \mathbb{A}\backslash\{\tau\}$ does not occur in H it follows $H \Longrightarrow H'$ for some H' such that X is totally unguarded in H'. Hence $H\{P/X\} \Longrightarrow H'\{P/X\} \stackrel{b}{\longrightarrow} P'$.

(0): Let $G\{P/X\} \not\longrightarrow$. Lemma 8 implies that $G \not\longrightarrow$ and that X is weakly guarded in G (recall that $P \longrightarrow$). Thus $G\{a.0/X\} \not\longrightarrow$, from which we obtain $H\{a.0/X\} \Longrightarrow H'\{a.0/X\} \not\longrightarrow$ for some H' with $H \Longrightarrow H'$. Thus X must be weakly guarded in H' (otherwise $H'\{a.0/X\} \stackrel{a}{\longrightarrow}$), and hence $H\{P/X\} \Longrightarrow H'\{P/X\} \not\longrightarrow$.

This finishes the proof that $\mathcal{R}$ is a *WB*^{0}. Hence $\mathcal{R} \subseteq \sim^0$. Now consider a pair $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$ such that not only $G\{a.0/X\} \sim^0 H\{a.0/X\}$ but $G\{a.0/X\} \simeq^0 H\{a.0/X\}$. By redoing the proof for the condition (*WB*) above and using $\mathcal{R} \subseteq \sim^0$, we see that this pair satisfies the root condition. Thus $G\{P/X\} \simeq^0 H\{P/X\}$. In particular, we obtain $E\{P/X\} \simeq^0 F\{P/X\}$. This finishes the proof for case 1.

Case 2. $P \not\longrightarrow$. We first claim that the symmetric closure of

$$\mathcal{R} = \{\langle G\{P/X\}, H\{P/X\}\rangle \mid G\{0/X\} \simeq^0 H\{0/X\}\}$$

is a *WB*^{0}. By symmetry it suffices to consider a pair $\langle G\{P/X\}, H\{P/X\}\rangle \in \mathcal{R}$.

(*WB*): Similarly to case 1 (note that case ii cannot occur, since $P \not\longrightarrow$).

(0): Let $G\{P/X\} \not\longrightarrow$. Then $G \not\longrightarrow$. Hence also $G\{0/X\} \not\longrightarrow$, which implies $H\{0/X\} \Longrightarrow H'\{0/X\} \not\longrightarrow$ for some H' with $H \Longrightarrow H'$. Since we assumed $P \not\longrightarrow$, it follows $H\{P/X\} \Longrightarrow H'\{P/X\} \not\longrightarrow$.

This finishes the proof that $\mathcal{R}$ is a *WB*^{0}. The rest of the argumentation is completely analogous to case 1. □