Confluence Problems for Trace Rewriting Systems

Markus Lohrey

Universität Stuttgart, Institut für Informatik,
Breitwiesenstr. 20–22, 70565 Stuttgart, Germany

E-mail: lohreyms@informatik.uni-stuttgart.de

Rewriting systems over trace monoids, briefly trace rewriting systems, generalize both semi-Thue systems and vector replacement systems. In [21], a particular trace monoid M is presented such that confluence is undecidable for the class of length–reducing trace rewriting systems over M. In this paper, we show that this result holds for every trace monoid, which is neither free nor free commutative. Furthermore we show that confluence for special trace rewriting systems over a fixed trace monoid is decidable in polynomial time.

Key Words: Confluence, trace monoids, undecidability results

1. INTRODUCTION

The theory of *free partially commutative monoids* generalizes both, the theory of *free monoids* and the theory of *free commutative monoids*. In computer science, free partially commutative monoids are commonly called *trace monoids* and their elements are called *traces*. Both notions are due to Mazurkiewicz [20], who recognized traces as a model of concurrent processes. [14] gives an extensive overview about current research trends in trace theory.

The relevance of trace theory for computer science can be explained as follows. Assume a finite alphabet Σ. An element of the free monoid over Σ, i.e., a finite word over Σ, may be viewed as the sequence of actions of a sequential process. In addition to a finite alphabet Σ, the specification of a trace monoid (over Σ) requires a binary and symmetric independence relation on Σ. If two symbols a and b are independent then they are allowed to commute. Thus, the two words $sabt$ and $sbat$, where s and t are arbitrary words, denote the same trace. This trace may be viewed as the sequence of actions of a concurrent process where the two independent actions a and b may occur concurrently and thus may be observed either in the order ab or in the order ba.

This point of view makes it interesting to consider identities between traces. If two traces of a concurrent process are semantically equivalent because they always transform the same initial state into the same final state, then these two traces should be equated. Then two traces that can be transformed into each other by a sequence of replacement steps, where in each step a subtrace is replaced by an equivalent trace, also represent semantically equivalent traces. For the algorithmic treatment of such trace identities it is useful to direct these identities. This leads to the notion of a *trace rewriting system* [12], which is thus a finite set of rules, where the left-hand side and right-hand side of each rule are traces. Trace rewriting systems are also interesting because they generalize both, semi–Thue systems (see [16, 7] for a detailed study) and vector replacement systems which are equivalent to Petri nets.

For all kinds of rewriting systems, termination and confluence are of central interest. Together, these two properties guarantee the existence of unique normal forms and thus the solvability of the word problem. Several decidability and undecidability results are known for the confluence problem for the different types of rewriting systems mentioned above. Let us just mention a few of these results. It is known that for terminating semi–Thue systems confluence is decidable [23]. In contrast to this result there exists a trace monoid such that confluence is undecidable for length–reducing trace rewriting systems over this trace monoid [21]. On the other hand in [12] several subclasses of trace rewriting systems were defined for which confluence is decidable. Finally for vector replacement systems it was shown in [24] that confluence is decidable for the class of all vector replacement systems.

In this paper we continue the investigation of the confluence problem for trace rewriting systems. First we will show that confluence is decidable for length-reducing trace rewriting systems over a trace monoid M if and only if M is free or free commutative. This sharpens the undecidability result of Narendran and Otto mentioned above. Moreover since there exists a trace monoid which is generated by three symbols and which is neither free nor free commutative, this result also solves the question for the minimal number of symbols for which the confluence problem for length-reducing systems becomes undecidable, see [12, p 117] and [4, Problem 6]. Our undecidability result also motivates the question for restricted subclasses of length-reducing trace rewriting systems for which confluence becomes decidable. In particular in [12, p 154] it was asked whether confluence is decidable for special trace rewriting systems, where special means that all right-hand sides are the empty trace. In Section 4 we answer this question positively. More precisely we prove that for a fixed trace monoid confluence is decidable in polynomial time for a class of trace rewriting systems that properly contains the class of special trace rewriting systems. Some results of this paper already appeared in a preliminary form in [18] and [19].

2. PRELIMINARIES

With $\mathbb{N}$ we denote the set of natural numbers $\{0, 1, 2, \ldots\}$. The *identity relation* $\{(a, a) \mid a \in A\}$ on a set A is denoted by Id_A. An alphabet is a finite non-empty set, whose elements are also called symbols. Let Σ be an alphabet. The set of all finite words over Σ is denoted by Σ^*. The empty word is denoted by 1. The set $\Sigma^* \backslash \{1\}$ of all non-empty finite words over Σ is denoted by Σ^+. The concatenation of two words $s, t \in \Sigma^*$ is denoted by st. The length of a word $s \in \Sigma^*$ is denoted by $|s|$. For $n \in \mathbb{N}$ we define $\Sigma^n = \{s \in \Sigma^* \mid |s| = n\}$. The set of all symbols from Σ that occur in the word $s \in \Sigma^*$ is denoted by $\mathrm{alph}(s)$. If $s = tu$ for $t, u \in \Sigma^*$ then we say that t is a *prefix* of s and u is a *suffix* of s, and we write $s = t\cdots$ or $s = \cdots u$. If furthermore $u \neq 1$ (respectively $t \neq 1$) then t (respectively u) is called a *proper prefix* (respectively *proper suffix*) of s.

In this paper a *deterministic Turing-machine* $\mathcal{M}$ is a tuple $(Q, \Sigma, \Box, \delta, q_0, q_f)$, where Q is the finite set of states, Σ is the finite tape alphabet, $\Box \in \Sigma$ is the blank symbol, $\delta : Q\backslash\{q_f\} \times \Sigma \to Q \times \Sigma \times \{L, R\}$ is the total transition function, $q_0 \in Q$ is the initial state, and $q_f \in Q$ is the unique final state. The symbols L and R indicate, whether the read-write head moves left or right. Configurations and transitions between configurations are defined as usual. An input for $\mathcal{M}$ is a word $w \in (\Sigma\backslash\{\Box\})^*$. The cells of the one-sided infinite tape of $\mathcal{M}$ can be identified with the natural numbers $\mathbb{N}$. In the initial configuration that corresponds to the input $w \in (\Sigma\backslash\{\Box\})^*$, the tape content is $w\Box\Box\cdots$, the state is q_0 and the head is scanning cell 0. We assume that $\mathcal{M}$ marks cell 0 in its first move somehow such that it never makes a left-move while scanning cell 0. The machine $\mathcal{M}$ terminates on the input w if and only if a final configuration (i.e., a configuration, where the state is q_f) is reached after a finite number of transitions from the initial configuration that corresponds to w.

2.1. Trace monoids and trace rewriting systems

For a good introduction into the theory of traces see [12] or [14]. An *independence alphabet* is a pair (Σ, I), where Σ is a finite alphabet and $I \subseteq \Sigma \times \Sigma$ is an irreflexive and symmetric binary relation, which is also called an *independence relation.* Thus an independence alphabet is an undirected graph without loops, and will be represented in this form in diagrams. In the following let (Σ, I) be an independence alphabet. The complement $(\Sigma \times \Sigma)\backslash I$ of I is also called a *dependence relation.* It is a reflexive and symmetric relation. The pair $(\Sigma, (\Sigma \times \Sigma)\backslash I)$ is also called a *dependence alphabet.* The smallest (with respect to set inclusion) congruence relation on Σ^*, which contains all pairs from $\{(ab, ba) \mid a\,I\,b\}$, is denoted by $\equiv_I$. Since I is symmetric, $\equiv_I$ is the reflexive and transitive closure of the relation $\{(sabt, sbat) \mid s, t \in \Sigma^*, a\,I\,b\}$. For $s \in \Sigma^*$ we denote by $[s]_I = \{t \in \Sigma^* \mid s \equiv_I t\}$ the equivalence class with respect to $\equiv_I$, which contains the word s. Such an equivalence class is called a *trace.* If $[s]_I = \{s\}$ for a word $s \in \Sigma^*$, which holds for instance for $s \in \{1\} \cup \Sigma$, then we identify the trace $[s]_I$ with the word s. The set of all traces is denoted by $\mathbb{M}(\Sigma, I) = \{[s]_I \mid s \in \Sigma^*\}$. Since $\equiv_I$ is a congruence relation on Σ^*, we can define the *concatenation* of two traces $[s]_I$ and $[t]_I$ by $[st]_I$. The concatenation of traces defines a monoid structure on the set $\mathbb{M}(\Sigma, I)$ of all traces, where the *empty trace* $[1]_I = \{1\}$ is the neutral element. This monoid is called the *trace monoid* generated by (Σ, I) and will be denoted by $\mathbb{M}(\Sigma, I)$ as well. Thus the trace monoid $\mathbb{M}(\Sigma, I)$ is the quotient monoid $\Sigma^*/{\equiv_I}$. If $I = \emptyset$ then $\mathbb{M}(\Sigma, I)$ is isomorphic to the *free monoid* Σ^*. On the other hand, if $I = (\Sigma \times \Sigma)\backslash \mathrm{Id}_\Sigma$ then $\mathbb{M}(\Sigma, I)$ is isomorphic to the *free commutative monoid* $\mathbb{N}^n$, where $n = |\Sigma|$. Finally if (Σ, I) is of the form $(\Sigma_1 \cup \Sigma_2,\ \Sigma_1 \times \Sigma_2 \cup \Sigma_2 \times \Sigma_1)$ then $\mathbb{M}(\Sigma, I)$ is isomorphic to the direct product of the free monoids Σ_1^* and Σ_2^*. In this case we identify traces from $\mathbb{M}(\Sigma, I)$ with pairs of words.

Since for all words $s, t \in \Sigma^*$ with $s \equiv_I t$ the identities $|s| = |t|$ and $\mathrm{alph}(s) = \mathrm{alph}(t)$ hold, we can define $|[s]_I| = |s|$ and $\mathrm{alph}([s]_I) = \mathrm{alph}(s)$. In the rest of this work we will use the following conventions: Words over some alphabet will be denoted by lower-case letters, possibly with a subscript or superscript. Traces will be denoted by bold lower-case letters, possibly with a subscript or superscript. The independence relation I can be lifted to the set $\mathbb{M}(\Sigma, I)$ in the following way: $\boldsymbol{u}\, I\, \boldsymbol{v}$ if $\mathrm{alph}(\boldsymbol{u}) \times \mathrm{alph}(\boldsymbol{v}) \subseteq I$. Obviously it holds $1\, I\, \boldsymbol{u}$ for every trace $\boldsymbol{u}$. For a trace $\boldsymbol{u} \in \mathbb{M}(\Sigma, I)$ we define $\min(\boldsymbol{u}) = \{a \in \Sigma \mid \exists s \in \Sigma^* : \boldsymbol{u} = [as]_I\}$ as the the set all *minimal symbols* of $\boldsymbol{u}$ and $\max(\boldsymbol{u}) = \{a \in \Sigma \mid \exists s \in \Sigma^* : \boldsymbol{u} = [sa]_I\}$ as the set all *maximal symbols* of $\boldsymbol{u}$. The following generalization of the well known Levi's lemma for traces [9] can be found for instance in [14, p 74].

LEMMA 2.1. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{M}(\Sigma, I)$. Then

$$\boldsymbol{u}_1 \boldsymbol{u}_2 \cdots \boldsymbol{u}_m = \boldsymbol{v}_1 \boldsymbol{v}_2 \cdots \boldsymbol{v}_n$$

if and only if there are traces $\boldsymbol{w}_{i,j} \in \mathbb{M}(\Sigma, I)$ $(i \in \{1, \dots, m\},\ j \in \{1, \dots, n\})$ such that

- $\boldsymbol{u}_i = \boldsymbol{w}_{i,1} \boldsymbol{w}_{i,2} \cdots \boldsymbol{w}_{i,n}$ for all $i \in \{1, \dots, m\}$,
- $\boldsymbol{v}_j = \boldsymbol{w}_{1,j} \boldsymbol{w}_{2,j} \cdots \boldsymbol{w}_{m,j}$ for all $j \in \{1, \dots, n\}$, and
- $\boldsymbol{w}_{i,j}\, I\, \boldsymbol{w}_{k,l}$ if $1 \le i < k \le m$ and $1 \le l < j \le n$.

The situation in Lemma 2.1 can be visualized by a diagram of the following form, where $m = 5$ and $n = 4$. The i-th column corresponds to $\boldsymbol{u}_i$, the j-th row corresponds to $\boldsymbol{v}_j$, and the square that results from intersecting the i-th column with the j-th row corresponds to the trace $\boldsymbol{w}_{i,j}$. Finally it holds $\boldsymbol{w}_{i,j}\, I\, \boldsymbol{w}_{k,l}$, if $\boldsymbol{w}_{i,j}$ is right-above of $\boldsymbol{w}_{k,l}$.

$\boldsymbol{v}_4$	$\boldsymbol{w}_{1,4}$	$\boldsymbol{w}_{2,4}$	$\boldsymbol{w}_{3,4}$	$\boldsymbol{w}_{4,4}$	$\boldsymbol{w}_{5,4}$
$\boldsymbol{v}_3$	$\boldsymbol{w}_{1,3}$	$\boldsymbol{w}_{2,3}$	$\boldsymbol{w}_{3,3}$	$\boldsymbol{w}_{4,3}$	$\boldsymbol{w}_{5,3}$
$\boldsymbol{v}_2$	$\boldsymbol{w}_{1,2}$	$\boldsymbol{w}_{2,2}$	$\boldsymbol{w}_{3,2}$	$\boldsymbol{w}_{4,2}$	$\boldsymbol{w}_{5,2}$
$\boldsymbol{v}_1$	$\boldsymbol{w}_{1,1}$	$\boldsymbol{w}_{2,1}$	$\boldsymbol{w}_{3,1}$	$\boldsymbol{w}_{4,1}$	$\boldsymbol{w}_{5,1}$
	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	$\boldsymbol{u}_3$	$\boldsymbol{u}_4$	$\boldsymbol{u}_5$

Trace rewriting systems were first considered in [10]. A *trace rewriting system*, briefly TRS, over the trace monoid $\mathbb{M}(\Sigma, I)$ is a finite subset of $\mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$. Trace rewriting systems will be denoted by the letters $\mathcal{R}$ and $\mathcal{P}$, possibly with a subscript. In the following let $\mathcal{R}$ be a TRS over the trace monoid $\mathbb{M}(\Sigma, I)$. Elements of $\mathcal{R}$ are also called *trace rewriting rules*, or briefly *rules*, over $\mathbb{M}(\Sigma, I)$. A rule $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ will be also denoted by $\boldsymbol{\ell} \to \boldsymbol{r}$. We define the set $\mathrm{dom}(\mathcal{R})$ of all *left-hand sides* of $\mathcal{R}$ by $\mathrm{dom}(\mathcal{R}) = \{\boldsymbol{\ell} \mid \exists \boldsymbol{r} \in \mathbb{M}(\Sigma, I) : (\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}\}$. The set $\mathrm{ran}(\mathcal{R})$ of all *right-hand sides* of $\mathcal{R}$ is defined by $\mathrm{ran}(\mathcal{R}) = \{\boldsymbol{r} \mid \exists \boldsymbol{\ell} \in \mathbb{M}(\Sigma, I) : (\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}\}$. Let $c = (\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ be a rule. The binary relation $\to_c$ is defined by

$$\to_c = \{(\boldsymbol{s}, \boldsymbol{t}) \in \mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I) \mid \exists \boldsymbol{u}, \boldsymbol{v} \in \mathbb{M}(\Sigma, I) : \boldsymbol{s} = \boldsymbol{u}\boldsymbol{\ell}\boldsymbol{v}, \boldsymbol{t} = \boldsymbol{u}\boldsymbol{r}\boldsymbol{v}\}.$$

The *one-step rewrite relation* $\to_{\mathcal{R}}$ is defined by $\to_{\mathcal{R}} = \bigcup_{c \in \mathcal{R}} \to_c$. If $I = \emptyset$, i.e., $\mathbb{M}(\Sigma, I) \simeq \Sigma^*$ then $\mathcal{R}$ is called a *semi-Thue system* over Σ. A detailed introduction into the theory of semi-Thue systems can be found in [16] and [7]. If on the other hand $I = (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$, i.e., $\mathbb{M}(\Sigma, I) \simeq \mathbb{N}^{|\Sigma|}$, then $\mathcal{R}$ is also called a *vector replacement system.*

In the rest of this section we omit the subscript $\mathcal{R}$ in the relation $\to_{\mathcal{R}}$. For $\to$ we usually use the infix notation, i.e., instead of $(\boldsymbol{s},\boldsymbol{t}) \in \to$ we write $\boldsymbol{s} \to \boldsymbol{t}$. The *transitive closure* and the *transitive reflexive closure* of $\to$ are denoted by $\to^+$ and $\to^*$, respectively. The TRS $\mathcal{R}$ is *terminating on* $\boldsymbol{s}$, if there does not exist an infinite chain of the form $\boldsymbol{s} = \boldsymbol{s}_1 \to \boldsymbol{s}_2 \to \boldsymbol{s}_3 \to \cdots$ in $\mathbb{M}(\Sigma, I)$. The TRS $\mathcal{R}$ is *terminating* if $\mathcal{R}$ is terminating on every $\boldsymbol{s} \in \mathbb{M}(\Sigma, I)$. A trace $\boldsymbol{s}$ is *irreducible* with respect to $\mathcal{R}$ if there does not exist a trace $\boldsymbol{t}$ with $\boldsymbol{s} \to \boldsymbol{t}$. The set of all traces that are irreducible with respect to $\mathcal{R}$ will be denoted by $\mathrm{IRR}(\mathcal{R})$. The trace $\boldsymbol{t}$ is a *normal form* of $\boldsymbol{s}$ (with respect to $\mathcal{R}$) if $\boldsymbol{s} \to^* \boldsymbol{t} \in \mathrm{IRR}(\mathcal{R})$. A pair $(\boldsymbol{s},\boldsymbol{t})$ of traces is *confluent* (with respect to $\mathcal{R}$) if there exists a trace $\boldsymbol{u}$ with $\boldsymbol{s} \to^* \boldsymbol{u}$ and $\boldsymbol{t} \to^* \boldsymbol{u}$. We say that $\mathcal{R}$ is confluent on $\boldsymbol{s}$ if for all $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{M}(\Sigma, I)$ with $\boldsymbol{s} \to^* \boldsymbol{t}$ and $\boldsymbol{s} \to^* \boldsymbol{u}$ there exists a $\boldsymbol{v}$ with $\boldsymbol{t} \to^* \boldsymbol{v}$ and $\boldsymbol{u} \to^* \boldsymbol{v}$. The TRS $\mathcal{R}$ is *confluent* if $\mathcal{R}$ is confluent on every $\boldsymbol{s}$. The TRS $\mathcal{R}$ is *locally confluent* if for all $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \in \mathbb{M}(\Sigma, I)$ with $\boldsymbol{s} \to \boldsymbol{t}$ and $\boldsymbol{s} \to \boldsymbol{u}$ there exists a $\boldsymbol{v}$ with $\boldsymbol{t} \to^* \boldsymbol{v}$ and $\boldsymbol{u} \to^* \boldsymbol{v}$. Newman's lemma [22] implies that if $\mathcal{R}$ is terminating then $\mathcal{R}$ is locally confluent if and only if $\mathcal{R}$ is confluent. If $\mathcal{R}$ is terminating and confluent then every trace has a unique normal form and the word problem can be decided by computing and normal forms. This well-known fact motivates the interest in terminating and confluent systems. The TRS $\mathcal{R}$ is called *length-reducing* if $|\boldsymbol{\ell}| > |\boldsymbol{r}|$ for all $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$. A length-reducing TRS is obviously terminating. The TRS $\mathcal{R}$ is called *special* if $\mathrm{ran}(\mathcal{R}) = \{1\}$ and $1 \notin \mathrm{dom}(\mathcal{R})$. Finally the *length of a TRS* $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| = \sum_{(\boldsymbol{\ell},\boldsymbol{r})\in\mathcal{R}} (|\boldsymbol{\ell}| + |\boldsymbol{r}|).$$

It is known that termination, confluence, and local confluence are all decidable properties for vector replacement systems: Termination can be easily decided by using Dickson's lemma, local confluence can be reduced to the decidable reachability problem, and confluence was shown to be decidable in [24]. On the other hand these three properties are undecidable for semi-Thue systems [15, 1]. If only terminating semi-Thue systems are considered, local confluence and hence confluence can be decided by considering so called critical pairs [23]. Furthermore for a length-reducing semi-Thue systems $\mathcal{R}$, it can be decided in time $O(\|\mathcal{R}\|^3)$ whether $\mathcal{R}$ is confluent [17]. Unfortunately these positive results cannot be extended to trace rewriting systems. In [21] a trace monoid $\mathbb{M}(\Sigma, I)$ was presented such that it is undecidable whether a length-reducing TRS over $\mathbb{M}(\Sigma, I)$ is confluent. In particular this implies that in contrast to semi-Thue systems, there does not exist a definition of critical pairs for trace rewriting systems that results in finite sets of critical pairs.

The main goal of this paper is to continue the investigation of the confluence problem for trace rewriting systems. For this, we define the following decision problems:

- $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is the following decision problem:

INPUT: A length-reducing TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$.
QUESTION: Is $\mathcal{R}$ confluent?

- $\mathrm{COSP}(\mathbb{M}(\Sigma, I))$ is the following decision problem:

INPUT: A special TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$.
QUESTION: Is $\mathcal{R}$ confluent?

In these problems the input length is the length $\|\mathcal{R}\|$ of the input TRS $\mathcal{R}$.

2.2. Critical pairs

For trace rewriting systems quite unusual phenomenons may be observed, which cannot occur for semi-Thue systems. One of these phenomenons concerns disjoint left-hand sides. Let $\mathcal{R}$ be a semi-Thue system over the alphabet Σ, and let $(\ell_0, r_0), (\ell_1, r_1) \in \mathcal{R}$ be two rules. Assume that the word $s \in \Sigma^*$ contains two disjoint occurrences of ℓ_0 and ℓ_1. Then s can be factorized as $s = t\ell_0 u\ell_1 v$. By applying the two rules we obtain the words $tr_0 u\ell_1 v$ and $t\ell_0 ur_1 v$. By applying to each of these words the other rules, both words can be rewritten into the word $tr_0 ur_1 v$. This trivial fact is in general no longer true for trace rewriting systems. The application of a rule $\boldsymbol{\ell}_0 \to \boldsymbol{r}_0$ may destroy the occurrence of another left-hand side $\boldsymbol{\ell}_1$ which is disjoint to the replaced occurrence of $\boldsymbol{\ell}_0$, see the following example. Here we will not give the formal (but obvious) definition of an occurrence of a trace in another trace.

EXAMPLE 2.1. Let $M = \mathbb{M}(\{a, b, c\}, \{(a, c), (c, a)\})$, and let $\mathcal{R}_1$ be the TRS $\mathcal{R}_1 = \{c \to b, aa \to 1\}$ over M. In the trace $[caa]_I = [aca]_I$ there exist unique disjoint occurrences of the left-hand sides c and aa. But it holds $[aca]_I \to_{\mathcal{R}_1} aba$ and $[aca]_I = [caa]_I \to_{\mathcal{R}_1} c$, and the pair (aba, c) is not confluent with respect to $\mathcal{R}_1$.

A second example is the one-rule TRS $\mathcal{R}_2 = \{[ac]_I \to b\}$ over the same trace monoid. Let $\boldsymbol{u} = [aacc]_I$. If we mark in $[aacc]_I$ the different occurrences of a and c in the form $[a_1 a_2 c_1 c_2]_I$, we see that in $\boldsymbol{u}$ there are four different occurrences of the left-hand side $[ac]_I$. But it holds $[a_1 a_2 c_1 c_2]_I \to_{\mathcal{R}_2} a_1 b c_2$ and $[a_1 a_2 c_1 c_2]_I = [c_1 a_1 c_2 a_2]_I \to_{\mathcal{R}_2} c_1 b a_2$, and again the pair (abc, cba) is not confluent with respect to $\mathcal{R}_2$.

In the following we will define a class of trace rewriting systems for which the phenomenon from Example 2.1 cannot occur. For this we define the following technical property (A).

A TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ satisfies condition (A) if the following holds:

(A1) For all $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ and all $a \in \Sigma$ with $a \, I \, \boldsymbol{\ell}$ it holds $a\boldsymbol{r} = \boldsymbol{r}a$.

(A2) For all $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and all factorizations $\boldsymbol{\ell}_0 = \boldsymbol{p}_0 \boldsymbol{q}_0$, $\boldsymbol{\ell}_1 = \boldsymbol{p}_1 \boldsymbol{q}_1$ such that $\boldsymbol{p}_i \neq 1 \neq \boldsymbol{q}_i$ for $i \in \{0, 1\}$, $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1$, and $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$ it holds:
There exist factorizations $\boldsymbol{r}_0 = \boldsymbol{s}_0 \boldsymbol{t}_0$, $\boldsymbol{r}_1 = \boldsymbol{s}_1 \boldsymbol{t}_1$ such that for all $a \in \Sigma$ and $i \in \{0, 1\}$ it holds: If $a \, I \, \boldsymbol{p}_i$ then $a \, I \, \boldsymbol{s}_i$, and if $a \, I \, \boldsymbol{q}_i$ then $a \, I \, \boldsymbol{t}_i$.

The TRS $\mathcal{R}_1$ from Example 2.1 does not satisfy condition (A1) for the rule $c \to b$: It holds $a \, I \, c$ and $[ab]_I \neq [ba]_I$. On the other hand, $\mathcal{R}_1$ satisfies condition (A2). The TRS $\mathcal{R}_2$ from the same example satisfies condition (A1) but it violates condition (A2): For instance if we choose $\boldsymbol{p}_0 = a$, $\boldsymbol{q}_0 = c$, $\boldsymbol{p}_1 = c$ and $\boldsymbol{q}_1 = a$ it holds $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1$ and $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$. Let $\boldsymbol{s}_0 \boldsymbol{t}_0$ be a factorization of the right-hand side b. Then either $\boldsymbol{s}_0 = b$ or $\boldsymbol{t}_0 = b$. In the first case it holds $c \, I \, \boldsymbol{p}_0$ but $c \, I \, \boldsymbol{s}_0$ does not hold. In the second case we have $a\,I\,\boldsymbol{q}_0$ but again $a\,I\,\boldsymbol{t}_0$ does not hold. On the other hand it is trivial that every special TRS satisfies condition (A). The importance of condition (A) results from the following technical lemma.

LEMMA 2.2. Let $\mathcal{R}$ be a TRS over $\mathbb{M}(\Sigma, I)$, which satisfies condition (A). Let $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{M}(\Sigma, I)$ and $(\boldsymbol{p}_0\boldsymbol{q}_0, \boldsymbol{r}_0), (\boldsymbol{p}_1\boldsymbol{q}_1, \boldsymbol{r}_1) \in \mathcal{R}$ such that

$$\boldsymbol{p}_0\,I\,\boldsymbol{p}_1, \quad \boldsymbol{q}_0\,I\,\boldsymbol{q}_1, \quad \boldsymbol{w}_0\,I\,\boldsymbol{w}_1, \quad \boldsymbol{w}_0\,I\,\boldsymbol{p}_1\boldsymbol{q}_0, \quad \boldsymbol{w}_1\,I\,\boldsymbol{p}_0\boldsymbol{q}_1.$$

Then the pair $(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1,\ \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0)$ is confluent with respect to $\mathcal{R}$.

Proof. First we consider the case that $\boldsymbol{p}_0 = 1$. We have to show that the pair $(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1,\ \boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0)$ is confluent. Because of $\boldsymbol{p}_0\boldsymbol{q}_0 = \boldsymbol{q}_0 \to_{\mathcal{R}} \boldsymbol{r}_0$ we have $\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0 \to_{\mathcal{R}} \boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{r}_0$. We claim that also $\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 \to_{\mathcal{R}} \boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{r}_0$ holds. Since $\mathcal{R}$ satisfies condition (A1) and $\boldsymbol{w}_0\boldsymbol{q}_1\,I\,\boldsymbol{q}_0$ and $(\boldsymbol{q}_0, \boldsymbol{r}_0) \in \mathcal{R}$ we have $\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 = \boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{r}_0$. Thus

$$\begin{aligned}
\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 &= \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{r}_0 && (\text{since } \boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 = \boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{r}_0)\\
&= \boldsymbol{w}_0\boldsymbol{p}_1\boldsymbol{q}_1\boldsymbol{w}_1\boldsymbol{r}_0 && (\text{since } \boldsymbol{w}_0\,I\,\boldsymbol{w}_1,\ \boldsymbol{w}_0\,I\,\boldsymbol{p}_1 \text{ and } \boldsymbol{w}_1\,I\,\boldsymbol{q}_1)\\
&\to_{\mathcal{R}} \boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{r}_0.
\end{aligned}$$

Analogous arguments apply if one of the traces $\boldsymbol{q}_0$, $\boldsymbol{p}_1$, or $\boldsymbol{q}_1$ is empty. Thus, in the following we may assume that $\boldsymbol{p}_i \neq 1 \neq \boldsymbol{q}_i$ for $i \in \{0,1\}$. Then condition (A2) implies that there exist factorizations $\boldsymbol{r}_0 = \boldsymbol{s}_0\boldsymbol{t}_0$ and $\boldsymbol{r}_1 = \boldsymbol{s}_1\boldsymbol{t}_1$ such that for all $a \in \Sigma$ and $i \in \{0,1\}$ it holds: If $a\,I\,\boldsymbol{p}_i$ then $a\,I\,\boldsymbol{s}_i$, and if $a\,I\,\boldsymbol{q}_i$ then $a\,I\,\boldsymbol{t}_i$. In particular we obtain $\boldsymbol{p}_1\,I\,\boldsymbol{s}_0,\ \boldsymbol{w}_1\,I\,\boldsymbol{s}_0,\ \boldsymbol{p}_0\,I\,\boldsymbol{s}_1,\ \boldsymbol{w}_0\,I\,\boldsymbol{s}_1,\ \boldsymbol{q}_1\,I\,\boldsymbol{t}_0,\ \boldsymbol{w}_0\,I\,\boldsymbol{t}_0,\ \boldsymbol{q}_0\,I\,\boldsymbol{t}_1,$ $\boldsymbol{w}_1\,I\,\boldsymbol{t}_1$. Furthermore it holds $\boldsymbol{s}_1\,I\,\boldsymbol{s}_0$ because of $\boldsymbol{p}_1\,I\,\boldsymbol{s}_0$, and similarly $\boldsymbol{t}_1\,I\,\boldsymbol{t}_0$ because of $\boldsymbol{q}_1\,I\,\boldsymbol{t}_0$. Together we obtain

$$\begin{aligned}
\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 = \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{s}_0\boldsymbol{t}_0\boldsymbol{w}_0\boldsymbol{q}_1 = \boldsymbol{s}_0\boldsymbol{w}_0\boldsymbol{p}_1\boldsymbol{q}_1\boldsymbol{w}_1\boldsymbol{t}_0 \to_{\mathcal{R}}&\\
\boldsymbol{s}_0\boldsymbol{w}_0\boldsymbol{s}_1\boldsymbol{t}_1\boldsymbol{w}_1\boldsymbol{t}_0 = \boldsymbol{s}_1\boldsymbol{w}_1\boldsymbol{s}_0\boldsymbol{t}_0\boldsymbol{w}_0\boldsymbol{t}_1&
\end{aligned}$$

and $\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{s}_1\boldsymbol{t}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{s}_1\boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{q}_0\boldsymbol{w}_0\boldsymbol{t}_1 \to_{\mathcal{R}} \boldsymbol{s}_1\boldsymbol{w}_1\boldsymbol{s}_0\boldsymbol{t}_0\boldsymbol{w}_0\boldsymbol{t}_1$. ■

For every semi-Thue system $\mathcal{R}$ there exists a finite set of so called critical pairs such that $\mathcal{R}$ is locally confluent if and only if all critical pairs of $\mathcal{R}$ are confluent [23]. These critical pairs result from overlapping left-hand sides of rules. In [11], see also [12, p 120], the notion of a critical pair was generalized to trace rewriting systems and it was shown that a TRS is confluent if and only if all its critical pairs are confluent. But with the definition in [11], the set of critical pairs associated to a TRS is in general an infinite set. But this is not an insufficiency of the definition, given in [11]. It is a principal limitation, since as already mentioned in Section 2.1 there exists a trace monoid such that confluence is already undecidable for length-reducing trace rewriting systems over this trace monoid. In this section we will present a definition of critical pairs for trace rewriting systems which differs in some details from the definition given in [11]. In contrast to [11] our critical pairs can be only used for trace rewriting systems, which satisfy condition (A), in order to check confluence. This restriction is motivated by our applications in the later sections of this paper. But also our definition will in general lead to infinite sets of critical pairs. In fact in Section 3 we will prove that confluence is also undecidable for length-reducing trace rewriting systems that satisfy the condition (A).

DEFINITION 2.1. Let $\mathcal{R}$ be a TRS over $\mathbb{M}(\Sigma, I)$. The set $\mathrm{CS}(\mathcal{R})$ of all *critical situations* of $\mathcal{R}$ is the set of all triples $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1)$ that satisfy the following condition: There exist rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and seven traces $\boldsymbol{p}_i, \boldsymbol{q}_i, \boldsymbol{w}_i$ ($i \in \{0,1\}$) and $\boldsymbol{s} \neq 1$ such that:

1. $\boldsymbol{\ell}_0 = \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0, \quad \boldsymbol{\ell}_1 = \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1,$
2. $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1, \quad \boldsymbol{q}_0 \, I \, \boldsymbol{q}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{w}_1, \quad \boldsymbol{s} \, I \, \boldsymbol{w}_0 \boldsymbol{w}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{q}_0 \boldsymbol{p}_1, \quad \boldsymbol{w}_1 \, I \, \boldsymbol{p}_0 \boldsymbol{q}_1$
3. For all $i \in \{0,1\}$ there does not exist an $a \in \min(\boldsymbol{w}_i)$ with $a \, I \, \boldsymbol{p}_i$, and there does not exist a $b \in \max(\boldsymbol{w}_i)$ with $b \, I \, \boldsymbol{q}_{1-i}$.
4. $\boldsymbol{t} = \boldsymbol{p}_1 \boldsymbol{w}_1 \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0 \boldsymbol{w}_0 \boldsymbol{q}_1 = \boldsymbol{p}_0 \boldsymbol{w}_0 \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1 \boldsymbol{w}_1 \boldsymbol{q}_0$,[1]
5. $\boldsymbol{t}_0 = \boldsymbol{p}_1 \boldsymbol{w}_1 \boldsymbol{r}_0 \boldsymbol{w}_0 \boldsymbol{q}_1, \quad \boldsymbol{t}_1 = \boldsymbol{p}_0 \boldsymbol{w}_0 \boldsymbol{r}_1 \boldsymbol{w}_1 \boldsymbol{q}_0$

This critical situation is *generated* by the rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0)$ and $(\boldsymbol{\ell}_1, \boldsymbol{r}_1)$. The set $\mathrm{CP}(\mathcal{R})$ of all *critical pairs* of $\mathcal{R}$ is $\mathrm{CP}(\mathcal{R}) = \{(\boldsymbol{t}_0, \boldsymbol{t}_1) \mid \exists \boldsymbol{t} : (\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})\}$. The set $\mathrm{CT}(\mathcal{R})$ of all *critical traces* of $\mathcal{R}$ is $\mathrm{CT}(\mathcal{R}) = \{\boldsymbol{t} \mid \exists \boldsymbol{t}_0, \boldsymbol{t}_1 : (\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})\}$.

We do not distinguish the critical situations $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1)$ and $(\boldsymbol{t}_1, \boldsymbol{t}, \boldsymbol{t}_0)$ as well as the critical pairs $(\boldsymbol{t}_0, \boldsymbol{t}_1)$ and $(\boldsymbol{t}_1, \boldsymbol{t}_0)$. The following lemma corresponds to Theorem 3.3 in [11].

LEMMA 2.3. Let $\mathcal{R}$ be a TRS over $\mathbb{M}(\Sigma, I)$, which satisfies condition (A). Then $\mathcal{R}$ is locally confluent if and only if all pairs in $\mathrm{CP}(\mathcal{R})$ are confluent.

Proof. Let $\mathcal{R}$ be a TRS over $\mathbb{M}(\Sigma, I)$, which satisfies condition (A). First note that $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_0$ and $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_1$ for all $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})$. This proves one direction of the lemma. For the other direction let us assume that all pairs in $\mathrm{CP}(\mathcal{R})$ are confluent and let $\boldsymbol{t}, \boldsymbol{t}_0, \boldsymbol{t}_1 \in \mathbb{M}(\Sigma, I)$ be such that $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_0$ and $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_1$. We have to show that the pair $(\boldsymbol{t}_0, \boldsymbol{t}_1)$ is confluent.

First there have to exist rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and traces $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{v}_0, \boldsymbol{v}_1 \in \mathbb{M}(\Sigma, I)$ with $\boldsymbol{t} = \boldsymbol{u}_0 \boldsymbol{\ell}_0 \boldsymbol{v}_0 = \boldsymbol{u}_1 \boldsymbol{\ell}_1 \boldsymbol{v}_1$, $\boldsymbol{t}_0 = \boldsymbol{u}_0 \boldsymbol{r}_0 \boldsymbol{v}_0$, and $\boldsymbol{t}_1 = \boldsymbol{u}_1 \boldsymbol{r}_1 \boldsymbol{v}_1$. Lemma 2.1 applied to the identity $\boldsymbol{u}_0 \boldsymbol{\ell}_0 \boldsymbol{v}_0 = \boldsymbol{u}_1 \boldsymbol{\ell}_1 \boldsymbol{v}_1$ gives nine traces $\boldsymbol{p}_i, \boldsymbol{q}_i, \boldsymbol{w}_i, \boldsymbol{y}_i$ ($i \in \{0,1\}$), $\boldsymbol{s}$ such that

- $\boldsymbol{\ell}_i = \boldsymbol{p}_i \boldsymbol{s} \boldsymbol{q}_i, \quad \boldsymbol{u}_i = \boldsymbol{y}_0 \boldsymbol{p}_{1-i} \boldsymbol{w}_{1-i}, \quad \boldsymbol{v}_i = \boldsymbol{w}_i \boldsymbol{q}_{1-i} \boldsymbol{y}_1 \quad (i \in \{0,1\}),$
- $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1, \quad \boldsymbol{q}_0 \, I \, \boldsymbol{q}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{w}_1, \quad \boldsymbol{s} \, I \, \boldsymbol{w}_0 \boldsymbol{w}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{p}_1 \boldsymbol{q}_0, \quad \boldsymbol{w}_1 \, I \, \boldsymbol{p}_0 \boldsymbol{q}_1,$

see also the following diagram:

$\boldsymbol{v}_1$	$\boldsymbol{w}_1$	$\boldsymbol{q}_0$	$\boldsymbol{y}_1$
$\boldsymbol{\ell}_1$	$\boldsymbol{p}_1$	$\boldsymbol{s}$	$\boldsymbol{q}_1$
$\boldsymbol{u}_1$	$\boldsymbol{y}_0$	$\boldsymbol{p}_0$	$\boldsymbol{w}_0$
	$\boldsymbol{u}_0$	$\boldsymbol{\ell}_0$	$\boldsymbol{v}_0$

[1]Note that the equality of these two factorizations of $\boldsymbol{t}$ follows from the independencies listed in the first point.

We have to show that the pair

$$(\boldsymbol{u}_0\boldsymbol{r}_0\boldsymbol{v}_0,\ \boldsymbol{u}_1\boldsymbol{r}_1\boldsymbol{v}_1) = (\boldsymbol{y}_0\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{y}_1,\ \boldsymbol{y}_0\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0\boldsymbol{y}_1) \tag{1}$$

is confluent. For this it suffices to show that the pair

$$(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1,\ \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0) \tag{2}$$

is confluent, because the confluence of the pair in (2) implies the confluence of the pair in (1). If $\boldsymbol{s} = 1$, i.e., $\boldsymbol{\ell}_i = \boldsymbol{p}_i\boldsymbol{q}_i$ for $i \in \{0,1\}$ then the pair (2) is confluent by Lemma 2.2. Hence let us assume that $\boldsymbol{s} \neq 1$. We will show that for all $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{M}(\Sigma, I)$ with $\boldsymbol{w}_0\, I\, \boldsymbol{w}_1$, $\boldsymbol{s}\, I\, \boldsymbol{w}_0\boldsymbol{w}_1$, $\boldsymbol{w}_0\, I\, \boldsymbol{p}_1\boldsymbol{q}_0$, and $\boldsymbol{w}_1\, I\, \boldsymbol{p}_0\boldsymbol{q}_1$ the pair in (2) is confluent. We will prove this by an induction on $|\boldsymbol{w}_0\boldsymbol{w}_1|$:

First let us assume that there does not exist an $a \in \min(\boldsymbol{w}_i)$ with $a\, I\, \boldsymbol{p}_i$ and there does not exist a $b \in \max(\boldsymbol{w}_i)$ with $b\, I\, \boldsymbol{q}_{1-i}$ for $i \in \{0,1\}$. Note that this case also includes the case $|\boldsymbol{w}_0\boldsymbol{w}_1| = 0$. Then the pair in (2) is contained in $\mathrm{CP}(\mathcal{R})$ and is therefore by assumption confluent. Next let us assume that for instance $\boldsymbol{w}_0 = a\boldsymbol{w}$ and $a\, I\, \boldsymbol{p}_0$ for some $a \in \Sigma$, the other cases can be dealt analogously. From $\boldsymbol{w}_0\, I\, \boldsymbol{s}\boldsymbol{q}_0$ and $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0$ it follows $a\, I\, \boldsymbol{\ell}_0$. Since $\mathcal{R}$ satisfies condition (A1) it follows $a\boldsymbol{r}_0 = \boldsymbol{r}_0 a$. Hence it holds $\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0 a\boldsymbol{w}\boldsymbol{q}_1 = a\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}\boldsymbol{q}_1$ and $\boldsymbol{p}_0 a\boldsymbol{w}\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0 = a\boldsymbol{p}_0\boldsymbol{w}\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0$. Since $\boldsymbol{w}$ satisfies at least the same independencies as $\boldsymbol{w}_0 = a\boldsymbol{w}$, the induction hypothesis implies that the pair $(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}\boldsymbol{q}_1,\ \boldsymbol{p}_0\boldsymbol{w}\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0)$ is confluent. But then also the pair $(a\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}\boldsymbol{q}_1, a\boldsymbol{p}_0\boldsymbol{w}\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0)$ is confluent. ■

Lemma 2.3 shows that in order to check whether a terminating TRS $\mathcal{R}$ that satisfies condition (A) is confluent, it suffices to check the confluence of all critical pairs of $\mathcal{R}$. Let $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1)$ be a critical situation of $\mathcal{R}$. Since $\mathcal{R}$ is terminating, we can calculate an (arbitrary) normal form $\boldsymbol{u}_i$ of $\boldsymbol{t}_i$. If $\boldsymbol{u}_0 = \boldsymbol{u}_1$ then the critical pair $(\boldsymbol{t}_0, \boldsymbol{t}_1)$ is confluent. On the other hand if $\boldsymbol{u}_0 \neq \boldsymbol{u}_1$ then $\mathcal{R}$ is not confluent, since

$$\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_0 \to^*_{\mathcal{R}} \boldsymbol{u}_0 \in \mathrm{IRR}(\mathcal{R}) \quad \text{and} \quad \boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_1 \to^*_{\mathcal{R}} \boldsymbol{u}_1 \in \mathrm{IRR}(\mathcal{R}).$$

EXAMPLE 2.2. We want to apply Lemma 2.3 in order to show that the special TRS $\mathcal{R} = \{ba \to 1, ab \to 1, c \to 1\}$ over $\mathbb{M}(\{a,b,c\}, \{(a,c),(c,a)\})$ is confluent. Since $\mathcal{R}$ satisfies condition (A), we can apply Lemma 2.3. Let $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0$ and $\boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1$ be left-hand sides of $\mathcal{R}$, where $\boldsymbol{s} \neq 1$, $\boldsymbol{p}_0\, I\, \boldsymbol{p}_1$, and $\boldsymbol{q}_0\, I\, \boldsymbol{q}_1$. If we exclude the trivial case $\boldsymbol{\ell}_0 = \boldsymbol{s} = \boldsymbol{\ell}_1$ then only the following two cases remain.

Case 1. $\boldsymbol{\ell}_0 = ab$, $\boldsymbol{\ell}_1 = ba$, $\boldsymbol{s} = b$, $\boldsymbol{p}_0 = a = \boldsymbol{q}_1$, and $\boldsymbol{p}_1 = 1 = \boldsymbol{q}_0$
We have to consider all pairs $(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{w}_0\boldsymbol{q}_1, \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{q}_0) = (\boldsymbol{w}_1\boldsymbol{w}_0 a, a\boldsymbol{w}_0\boldsymbol{w}_1)$, where (among other independencies) $\boldsymbol{s}\, I\, \boldsymbol{w}_0\boldsymbol{w}_1$. From $\boldsymbol{s} = b$ it follows $\boldsymbol{w}_0 = 1 = \boldsymbol{w}_1$. Hence we obtain the confluent pair (a,a).

Case 2. $\boldsymbol{\ell}_0 = ba$, $\boldsymbol{\ell}_1 = ab$, $\boldsymbol{s} = a$, $\boldsymbol{p}_0 = b = \boldsymbol{q}_1$, and $\boldsymbol{p}_1 = 1 = \boldsymbol{q}_0$
We have to consider all pairs $(\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{w}_0\boldsymbol{q}_1, \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{q}_0) = (\boldsymbol{w}_1\boldsymbol{w}_0 b, b\boldsymbol{w}_0\boldsymbol{w}_1)$, where (among other independencies) $\boldsymbol{s}\, I\, \boldsymbol{w}_0$ and $\boldsymbol{w}_1\, I\, \boldsymbol{p}_0$. From $\boldsymbol{w}_1\, I\, \boldsymbol{p}_0$, i.e., $\boldsymbol{w}_1\, I\, b$, it follows $\boldsymbol{w}_1 = 1$. From $\boldsymbol{s}\, I\, \boldsymbol{w}_0$, i.e., $a\, I\, \boldsymbol{w}_0$ it follows $\boldsymbol{w}_0 = c^n$ for some $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$ we have to consider the pair $(c^n b, bc^n)$, which is confluent due to the rule $c \to 1$.

2.3. Coding of trace rewriting systems

If $\sigma : M \longrightarrow M'$ is a monoid morphism between the trace monoids M and M', and $\mathcal{R}$ is a TRS over M, then we can define a TRS $\sigma(\mathcal{R})$ over M' by $\sigma(\mathcal{R}) = \{\sigma(\boldsymbol{\ell}) \to \sigma(\boldsymbol{r}) \mid (\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}\}$. In general it is of course possible that $\mathcal{R}$ is confluent but $\sigma(\mathcal{R})$ is not confluent or vice versa. For instance for the terminating and confluent semi-Thue system $\mathcal{R} = \{a \to b\}$ and the injective morphism σ with $\sigma(a) = aa$ and $\sigma(b) = b$ the semi-Thue system $\sigma(\mathcal{R}) = \{aa \to b\}$ is not confluent. On the other hand if the morphism σ maps every symbol to the empty trace then $\sigma(\mathcal{R})$ is confluent for every (also non-confluent) TRS $\mathcal{R}$. The following lemma gives conditions that exclude these possibilities.

LEMMA 2.4. Let $\sigma : M \longrightarrow M'$ be a monoid morphism between the trace monoids M and M', and let $\mathcal{R}$ be a TRS over M. Furthermore assume that the following four conditions hold.

1. σ is injective.
2. $\sigma(\mathcal{R})$ is terminating and satisfies condition (A).
3. If $\boldsymbol{\ell} \in \mathrm{dom}(\mathcal{R})$ and $\sigma(\boldsymbol{s}) = \boldsymbol{u}'\sigma(\boldsymbol{\ell})\boldsymbol{v}'$ then there exist $\boldsymbol{u}, \boldsymbol{v} \in M$ with $\boldsymbol{u}' = \sigma(\boldsymbol{u})$ and $\boldsymbol{v}' = \sigma(\boldsymbol{v})$.
4. If $\boldsymbol{t}' \in \mathrm{CT}(\sigma(\mathcal{R}))$ then there exists a $\boldsymbol{t} \in M$ with $\boldsymbol{t}' = \sigma(\boldsymbol{t})$.

Then $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent.

Proof. Let $\sigma : M \longrightarrow M'$ be a monoid morphism between the trace monoids M and M', and let $\mathcal{R}$ be a TRS over M, which satisfies the four conditions from the lemma. First we show the following claim:

$$\text{If } \sigma(\boldsymbol{s}) \to_{\sigma(\mathcal{R})} \boldsymbol{t}' \text{ then } \boldsymbol{t}' = \sigma(\boldsymbol{t}) \text{ and } \boldsymbol{s} \to_{\mathcal{R}} \boldsymbol{t} \text{ for some } \boldsymbol{t} \in M. \tag{3}$$

In order to prove this claim, assume that $\sigma(\boldsymbol{s}) = \boldsymbol{u}'\sigma(\boldsymbol{\ell})\boldsymbol{v}'$ and $\boldsymbol{t}' = \boldsymbol{u}'\sigma(\boldsymbol{r})\boldsymbol{v}'$ for a rule $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$. Condition (3) from the lemma implies that there exist $\boldsymbol{u}, \boldsymbol{v} \in M$ with $\boldsymbol{u}' = \sigma(\boldsymbol{u})$ and $\boldsymbol{v}' = \sigma(\boldsymbol{v})$. Thus $\sigma(\boldsymbol{s}) = \sigma(\boldsymbol{u\ell v})$ and therefore $\boldsymbol{s} = \boldsymbol{u\ell v}$, since σ is injective. It follows $\boldsymbol{s} \to_{\mathcal{R}} \boldsymbol{urv}$ and $\sigma(\boldsymbol{urv}) = \boldsymbol{u}'\sigma(\boldsymbol{r})\boldsymbol{v}' = \boldsymbol{t}'$, which proves (3).

Now we prove the statement of the lemma. First let $\sigma(\mathcal{R})$ be confluent and let $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \in M$ such that $\boldsymbol{s} \to_{\mathcal{R}}^* \boldsymbol{t}$ and $\boldsymbol{s} \to_{\mathcal{R}}^* \boldsymbol{u}$. Since σ is a monoid morphism, it follows $\sigma(\boldsymbol{s}) \to_{\sigma(\mathcal{R})}^* \sigma(\boldsymbol{t})$ and $\sigma(\boldsymbol{s}) \to_{\sigma(\mathcal{R})}^* \sigma(\boldsymbol{u})$. Since $\sigma(\mathcal{R})$ is confluent, there exists a $\boldsymbol{v}' \in M'$ with $\sigma(\boldsymbol{t}) \to_{\sigma(\mathcal{R})}^* \boldsymbol{v}'$ and $\sigma(\boldsymbol{u}) \to_{\sigma(\mathcal{R})}^* \boldsymbol{v}'$. An inductive extension of (3) gives $\boldsymbol{v}_1, \boldsymbol{v}_2 \in M$ with $\boldsymbol{v}' = \sigma(\boldsymbol{v}_1)$, $\boldsymbol{t} \to_{\mathcal{R}}^* \boldsymbol{v}_1$ and $\boldsymbol{v}' = \sigma(\boldsymbol{v}_2)$, $\boldsymbol{u} \to_{\mathcal{R}}^* \boldsymbol{v}_2$. Finally $\boldsymbol{v}_1 = \boldsymbol{v}_2$ follows from $\sigma(\boldsymbol{v}_1) = \boldsymbol{v}' = \sigma(\boldsymbol{v}_2)$ and the injectivity of σ. Hence $\mathcal{R}$ is confluent. Note that up to now we did not use the assumptions (2) and (4) from the lemma.

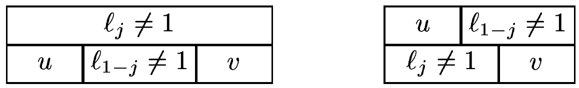
Now let $\mathcal{R}$ be confluent. We have to show that also $\sigma(\mathcal{R})$ is confluent. Since $\sigma(\mathcal{R})$ is terminating, it suffices to show that $\sigma(\mathcal{R})$ is locally confluent. Furthermore since $\sigma(\mathcal{R})$ satisfies condition (A), it suffices to show that all critical pairs are confluent. Let $(\boldsymbol{u}', \boldsymbol{t}', \boldsymbol{v}') \in \mathrm{CS}(\sigma(\mathcal{R}))$. Because of the fourth condition from the lemma there exists a $\boldsymbol{t} \in M$ with $\boldsymbol{t}' = \sigma(\boldsymbol{t})$. Thus $\sigma(\boldsymbol{t}) \to_{\sigma(\mathcal{R})} \boldsymbol{u}'$ and $\sigma(\boldsymbol{t}) \to_{\sigma(\mathcal{R})} \boldsymbol{v}'$. From (3) it follows that there exist $\boldsymbol{u}, \boldsymbol{v} \in M$ with $\boldsymbol{u}' = \sigma(\boldsymbol{u})$, $\boldsymbol{v}' = \sigma(\boldsymbol{v})$ and $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{u}$, $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{v}$.

Since $\mathcal{R}$ is confluent, there exists a $\boldsymbol{w} \in M$ with $\boldsymbol{u} \rightarrow_{\mathcal{R}}^{*} \boldsymbol{w}$ and $\boldsymbol{v} \rightarrow_{\mathcal{R}}^{*} \boldsymbol{w}$. This implies $\boldsymbol{u}' = \sigma(\boldsymbol{u}) \rightarrow_{\sigma(\mathcal{R})}^{*} \sigma(\boldsymbol{w})$ and $\boldsymbol{v}' = \sigma(\boldsymbol{v}) \rightarrow_{\sigma(\mathcal{R})}^{*} \sigma(\boldsymbol{w})$. Hence $\sigma(\mathcal{R})$ is confluent. ■

The last lemma of this section only deals with free monoids. The coding function ϕ from the following lemma will be used twice in the next section. Let $\ell_0, \ell_1 \in \Sigma^*$ be two words with $\ell_0 \neq 1 \neq \ell_1$. We say that a word t is an *overlapping* of ℓ_0 and ℓ_1 if one of the following two cases holds for $j = 0$ or $j = 1$.

- $\exists u, v \in \Sigma^* : t = \ell_j = u\ell_{1-j}v$
- $\exists u, v \in \Sigma^* : t = \ell_j v = u\ell_{1-j}$ and $|\ell_j| > |u|$ (and thus $|\ell_{1-j}| > |v|$).

See also the following picture:



LEMMA 2.5. Let $\Sigma = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ and $\Gamma = \{a_1, \ldots, a_m, b_1, b_2\}$, where $m \in \mathbb{N}$ and $n \geq 2$. Define the monoid morphism $\phi : \Sigma^* \longrightarrow \Gamma^*$ by

$$\phi(a_i) = a_i \text{ for } i \in \{1, \ldots, m\} \text{ and } \phi(b_i) = b_1 b_2^i b_1 b_2^{2n+1-i} \text{ for } i \in \{1, \ldots, n\}.$$

Then the following holds:

1. ϕ is injective.
2. If $\phi(s) = s_1\phi(\ell)s_2$ and $\ell \neq 1$ then there exist $u_1, u_2 \in \Sigma^*$ with $s_1 = \phi(u_1)$, $s_2 = \phi(u_2)$, and $s = u_1 \ell u_2$.
3. If $\phi(\ell_1) = s_1 s$ and $\phi(\ell_2) = s s_2$ then there exist $u, u_1, u_2 \in \Sigma^*$ with $\ell_1 = u_1 u$, $\ell_2 = u u_2$, $\phi(u) = s$, $\phi(u_1) = s_1$, and $\phi(u_2) = s_2$.

Note that ϕ was chosen such that $|\phi(b_i)| = 2n + 3$ for all $i \in \{1, \ldots, n\}$.

Proof. Since the set $\phi(\Sigma)$ is a biprefix code [2], the injectivity of ϕ is clear. The other two statements of the lemma follow immediately from the following statement, see also [7, p 60]:

For all $c, d \in \Sigma$, if t is an overlapping of $\phi(c)$ and $\phi(d)$ then $c = d$.

It should be noted that from this fact it easily follows that the set $\phi(\Sigma)$ is a comma-free code [2, p 336]. ■

3. LENGTH-REDUCING SYSTEMS

In this section we prove that $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is undecidable if neither $I = \emptyset$ (i.e. $\mathbb{M}(\Sigma, I) \simeq \Sigma^*$) nor $I = (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$ (i.e. $\mathbb{M}(\Sigma, I) \simeq \mathbb{N}^{|\Sigma|}$) holds. For this let $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma, I))$ be the following stronger version of $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$:

INPUT: A length-reducing TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ such that $1 \notin \mathrm{ran}(\mathcal{R})$.

QUESTION: Is $\mathcal{R}$ confluent?

Obviously if $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma, I))$ is undecidable then also $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is undecidable. Our proof will consist of two main steps. In Section 3.1 we will prove that $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\{a, b, c\}, I))$ is undecidable for $I = \{(a, c), (c, a), (b, c), (c, b)\}$ and

$I = \{(a,c),(c,a)\}$. The corresponding trace monoids are the smallest trace monoids (measured in $|\Sigma|$) that are neither free nor free commutative. In a second step we prove in Section 3.2 that the undecidability of $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Gamma,(\Gamma\times\Gamma)\cap I))$ for some $\Gamma \subseteq \Sigma$ implies that also $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma, I))$ is undecidable. Only for this last step the condition $1 \notin \mathrm{ran}(\mathcal{R})$ is important.

3.1. Independence alphabets with three symbols

If (Σ, I) is an independence alphabet with $|\Sigma| = 2$ then either $\mathbb{M}(\Sigma, I) \simeq \{a,b\}^*$ or $\mathbb{M}(\Sigma, I) \simeq \mathbb{N}^2$. In both cases we can decide confluence for terminating trace rewriting systems. If $|\Sigma| = 3$ then there exist up to isomorphism two independence alphabets whose corresponding trace monoids are neither free nor free commutative. These are the following two independence alphabets:

$$a\text{—}c\text{—}b \qquad\qquad a\text{—}c \quad b$$

In the next two sections we will consider these independence alphabets.

3.1.1. The case $a-c-b$

Let $(\Sigma, I) = (\{a,b,c\},\{(a,c),(c,a),(b,c),(c,b)\})$. Then the trace monoid $\mathbb{M}(\Sigma, I)$ is isomorphic to $\{a,b\}^* \times \{c\}^*$. In this section we will prove that the problem $\mathrm{COLR}_{\neq 1}(\{a,b\}^* \times \{c\}^*)$ is undecidable. First we prove that $\mathrm{COLR}_{\neq 1}(\Gamma^* \times \{c\}^*)$ is undecidable for a particular alphabet Γ, which contains more than two symbols. In a second step we show that the alphabet Γ can be coded into the alphabet $\{a,b\}$.

First we study the structure of critical situations from $\mathrm{CS}(\mathcal{R})$, if $\mathcal{R}$ is a TRS over a direct product $\Sigma_1^* \times \Sigma_2^*$ of free monoids. In the following let Σ_1 and Σ_2 be two non-empty finite alphabets. For a trace $\boldsymbol{u} = (u_1,u_2) \in \Sigma_1^* \times \Sigma_2^*$ we write $\boldsymbol{u}^{(1)} = u_1$ and $\boldsymbol{u}^{(2)} = u_2$ in the following. The next lemma is obvious.

LEMMA 3.1. Let $\mathcal{R}$ be a TRS over $\Sigma_1^* \times \Sigma_2^*$. Then $\mathcal{R}$ satisfies condition (A) (from Section 2.2) if $\boldsymbol{\ell}^{(i)} = 1$ implies $\boldsymbol{r}^{(i)} = 1$ for all rules $(\boldsymbol{\ell},\boldsymbol{r}) \in \mathcal{R}$ and all $i \in \{1,2\}$.

Let $\ell_0, \ell_1 \in \Sigma_i^*$ be two words with $\ell_0 \neq 1 \neq \ell_1$. We say that a word $t \in \Sigma_i^*$ *is generated disjointly by* ℓ_0 *and* ℓ_1 if there exists a word $u \in \Sigma_i^*$ such that $t = \ell_0 u \ell_1$ or $t = \ell_1 u \ell_0$. The next lemma is intuitively quite obvious. It says that for a critical trace $(\boldsymbol{t}^{(1)}, \boldsymbol{t}^{(2)})$ with respect to a TRS over $\Sigma_1^* \times \Sigma_2^*$ there exist left-hand sides $\boldsymbol{\ell}_0$ and $\boldsymbol{\ell}_1$ such that at least one component $\boldsymbol{t}^{(i)} \in \Sigma_i^*$ ($i \in \{1,2\}$) is an overlapping of $\boldsymbol{\ell}_0^{(i)}$ and $\boldsymbol{\ell}_1^{(i)}$ (in particular $\boldsymbol{\ell}_0^{(i)} \neq 1 \neq \boldsymbol{\ell}_1^{(i)}$). If this holds for instance for $i = 1$ then the second component $\boldsymbol{t}^{(2)}$ need not necessarily be an overlapping of $\boldsymbol{\ell}_0^{(2)}$ and $\boldsymbol{\ell}_1^{(2)}$. If $\boldsymbol{\ell}_0^{(2)} \neq 1 \neq \boldsymbol{\ell}_1^{(2)}$ then it suffices that $\boldsymbol{t}^{(2)}$ is generated disjointly by $\boldsymbol{\ell}_0^{(2)}$ and $\boldsymbol{\ell}_1^{(2)}$. On the other hand if say $\boldsymbol{\ell}_0^{(2)} = 1$ then we can restrict to the case $\boldsymbol{t}^{(2)} = \boldsymbol{\ell}_1^{(2)}$.

LEMMA 3.2. Let $\mathcal{R}$ be a TRS over $\Sigma_1^* \times \Sigma_2^*$ such that $\boldsymbol{\ell}^{(i)} = 1$ implies $\boldsymbol{r}^{(i)} = 1$ for all rules $(\boldsymbol{\ell},\boldsymbol{r}) \in \mathcal{R}$ and all $i \in \{1,2\}$. Let $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})$. Then there exist rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ such that for both $i = 1$ and $i = 2$ one of the following four cases (1) to (4) holds.

		$t^{(i)} =$	$t_0^{(i)} =$	$t_1^{(i)} =$
(1)	$\ell_0^{(i)} \neq 1 \neq \ell_1^{(i)}$	$\ell_0^{(i)} u \ell_1^{(i)}$	$r_0^{(i)} u \ell_1^{(i)}$	$\ell_0^{(i)} u r_1^{(i)}$
(2)	$\ell_0^{(i)} \neq 1 \neq \ell_1^{(i)}$	$\ell_0^{(i)} = u \ell_1^{(i)} v$	$r_0^{(i)}$	$u r_1^{(i)} v$
(3)	$\lvert\ell_0^{(i)}\rvert > \lvert u\rvert$, $\lvert\ell_1^{(i)}\rvert > \lvert v\rvert$	$\ell_0^{(i)} v = u \ell_1^{(i)}$	$r_0^{(i)} v$	$u r_1^{(i)}$
(4)	$\ell_1^{(i)} = 1$	$\ell_0^{(i)}$	$r_0^{(i)}$	$\ell_0^{(i)}$

Furthermore for $i = 1$ or $i = 2$ either case (2) or case (3) has to hold.

Note that in case (1) $t^{(i)}$ is generated disjointly by $\ell_0^{(i)} \neq 1$ and $\ell_1^{(i)} \neq 1$, whereas in case (2) and (3) $t^{(i)}$ is an overlapping of $\ell_0^{(i)} \neq 1$ and $\ell_1^{(i)} \neq 1$. Furthermore if one of the four cases above holds then $t \to_{\mathcal{R}} t_i$ for $i \in \{0, 1\}$.

Proof. Let $(t_0, t, t_1) \in \mathrm{CS}(\mathcal{R})$. According to Definition 2.1 there exist rules $(\ell_0, r_0), (\ell_1, r_1) \in \mathcal{R}$ and pairs $p_j, q_j, w_j, s \in \Sigma_1^* \times \Sigma_2^*$ ($j \in \{0, 1\}$) such that

- $s \neq 1, \quad \ell_0 = p_0 s q_0, \quad \ell_1 = p_1 s q_1,$
- $p_0 \, I \, p_1, \quad q_0 \, I \, q_1, \quad w_0 \, I \, w_1, \quad s \, I \, w_0 w_1, \quad w_0 \, I \, p_1 q_0, \quad w_1 \, I \, p_0 q_1,$
- $p_j \, I \, w_j$ or $q_{1-j} \, I \, w_j$ implies $w_j = 1$ for $j \in \{0, 1\}$,[2]
- $t = p_1 w_1 p_0 s q_0 w_0 q_1 = p_0 w_0 p_1 s q_1 w_1 q_0,$
- $t_0 = p_1 w_1 r_0 w_0 q_1, \quad t_1 = p_0 w_0 r_1 w_1 q_0.$

Because of $s = (s^{(1)}, s^{(2)}) \neq (1, 1)$, either $s^{(1)} \neq 1$ or $s^{(2)} \neq 1$. W.l.o.g. assume that $s^{(1)} \neq 1$. It follows $\ell_0^{(1)} \neq 1 \neq \ell_1^{(1)}$. Next from $s \, I \, w_0 w_1$ it follows $w_0^{(1)} = w_1^{(1)} = 1$ and therefore $t^{(1)} = p_1^{(1)} \ell_0^{(1)} q_1^{(1)} = p_0^{(1)} \ell_1^{(1)} q_0^{(1)}$. Furthermore $p_0 \, I \, p_1$ implies $p_j^{(1)} = 1$ for some $j \in \{0, 1\}$. Analogously $q_j^{(1)} = 1$ for some $j \in \{0, 1\}$. For each of the four possible combinations it is easy to see that $t^{(1)}$ is an overlapping of $\ell_0^{(1)}$ and $\ell_1^{(1)}$, i.e., for $i = 1$ case (2) or case (3) from the lemma holds. If also $s^{(2)} \neq 1$ then also $t^{(2)}$ is an overlapping of $\ell_0^{(2)}$ and $\ell_1^{(2)}$. Now assume that $s^{(2)} = 1$. Since $w_0^{(2)} \neq 1 \neq w_1^{(2)}$ contradicts $w_0 \, I \, w_1$, we can w.l.o.g. assume that $w_1^{(2)} = 1$.

Case 1. $w_0^{(2)} \neq 1$: It follows that neither $p_0 \, I \, w_0$ nor $q_1 \, I \, w_0$ holds (for instance by the third point above $p_0 \, I \, w_0$ would imply $w_0 = 1$). Because of $w_0^{(1)} = 1$, we obtain $p_0^{(2)} \neq 1 \neq q_1^{(2)}$. Finally from $p_0 \, I \, p_1$ and $q_0 \, I \, q_1$ it follows $p_1^{(2)} = 1 = q_0^{(2)}$. Therefore

$$\ell_0^{(2)} = p_0^{(2)} s^{(2)} q_0^{(2)} = p_0^{(2)} \neq 1 \neq q_1^{(2)} = p_1^{(2)} s^{(2)} q_1^{(2)} = \ell_1^{(2)}$$

and hence

$$t^{(2)} = p_0^{(2)} w_0^{(2)} p_1^{(2)} s^{(2)} q_1^{(2)} w_1^{(2)} q_0^{(2)} = p_0^{(2)} w_0^{(2)} q_1^{(2)} = \ell_0^{(2)} w_0^{(2)} \ell_1^{(2)}.$$

Thus $t^{(2)}$ is generated disjointly by $\ell_0^{(2)}$ and $\ell_1^{(2)}$, i.e., case (1) from the lemma holds for $i = 2$.

Case 2. $w_0^{(2)} = 1$, i.e., $w_0 = (1, 1) = w_1$: Because of $p_0 \, I \, p_1$, w.l.o.g. we may assume $p_1^{(2)} = 1$, i,e, $t^{(2)} = p_0^{(2)} q_1^{(2)} q_0^{(2)}$. If also $q_1^{(2)} = 1$ then $\ell_1^{(2)} =$

[2]This weakening of the third condition from Definition 2.1 is sufficient for the following considerations.

(1a) $x0 \to 0$ for $x \in \Gamma$	(2a) $\triangleleft y \to 0$ for $y \in \Gamma \backslash \{\$\}$
(1b) $0x \to 0$ for $x \in \Gamma$	(2b) $\triangleleft \$ y \to 0$ for $y \in \Gamma \backslash \{\$\}$
(1c) $(0, c) \to 0$	(2c) $x \triangleright \to 0$ for $x \in \Gamma$
(3a) $(\triangleright A^{\lvert w \rvert + 2}, c) \to \triangleright q_0 w \triangleleft$	(4a) $a\beta\$\$ \to a\$\beta$ for $a \in \Sigma$, $\beta \in \Sigma \cup \{\triangleleft\}$
(3b) $(B, c) \to 0$	(4b) $a\$\beta\$\$ \to a\$\$\beta$ for $a \in \Sigma$, $\beta \in \Sigma \cup \{\triangleleft\}$
(5a) $q \triangleleft \$\$ \to a'p\triangleleft$ if $\delta(q, \Box) = (p, a', R)$	
(5b) $bq \triangleleft \$\$ \to pba'\triangleleft$ if $\delta(q, \Box) = (p, a', L)$, $b \in \Sigma$	
(5c) $qa\$\$ \to a'p$ if $\delta(q, a) = (p, a', R)$	
(5d) $bqa\$\$ \to pba'$ if $\delta(q, a) = (p, a', L)$, $b \in \Sigma$	

FIG. 1. The TRS $\mathcal{R}$ from the proof of Lemma 3.3.

1 and $\boldsymbol{t}^{(2)} = \boldsymbol{p}_0^{(2)} \boldsymbol{q}_0^{(2)} = \boldsymbol{\ell}_0^{(2)}$. Thus we obtain case (4) from the lemma. On the other hand if $\boldsymbol{q}_1^{(2)} \neq 1$ then $\boldsymbol{q}_0^{(2)} = 1$ follows from $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$. Thus $\boldsymbol{t}^{(2)} = \boldsymbol{p}_0^{(2)} \boldsymbol{q}_1^{(2)} = \boldsymbol{\ell}_0^{(2)} \boldsymbol{\ell}_1^{(2)}$. Hence we obtain either case (1) (with $u = 1$) from the lemma (if $\boldsymbol{\ell}_0^{(2)} = \boldsymbol{p}_0^{(2)} \neq 1$) or case (4) from the lemma (if $\boldsymbol{\ell}_0^{(2)} = \boldsymbol{p}_0^{(2)} = 1$). ■

We now start to prove the undecidability of $\mathrm{COLR}_{\neq 1}(\{a, b\}^* \times \{c\}^*)$. In the following let $\mathcal{M} = (Q, \Sigma, \Box, \delta, q_0, q_f)$ be an universal deterministic Turing-machine. Since $\mathcal{M}$ is universal, it is undecidable whether $\mathcal{M}$ terminates on a given input word $w \in (\Sigma \backslash \{\Box\})^*$. Since $\mathcal{M}$ terminates if and only if $\mathcal{M}$ reaches the final state q_f, it is undecidable whether $\mathcal{M}$ reaches the final state q_f after a finite number of steps when started with a given input word $w \in (\Sigma \backslash \{\Box\})^*$. For the further considerations we fix an input $w \in (\Sigma \backslash \{\Box\})^*$ for $\mathcal{M}$. First we will construct a TRS $\mathcal{R}$ over a trace monoid of the form $\Gamma^* \times \{c\}^*$, which is confluent if and only if $\mathcal{M}$ terminates on the input w.

The following TRS $\mathcal{R}$ is a variante of a TRS, presented in [21]. Let $\Gamma = Q \cup \Sigma \cup \{0, \triangleright, \triangleleft, A, B, \$\}$, where $0, \triangleright, \triangleleft, A, B, \$ \notin Q \cup \Sigma$ are new symbols. Furthermore let $c \notin \Gamma$ be another symbol. In the following we will work in the trace monoid $\Gamma^* \times \{c\}^*$. A pair of the form $(s, 1)$ with $s \in \Gamma^*$ will be briefly denoted by s. We define the TRS $\mathcal{R}$ over $\Gamma^* \times \{c\}^*$ by the rules in Figure 1. Note that $\mathcal{R}$ is length-reducing and that $1 \notin \mathrm{ran}(\mathcal{R})$ holds. The rules in (1a), (1b), and (1c) make the symbol 0 absorbing, whereas the rules in (5a) to (5d) simulate the Turing-machine $\mathcal{M}$. Here the symbols $\triangleright$ and $\triangleleft$ operate as a left- and a right-end marker, respectively. The additional two \$-symbols in the rules in (5a) to (5d) make these rules length-reducing. Furthermore note that since $\mathcal{M}$ is deterministic, there do not exist overlappings between the left-hand sides of these rules. In order to apply the simulation-rules in (5a) to (5d), the transport-rules in (4a) and (4b) are necessary. They shift \$-symbols to the left in configurations until a simulation-rule can be applied. In order to make the transport-rules itself length-reducing, they consume one \$ when shifting one \$ to the left. Finally, the rules (3a) and (3b) create the critical trace $(\triangleright A^{\lvert w \rvert + 2} v B, c)$ for every $v \in \Gamma^*$ by sharing the c in the second component. In Lemma 3.3 we will prove that $\mathcal{R}$ is confluent on all these critical traces if and only if the machine $\mathcal{M}$ does not terminate on the input w. But the rules (3a) and (3b) also generate unwanted critical traces like for instance $(Bv \triangleright A^{|w|+2}, c)$. Some of the resulting unwanted critical pairs are made confluent with the rules in (2c). Finally the rules in (2a) and (2b) generate the absorbing symbol 0 if the transport-rules in (4a) and (4b) have consumed sufficiently many \$-symbols.

Let $\mathcal{R}_1$ be the TRS, which consists of the rules in (1a), (1b), and (1c), and let $\mathcal{R}_{4,5}$ be the TRS, which consists of the rules in (4a), (4b), and (5a) to (5d).

LEMMA 3.3. $\mathcal{R}$ is confluent if and only if $\mathcal{M}$ does not terminate on the input w.

Proof. First we assume that $\mathcal{M}$ terminates on the input w. Then there exist $m > 0$, $u \in \Sigma^*$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $k \geq 2$ such that

$$(\triangleright A^{|w|+2}\$^m B, c) \rightarrow_{(3a)} \triangleright q_0 w \triangleleft \$^m B \rightarrow^+_{\mathcal{R}_{4,5}} \triangleright u q_f a_1 \$^2 a_2 \$^2 \cdots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B = t \quad (4)$$

holds. Since $\mathcal{M}$ cannot move out of the final state q_f and because of $k \geq 2$, the word t is irreducible with respect to $\mathcal{R}$. On the other hand, it also holds $(\triangleright A^{|w|+2}\$^m B, c) \rightarrow_{(3b)} \triangleright A^{|w|+2}\$^m 0 \rightarrow^+_{(1a)} 0$, and the word 0 is also irreducible. Thus $\mathcal{R}$ is not confluent.

Now we assume that $\mathcal{M}$ does not terminate on the input w. We will show that $\mathcal{R}$ is confluent. Since $\mathcal{R}$ is terminating and satisfies condition (A) by Lemma 3.1, it suffices to show that all pairs in $\mathrm{CP}(\mathcal{R})$ are confluent. Thus, by Lemma 3.2 it suffices to consider all overlappings in at least one component that can occur between left-hand sides of $\mathcal{R}$. The following critical situations exist:

1. Since the symbol 0 cannot be deleted by rules from $\mathcal{R}$, $(s,t) \rightarrow_{\mathcal{R}} (u,v)$ and $0 \in \mathrm{alph}(s)$ implies $0 \in \mathrm{alph}(u)$. Since each pair of the form $(s0t, u)$ can be reduced to the absorbing symbol 0 with the rules in (1a), (1b), and (1c), the following holds: If $(s,t) \rightarrow_{d_1} (s_1,t_1)$ and $(s,t) \rightarrow_{d_2} (s_2,t_2)$, and $\{d_1,d_2\} \cap \{(1a),(1b),(1c)\} \neq \emptyset$ then the pair $((s_1,t_1),(s_2,t_2))$ is confluent. The same observation also holds if $d_1, d_2 \in \{(2a),(2b),(2c)\}$.

It remains to consider all critical situations that are generated by two rules from $\mathcal{R}\backslash\mathcal{R}_1$, where not both rules belong to $\{(2a),(2b),(2c)\}$. The following four cases deal with overlappings in the first component.

2. For each rule $d \in \mathcal{R}\backslash\mathcal{R}_1$ of the form $(\ell, c) \rightarrow r$ (i.e. for the rules (3a) and (3b)) and for all $v \in \{c\}^*$ it holds $(\ell, cvc) \rightarrow_d (r, vc)$ and $(\ell, cvc) \rightarrow_d (r, cv)$. The resulting critical pair $((r,vc),(r,cv))$ is trivially confluent because of $cv = vc$.

3. Let $d \in \mathcal{R}$ be a rule of the form $(y\ell, c^i) \rightarrow r$, where $y \in \Gamma\backslash\{\$\}$ and $i \in \{0,1\}$ holds. It follows $(\triangleleft y\ell, c^i) \rightarrow_d \triangleleft r$ and $(\triangleleft y\ell, c^i) \rightarrow_{(2a)} (0\ell, c^i)$, and we obtain the critical pair $((\triangleleft r, 1),(0\ell, c^i))$. This pair is confluent because on the one hand it holds $(0\ell, c^i) \rightarrow^+_{\mathcal{R}_1} 0$. On the other hand note that r is of the form zs for some $z \in \Gamma\backslash\{\$\}$. Thus it follows $\triangleleft r = \triangleleft zs \rightarrow_{(2a)} 0s \rightarrow^+_{(1b)} 0$. The same arguments can be used if a rule in (2b) instead of a rule in (2a) is applied.

4. Let d be an arbitrary rule from $\mathcal{R}$ of the form $(\ell x, c^i) \rightarrow r$, where $x \in \Gamma$ and $i \in \{0,1\}$. Then it holds $(\ell x \triangleright, c^i) \rightarrow_d r\triangleright$ and $(\ell x \triangleright, c^i) \rightarrow_{(2c)} (\ell 0, c^i)$. Thus we obtain the critical pair $((r\triangleright, 1),(\ell 0, c^i))$. Since r is of the form sz for some $z \in \Gamma$, this pair is confluent.

5. $(x \rhd A^{|w|+2}, c) \to_{(3a)} x \rhd q_0 w \lhd$ and $(x \rhd A^{|w|+2}, c) \to_{(2c)} (0A^{|w|+2}, c)$ for some $x \in \Gamma$: We obtain the critical pair $((x \rhd q_0 w \lhd, 1), (0A^{|w|+2}, c))$, which is confluent because of $(0A^{|w|+2}, c) \to^+_{\mathcal{R}_1} 0$ and $x \rhd q_0 w \lhd \to_{(2c)} 0q_0 w \lhd \to^+_{(1b)} 0$.

The remaining four types of critical pairs are generated by the two main rules (3a) and (3b) by sharing the c in two left-hand sides of these rules.

6. $(\rhd A^{|w|+2} v \rhd A^{|w|+2}, c) \to_{(3a)} \rhd A^{|w|+2} v \rhd q_0 w \lhd$ and $(\rhd A^{|w|+2} v \rhd A^{|w|+2}, c) \to_{(3a)} \rhd q_0 w \lhd v \rhd A^{|w|+2}$, where $v \in \Gamma^*$ is arbitrary: Since the words $\rhd A^{|w|+2} v \rhd q_0 w \lhd$ and $\rhd q_0 w \lhd v \rhd A^{|w|+2}$ both contain a factor of the form $x \rhd$ ($x \in \Gamma$), we can apply a rule in (2c) to both words. The resulting words can be both reduced to 0 with the rules in (1a) and (1b).

7. $(Bv \rhd A^{|w|+2}, c) \to_{(3a)} Bv \rhd q_0 w \lhd$ and $(Bv \rhd A^{|w|+2}, c) \to_{(3b)} 0v \rhd A^{|w|+2}$, where $v \in \Gamma^*$ is arbitrary: Again in the word $Bv \rhd q_0 w \lhd$ there exists a factor of the form $x \rhd$, hence this word can be reduced to 0. But of course the same also holds for the word $0v \rhd A^{|w|+2}$.

8. $(\rhd A^{|w|+2} vB, c) \to_{(3a)} \rhd q_0 w \lhd vB$ and $(\rhd A^{|w|+2} vB, c) \to_{(3b)} \rhd A^{|w|+2} v0$, where $v \in \Gamma^*$ is arbitrary: This is the main case, whose consideration will be postponed.

9. $(BvB, c) \to_{(3b)} 0vB$ and $(BvB, c) \to_{(3b)} Bv0$, where $v \in \Gamma^*$ is arbitrary: trivial

Other critical pairs, which are not confluent for trivial reasons, do not exist. In particular the rules in (5a) to (5d) do not generate further critical situations, since $\mathcal{M}$ is deterministic. Thus it remains to show that for all $v \in \Gamma^*$ the critical pair $(\rhd q_0 w \lhd vB, \rhd A^{|w|+2} v0)$ is confluent. Since $\rhd A^{|w|+2} v0 \to^+_{(1a)} 0$, it suffices to prove the following claim.

Claim. If $\mathcal{M}$ does not terminate on the input w then $\rhd q_0 w \lhd vB \to^+_{\mathcal{R}} 0$ for all $v \in \Gamma^*$.

Case 1. $v = \$^m$ for some $m \geq 0$: By simulating $\mathcal{M}$ long enough and thus consuming enough \$-symbols, we obtain

$$\rhd q_0 w \lhd \$^m B \to^*_{\mathcal{R}_{4,5}} \rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} \lhd \$^{i_{l+1}} B,$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $i_1, \ldots, i_{l+1} \in \{0, 1\}$. Thus

$$\begin{aligned} &\rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} \lhd \$^{i_{l+1}} B \to_{\{(2a),(2b)\}} \\ &\rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} 0 \to^+_{(1a)} 0. \end{aligned}$$

Case 2. $v = \$^m yv'$, where $m \geq 0, y \in \Gamma \backslash \{\$\}$, and $v' \in \Gamma^*$: We obtain

$$\rhd q_0 w \lhd \$^m yv'B \to^*_{\mathcal{R}_{4,5}} \rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} \lhd \$^{i_{l+1}} yv'B,$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $i_1, \ldots, i_{l+1} \in \{0, 1\}$. Since $y \in \Gamma \backslash \{\$\}$, it follows

$$\begin{aligned} &\rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} \lhd \$^{i_{l+1}} yv'B \to_{\{(2a),(2b)\}} \\ &\rhd uqa_1\$^{i_1} a_2 \$^{i_2} \cdots a_{l-1}\$^{i_{l-1}} a_l \$^{i_l} 0v'B \to^+_{\{(1a),(1b)\}} 0. \end{aligned}$$

Now we have considered all critical situations of $\mathcal{R}$ and the proof of the lemma is complete. ■

The previous lemma implies that $\mathrm{COLR}_{\neq 1}(\Gamma^* \times \{c\}^*)$ is undecidable. This result is sharpened by the following lemma. Furthermore this lemma solves an open question from [12], p 117, namely whether confluence for length-reducing trace rewriting system is already undecidable for independence alphabets with only three symbols.

LEMMA 3.4. $\mathrm{COLR}_{\neq 1}(\{a,b\}^* \times \{c\}^*)$ is undecidable.

Proof. Take $\mathcal{R}$ and Γ from the previous proof. We will make use of the coding function from Lemma 2.5. Let $\Gamma = \{b_1, \ldots, b_n\}$. Then define $\phi(b_i) = ab^iab^{2n+1-i}$ for $i \in \{1, \ldots, n\}$ and $\sigma(s,t) = (\phi(s), t)$ for $(s,t) \in \Gamma^* \times \{c\}^*$. But now we cannot immediately apply Lemma 2.4, since the fourth condition from this lemma is not fulfilled for the TRS $\sigma(\mathcal{R})$. For instance, it holds $(\phi(B)s\phi(B), c) \in \mathrm{CT}(\sigma(\mathcal{R}))$ for all $s \in \{a,b\}^*$, but if s is not contained in the image of ϕ, then also $(\phi(B)s\phi(B), c)$ is not contained in the image of σ. We solve this problem by introducing additional rules. Let $\mathcal{P}$ be the TRS that consists of the rules in $\sigma(\mathcal{R})$ plus the following rules, where $x \in \{a,b\}$ and $s \in \{a,b\}^{2n+3} \setminus \{\phi(x) \mid x \in \Gamma\}$ are arbitrary: [3]

(6a) $x\phi(0) \to \phi(0)$	(6c) $\phi(\triangleleft)s \to \phi(0)$	(6e) $x\phi(\triangleright) \to \phi(0)$
(6b) $\phi(0)x \to \phi(0)$	(6d) $\phi(\triangleleft\$)s \to \phi(0)$	

If a rule $d \in \mathcal{R}$ has the form $(\ell, c^i) \to r$ ($i \in \{0,1\}$) then we denote the corresponding rule $(\phi(\ell), c^i) \to \phi(r)$ of $\mathcal{P}$ by $\sigma(d)$. For instance $\sigma(3a)$ is the rule $(\phi(\triangleright A^{|w|+2}), c) \to \phi(\triangleright q_0 w \triangleleft)$. The rules in (6a), (6b), and (6e) correspond to the rules σ(1a), σ(1b), and σ(2c) from $\sigma(\mathcal{R})$. In fact the latter three rules are superfluous in $\mathcal{P}$ (but they do not lead to problems). Obviously the TRS $\mathcal{P}$ is length-reducing, and it holds $1 \notin \mathrm{ran}(\mathcal{P})$. Furthermore $\mathcal{P}$ satisfies condition (A). Now the lemma immediately follows from the following claim:

Claim. $\mathcal{P}$ is confluent if and only if the Turing-machine $\mathcal{M}$ does not terminate on the input w.

First we assume that the machine $\mathcal{M}$ terminates on the input w. By (4) from the proof of Lemma 3.3 there exists an $m > 0$ with

$$(\triangleright A^{|w|+2}\$^m B, c) \to_{\mathcal{R}}^{+} \triangleright u q_f a_1 \$^2 a_2 \$^2 \cdots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B = t,$$

where $u \in \Sigma^*, l \geq 0, a_1, \ldots, a_l \in \Sigma$, $k \geq 2$, and $t \in \mathrm{IRR}(\mathcal{R})$. An application of σ gives

$$(\phi(\triangleright A^{|w|+2}\$^m B), c) \to_{\mathcal{P}}^{+} \phi(\triangleright u q_f a_1 \$^2 a_2 \$^2 \cdots a_{l-1} \$^2 a_l \$^2 \triangleleft \$^k B) = \phi(t).$$

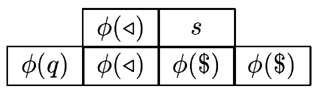
We claim that $\phi(t) \in \mathrm{IRR}(\mathcal{P})$. Irreducibility with respect to the rules from $\sigma(\mathcal{R}) \subset \mathcal{P}$ follows from the irreducibility of t with respect to $\mathcal{R}$ and statement (2) in Lemma

[3] Note that each word $\phi(x)$ for $x \in \Gamma$ has the same length $2n+3$.

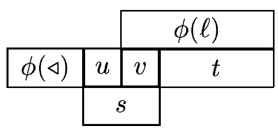
2.5. Furthermore, the word $\phi(t)$ is also irreducible with respect to the additional rules (6a) to (6e), since $\phi(t)$ does not contain a factor of the form $\phi(0)$, $x\phi(\triangleright)$, or $\phi(\triangleleft\$^i)s$, where $x \in \{a,b\}$, $i \in \{0,1\}$, and $s \in \{a,b\}^{2n+3}\backslash\{\phi(x) \mid x \in \Gamma\}$. This follows again from the statement (2) in Lemma 2.5. Since $(\triangleright A^{|w|+2}\$^m B, c) \to_{\mathcal{R}}^{+} 0$, i.e., $(\phi(\triangleright A^{|w|+2}\$^m B), c) \to_{\mathcal{P}}^{+} \phi(0)$, and since also $\phi(0) \in \mathrm{IRR}(\mathcal{P})$, the TRS $\mathcal{P}$ is not confluent.

Now we assume that the machine $\mathcal{M}$ does not terminate on the input w. Then by Lemma 3.3 the TRS $\mathcal{R}$ is confluent. Let us take an arbitrary critical situation $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{P})$ where $\boldsymbol{t}, \boldsymbol{t}_0, \boldsymbol{t}_1 \in \{a,b\}^* \times \{c\}^*$. Let d_0 and d_1 be the rules from $\mathcal{P}$ that generate this critical situation. The case that $d_0, d_1 \in \{\sigma(1a), \sigma(1b), \sigma(1c), \sigma(2a), \sigma(2b), \sigma(2c), (6a), \ldots, (6e)\}$ is clear, since in this case $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$ both contain $\phi(0)$ as a factor and therefore can be reduced to $\phi(0)$ with the rules in (6a) and (6b).

Next we consider the case that for instance $d_0 \in \{(6a), \ldots, (6e)\}$, whereas $d_1 \in \{\sigma(3a), \sigma(3b), \sigma(4a), \sigma(4b), \sigma(5a), \ldots, \sigma(5e)\}$. We only consider the case that d_0 is the rule (6c), the other cases can be dealt similarly. Let d_1 be of the form $(\phi(\ell), c^i) \to \phi(r)$, where $\ell, r \in \Gamma^+$ and $i \in \{0,1\}$. We have to consider all possible overlappings between $\phi(\ell)$ and $\phi(\triangleleft)s$, where $|s| = 2n+3$ and $s \notin \{\phi(x) \mid x \in \Gamma\}$. Note that $\phi(\triangleleft)$ is not a suffix of $\phi(\ell)$. Therefore the case that the prefix $\phi(\triangleleft)$ of $\phi(\triangleleft)s$ can be matched with an occurrence of $\phi(\triangleleft)$ in $\phi(\ell)$ cannot occur, because otherwise s would be a word of the form $\phi(x)$ for some $x \in \Gamma$, see the following picture for the case $\ell = q \triangleleft \$\$$ (i.e., d_1 is the rule $\sigma(5a)$):



Therefore because of Lemma 2.5 the only possible overlappings between $\phi(\triangleleft)s$ and $\phi(\ell)$ are of the form $\phi(\triangleleft)u\phi(\ell)$, where $s = uv$, $\phi(\ell) = v\ell'$, and $u \neq 1 \neq v$, see also the following picture:



We obtain the critical pair $((\phi(0)t, c^i), (\phi(\triangleleft)u\phi(r), 1))$. On the one hand it holds $(\phi(0)t, c^i) \to_{\{\sigma(1c),(6b)\}}^{+} \phi(0)$. Furthermore Lemma 2.5(3) and $0 < |u| < 2n+3$ imply that the prefix of $u\phi(r)$ of length $2n+3$ (which exists, since $r \neq 1$) does not belong to $\{\phi(x) \mid x \in \Gamma\}$, because otherwise the prefix of $\phi(r)$ of length $2n+3-|u|$ would belong to the image of ϕ. This implies also $\phi(\triangleleft)u\phi(r) \to_{(6c)} \to_{(6b)}^{*} \phi(0)$.

Finally we consider the case that the situation $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1)$ is generated by two rules $d_0, d_1 \in \sigma(\mathcal{R})$. Let d_i be of the form $(\phi(\ell_i), c^{k_i}) \to \phi(r_i)$, where $k_i \in \{0,1\}$. Let $\boldsymbol{t} = (t, c^k)$ for some $k \geq 0$. If t is an overlapping of $\phi(\ell_0)$ and $\phi(\ell_1)$ then Lemma 2.5 implies that this overlapping results from an overlapping of ℓ_0 and ℓ_1 and thus $t = \phi(s)$ for some $s \in \Gamma^*$. Thus there exist $\boldsymbol{s}, \boldsymbol{s}_0, \boldsymbol{s}_1$ with $\boldsymbol{t} = \sigma(\boldsymbol{s})$, $\boldsymbol{t}_0 = \sigma(\boldsymbol{s}_0)$, $\boldsymbol{t}_1 = \sigma(\boldsymbol{s}_1)$, and $\boldsymbol{s} \to_{\mathcal{R}} \boldsymbol{s}_i$ for $i \in \{1,2\}$. Since $\mathcal{R}$ is confluent, there exists a $\boldsymbol{u}$ with $\boldsymbol{s}_0 \to_{\mathcal{R}}^{*} \boldsymbol{u}$ and $\boldsymbol{s}_1 \to_{\mathcal{R}}^{*} \boldsymbol{u}$. An application of σ gives $\boldsymbol{t}_0 = \sigma(\boldsymbol{s}_0) \to_{\mathcal{P}}^{*} \sigma(\boldsymbol{u})$ and $\boldsymbol{t}_1 = \sigma(\boldsymbol{s}_1) \to_{\mathcal{P}}^{*} \sigma(\boldsymbol{u})$. Hence it suffices to consider the case that $k_0 = k_1 = k = 1$ and t is disjointly generated by $\phi(\ell_0)$ and $\phi(\ell_1)$. The case $\sigma(1c) \in \{d_0, d_1\}$ is clear, since in this case both $\boldsymbol{t}_0$ and $\boldsymbol{t}_1$ contain $\phi(0)$ in its first component. Thus only the case $d_0, d_1 \in \{\sigma(3\text{a}), \sigma(3\text{b})\}$ remains. The resulting critical situations correspond to the cases (6) to (9) in the proof of Lemma 3.3. The cases that correspond to (6), (7), or (9) can be dealt completely analogously to the corresponding cases from the proof of Lemma 3.3 by applying the rules in (6a), (6b), and (6e). Thus the only remaining case is the critical pair $(\phi(\triangleright q_0 w \triangleleft) v \phi(B),\ \phi(\triangleright A^{|w|+2}) v \phi(0))$, where $v \in \{a,b\}^*$ is arbitrary. The case that $v = \phi(u)$ for some word $u \in \Gamma^*$ is clear, since then $t_0 = \phi(s_0)$ and $t_1 = \phi(s_1)$ for a $(s_0, s_1) \in \text{CP}(\mathcal{R})$. Thus we can assume that v is not contained in the image of ϕ. Let $v = \phi(v')s$ for some $v' \in \Gamma^*$ and some $s \in \{a,b\}^+$ such that s does not have a prefix of the form $\phi(x)$ with $x \in \Gamma$. Since $\phi(\triangleright A^{|w|+2}) v \phi(0) \rightarrow^+_{(6a)} \phi(0)$, we have to show that also $\phi(\triangleright q_0 w \triangleleft v') s \phi(B) \rightarrow^*_{\mathcal{P}} \phi(0)$. If $v' = \$^m y v''$ for some $m \geq 0$, $y \in \Gamma \backslash \{\$\}$, and $v'' \in \Gamma^*$ then we can use the arguments from case 2 at the end of the proof of Lemma 3.3. Thus it suffices to consider all words of the form $\phi(\triangleright q_0 w \triangleleft \$^m) s \phi(B)$ with $m \geq 0$. By simulating the machine $\mathcal{M}$ long enough, we obtain

$$\phi(\triangleright q_0 w \triangleleft \$^m) s \phi(B) \rightarrow^*_{\mathcal{P}} \phi(\triangleright u q a_1 \$^{i_1} a_2 \$^{i_2} \cdots a_{l-1} \$^{i_{l-1}} a_l \$^{i_l} \triangleleft \$^{i_{l+1}}) s \phi(B),$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $i_1, \ldots, i_{l+1} \in \{0,1\}$. Since $s \neq 1$ does not have a prefix of the form $\phi(x)$ for some $x \in \Gamma$, the prefix of $s\phi(B)$ of length $2n+3$ does not belong to $\{\phi(x) \mid x \in \Gamma\}$. Thus we can apply a rule in (6c) or (6d), which produces the factor $\phi(0)$. The resulting word can be reduced to $\phi(0)$ with the rules in (6a) and (6b). ■

3.1.2. *The case* $a - c \quad b$

In this section we will deal with independence alphabets of the form

$$(\Sigma_n, I_n) = (\{a, c, b_1, \ldots, b_n\}, \{(a,c), (c,a)\}),$$

where $n > 0$. The goal of this section is to show that $\text{COLR}_{\neq 1}(\mathbb{M}(\Sigma_1, I_1))$ is undecidable. This result will we proven in three steps. First we will show that a stronger version of $\text{COLR}_{\neq 1}(\mathbb{M}(\Sigma_n, I_n))$ for a particular $n > 2$ is undecidable (Lemma 3.6). In a second step we will code the $n > 2$ many symbols $b_1, \ldots, b_n$ into the two symbols b_1 and b_2 (Lemma 3.7) and thus prove the undecidability of a stronger version of $\text{COLR}_{\neq 1}(\mathbb{M}(\Sigma_2, I_2))$. The coding we are going to use for this step is the morphism from Lemma 2.5 if we set $m = 2$, $a_1 = a$, and $a_2 = c$ in this lemma. Finally in the last step we will code the two symbols b_1 and b_2 into b_1 via the morphism defined by $b_1 \mapsto b_1 b_1 a b_1$, $b_2 \mapsto b_1 b_1 c b_1$, $a \mapsto a$, and $c \mapsto c$ (Lemma 3.8). For this last step it will be important that in the two previous steps we considered stronger versions of $\text{COLR}_{\neq 1}(\mathbb{M}(\Sigma_n, I_n))$ and $\text{COLR}_{\neq 1}(\mathbb{M}(\Sigma_2, I_2))$, respectively. The splitting of the whole coding into two steps makes the proof more comprehensible. In both steps we will use Lemma 2.4 as well as the next Lemma 3.5, which applies to trace rewriting systems that fulfill the following property (B).

A TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma_n, I_n)$ $(n > 0)$ satisfies condition (B), if for all $\boldsymbol{\ell} \in \text{dom}(\mathcal{R})$ it holds: $\{b_1, \ldots, b_n\} \cap \text{alph}(\boldsymbol{\ell}) \neq \emptyset$.

It is easy to see that a TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma_n, I_n)$, which satisfies condition (B), also satisfies condition (A): (i) There cannot exist an $\boldsymbol{\ell} \in \text{dom}(\mathcal{R})$ and a symbol $b \in \Sigma_n$

with $b\,I\,\boldsymbol{\ell}$. Thus $\mathcal{R}$ fulfills condition (A1). (ii) For all $\boldsymbol{\ell}_0, \boldsymbol{\ell}_1 \in \mathrm{dom}(\mathcal{R})$ it is not possible that there exist factorizations $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{q}_0$, $\boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{q}_1$ with $\boldsymbol{p}_i \neq 1 \neq \boldsymbol{q}_i$ for $i \in \{0,1\}$ and $\boldsymbol{p}_0\, I\, \boldsymbol{p}_1$, $\boldsymbol{q}_0\, I\, \boldsymbol{q}_1$, because either $\boldsymbol{p}_0$ or $\boldsymbol{q}_0$ must contain a symbol from $\{b_1, \dots, b_n\}$, which is therefore dependent from all other symbols. In the following let $\overline{a} = c$ and $\overline{c} = a$. Thus for $x \in \{a, c\}$ it holds $x\, I_n\, \overline{x}$.

LEMMA 3.5. Let $n > 0$. Let $\mathcal{R}$ be a TRS over $\mathbb{M}(\Sigma_n, I_n)$, which satisfies condition (B). If $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})$ then there exist rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$, natural numbers $\alpha, \beta, \gamma, \zeta > 0$, traces $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, $\boldsymbol{s}$ with $\boldsymbol{s} \neq 1$, and $x, y \in \{a, c\}$ such that one of the following six cases holds:

	$\boldsymbol{\ell}_0 =$	$\boldsymbol{\ell}_1 =$	$\boldsymbol{t} =$	$\boldsymbol{t}_0 =$	$\boldsymbol{t}_1 =$
(1)	$x^\alpha \boldsymbol{s} y^\gamma$	$\overline{x}^\beta \boldsymbol{s} \overline{y}^\zeta$	$\overline{x}^\beta x^\alpha \boldsymbol{s} y^\gamma \overline{y}^\zeta = x^\alpha \overline{x}^\beta \boldsymbol{s} \overline{y}^\zeta y^\gamma$	$\overline{x}^\beta \boldsymbol{r}_0 \overline{y}^\zeta$	$\boldsymbol{t}_1 = x^\alpha \boldsymbol{r}_1 y^\gamma$
(2)	$x^\alpha \boldsymbol{s}$	$\overline{x}^\beta \boldsymbol{s} \boldsymbol{q}_1$	$\overline{x}^\beta x^\alpha \boldsymbol{s} \boldsymbol{q}_1 = x^\alpha \overline{x}^\beta \boldsymbol{s} \boldsymbol{q}_1$	$\overline{x}^\beta \boldsymbol{r}_0 \boldsymbol{q}_1$	$x^\alpha \boldsymbol{r}_1$
(3)	$\boldsymbol{s} x^\alpha$	$\boldsymbol{p}_1 \boldsymbol{s} \overline{x}^\beta$	$\boldsymbol{p}_1 \boldsymbol{s} x^\alpha \overline{x}^\beta = \boldsymbol{p}_1 \boldsymbol{s} \overline{x}^\beta x^\alpha$	$\boldsymbol{p}_1 \boldsymbol{r}_0 \overline{x}^\beta$	$x^\alpha \boldsymbol{r}_1$
(4)	$\boldsymbol{s} \boldsymbol{q}_0$	$\boldsymbol{p}_1 \boldsymbol{s}$	$\boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_0$	$\boldsymbol{p}_1 \boldsymbol{r}_0$	$\boldsymbol{r}_1 \boldsymbol{q}_0$
(5)	$\boldsymbol{s}$	$\boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1$	$\boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1$	$\boldsymbol{p}_1 \boldsymbol{r}_0 \boldsymbol{q}_1$	$\boldsymbol{r}_1 \boldsymbol{q}_0$
(6)	$\boldsymbol{p}_0 x^\alpha$	$x^\alpha \boldsymbol{q}_1$	$\boldsymbol{p}_0 x^\alpha \overline{x}^\beta \boldsymbol{q}_1 = \boldsymbol{p}_0 \overline{x}^\beta x^\alpha \boldsymbol{q}_1$	$\boldsymbol{r}_0 \overline{x}^\beta \boldsymbol{q}_1$	$\boldsymbol{p}_0 \overline{x}^\beta \boldsymbol{r}_1$

Note that the cases (4) and (6) do not exclude each other. Furthermore if one of the six cases above holds then $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_i$ for $i \in \{0,1\}$.

Proof. Let $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})$. Thus there exist rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and traces $\boldsymbol{p}_i$, $\boldsymbol{q}_i$, $\boldsymbol{w}_i$ $(i \in \{0,1\})$, $\boldsymbol{s} \neq 1$ such that:

- $\boldsymbol{\ell}_0 = \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0, \quad \boldsymbol{\ell}_1 = \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1$
- $\boldsymbol{p}_0\, I_n\, \boldsymbol{p}_1, \quad \boldsymbol{q}_0\, I_n\, \boldsymbol{q}_1, \quad \boldsymbol{w}_0\, I_n\, \boldsymbol{w}_1, \quad \boldsymbol{s}\, I_n\, \boldsymbol{w}_0\boldsymbol{w}_1, \quad \boldsymbol{w}_0\, I_n\, \boldsymbol{p}_1\boldsymbol{q}_0, \quad \boldsymbol{w}_1\, I_n\, \boldsymbol{p}_0\boldsymbol{q}_1$
- $\boldsymbol{t} = \boldsymbol{p}_1 \boldsymbol{w}_1 \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0 \boldsymbol{w}_0 \boldsymbol{q}_1 = \boldsymbol{p}_0 \boldsymbol{w}_0 \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1 \boldsymbol{w}_1 \boldsymbol{q}_0,$
- $\boldsymbol{t}_0 = \boldsymbol{p}_1 \boldsymbol{w}_1 \boldsymbol{r}_0 \boldsymbol{w}_0 \boldsymbol{q}_1, \quad \boldsymbol{t}_1 = \boldsymbol{p}_0 \boldsymbol{w}_0 \boldsymbol{r}_1 \boldsymbol{w}_1 \boldsymbol{q}_0$

Now we can separate the following cases:

Case 1. $\boldsymbol{w}_0 = 1 = \boldsymbol{w}_1$: Then $\boldsymbol{t} = \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{q}_0$, $\boldsymbol{t}_0 = \boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1$, and $\boldsymbol{t}_1 = \boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0$.

Case 1.1. $\boldsymbol{p}_0 \neq 1 \neq \boldsymbol{p}_1$: Because of $\boldsymbol{p}_0\, I_n\, \boldsymbol{p}_1$ there exist $x \in \{a, c\}$ and $\alpha, \beta > 0$ such that $\boldsymbol{p}_0 = x^\alpha$ and $\boldsymbol{p}_1 = \overline{x}^\beta$. *Case 1.1.1* $\boldsymbol{q}_0 \neq 1 \neq \boldsymbol{q}_1$: Then $\boldsymbol{q}_0 = y^\gamma$ and $\boldsymbol{q}_1 = \overline{y}^\zeta$ for some $y \in \{a, c\}$ and $\gamma, \zeta > 0$, and we obtain type (1) from the lemma.

Case 1.1.2. $\boldsymbol{q}_0 = 1$: We obtain type (2). The case $\boldsymbol{q}_1 = 1$ is symmetric.

Case 1.2. $\boldsymbol{p}_0 = 1$ (The case $\boldsymbol{p}_1 = 1$ is symmetric.):

Case 1.2.1. $\boldsymbol{q}_0 \neq 1 \neq \boldsymbol{q}_1$: It follows $\boldsymbol{q}_0 = x^\alpha$ and $\boldsymbol{q}_1 = \overline{x}^\beta$ for some $x \in \{a, c\}$ and $\alpha, \beta > 0$. We obtain type (3).

Case 1.2.2. $\boldsymbol{q}_1 = 1$: We obtain type (4).

Case 1.2.3. $\boldsymbol{q}_0 = 1$: We obtain type (5).

Case 2. $\boldsymbol{w}_0 \neq 1$: Because of $\boldsymbol{s} \neq 1$ and $\boldsymbol{s}\, I_n\, \boldsymbol{w}_0$, there exist $x \in \{a, c\}$ and $\alpha, \beta > 0$ such that $\boldsymbol{s} = x^\alpha$ and $\boldsymbol{w}_0 = \overline{x}^\beta$. But then $\boldsymbol{s}\boldsymbol{w}_0\, I_n\, \boldsymbol{w}_1$ implies $\boldsymbol{w}_1 = 1$. We claim that also $\boldsymbol{p}_1 = \boldsymbol{q}_0 = 1$. Assume that $\boldsymbol{p}_1 \neq 1$. From $\boldsymbol{w}_0\, I_n\, \boldsymbol{p}_1$ and $\boldsymbol{w}_0 = \overline{x}^\beta$ it follows $\boldsymbol{p}_1 = x^\gamma$ for some $\gamma > 0$. Thus $\boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1 = x^\gamma x^\alpha \boldsymbol{q}_1$. But because of condition (B), $\boldsymbol{\ell}_1$ must contain a symbol from $\{b_1, \dots, b_n\}$. Therefore $\boldsymbol{q}_1 \neq 1$. Similarly, because of $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0 = \boldsymbol{p}_0 x^\alpha \boldsymbol{q}_0$ either $\boldsymbol{p}_0$ or $\boldsymbol{q}_0$ must contain a symbol

(1) $x0\$ \to 0\$$ for $x \in \Gamma$	(2) $\lhd \$^k C\$ \to 0\$$ for $k \in \{0, \ldots, \omega - 1\}$
(3a) $A^{\lvert w \rvert +3} B \to \rhd q_0 w \lhd$ (3b) $BC\$ \to 0\$$	(4) $a\$^k \beta \$^\omega \to a\$^{k+1}\beta$ for $k \in \{0, \ldots, \omega - 1\}$, $a \in \Sigma$, and $\beta \in \Sigma \cup \{\lhd\}$
(5a) $q \lhd \$^\omega \to a'p\lhd$ if $\delta(q, \Box) = (p, a', R)$ (5b) $bq \lhd \$^\omega \to pba'\lhd$ if $\delta(q, \Box) = (p, a', L)$, $b \in \Sigma$ (5c) $qa\$^\omega \to a'p$ if $\delta(q, a) = (p, a', R)$ (5d) $bqa\$^\omega \to pba'$ if $\delta(q, a) = (p, a', L)$, $b \in \Sigma$	

FIG. 2. The TRS $\mathcal{R}$ from Lemma 3.6.

from $\{b_1, \ldots, b_n\}$. The first possibility contradicts $\boldsymbol{p}_0 \, I_n \, \boldsymbol{p}_1$ and $\boldsymbol{p}_1 \neq 1$, whereas the second possibility contradicts $\boldsymbol{q}_0 \, I_n \, \boldsymbol{q}_1$ and $\boldsymbol{q}_1 \neq 1$. Hence $\boldsymbol{p}_1 = 1$. Analogously we can prove that $\boldsymbol{q}_0 = 1$. Thus we obtain type (6). The case $\boldsymbol{w}_1 \neq 1$ is symmetric. ■

Next we will show that a stronger version of $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma_n, I_n))$ is undecidable for a particular $n > 2$. For this let $\mathcal{M} = (Q, \Sigma, \Box, \delta, q_0, q_f)$ be the deterministic universal Turing-machine from the last section, and let $w \in (\Sigma \backslash \{\Box\})^*$ be an input for $\mathcal{M}$. Let $\Gamma = Q \cup \Sigma \cup \{0, \rhd, \lhd, A, B, C, \$\}$. We define an independence relation $I \subseteq \Gamma \times \Gamma$ by $I = \{(\$, B), (B, \$)\})$. Note that the independence alphabet (Γ, I) is of the form (Σ_n, I_n) for an $n > 2$. We define the TRS $\mathcal{R}$ over $\mathbb{M}(\Gamma, I)$ by the rules in Figure 2, where the exact value of the constant $\omega \geq 2$ will be fixed later. The following Lemma 3.6 holds for every $\omega \geq 2$. Obviously the following properties hold:

- $\mathcal{R}$ satisfies condition (B).
- $\mathcal{R}$ is length-reducing.
- For all $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ it holds $\max(\boldsymbol{\ell}) \subseteq \{B, \$\}$ and $\boldsymbol{r} \neq 1$.

LEMMA 3.6. *$\mathcal{R}$ is confluent if and only if $\mathcal{M}$ does not terminate on the input w.*

Proof. The proof is similar to the proof of Lemma 3.3. Let $\mathcal{R}_{4,5}$ be the TRS over $\mathbb{M}(\Gamma, I)$ that consists of the rules in (4) and (5a) to (5d). Assume that the machine $\mathcal{M}$ terminates on the input w. Then there exist $m \geq 0$, $u \in \Sigma^*$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $k \geq \omega$ such that

$$[A^{|w|+3} B \$^m C \$]_I \to_{(3a)} \rhd q_0 w \lhd \$^m C\$ \to^+_{\mathcal{R}_{4,5}} \rhd u q_f a_1 \$^\omega a_2 \$^\omega \cdots a_{l-1} \$^\omega a_l \$^\omega \lhd \$^k C\$ = \boldsymbol{v}.$$

Since $\mathcal{M}$ cannot move out of the final state q_f, none of the rules in (5a) to (5d) can be applied to the trace $\boldsymbol{v}$. Furthermore, since $k \geq \omega$, rules in (2) cannot be applied to $\boldsymbol{v}$. Finally, since also the rules in (1), (3a), (3b), and (4) cannot be applied to $\boldsymbol{v}$, we have $\boldsymbol{v} \in \mathrm{IRR}(\mathcal{R})$. On the other hand, since also $[A^{|w|+3} B \$^m C \$]_I = [A^{|w|+3} \$^m BC\$]_I \to_{(3b)} A^{|w|+3} \$^m 0\$ \to^+_{(1)} 0\$ \in \mathrm{IRR}(\mathcal{R})$, the TRS $\mathcal{R}$ is not confluent.

Now we assume that the machine $\mathcal{M}$ does not terminate on the input w. We will to show that $\mathcal{R}$ is confluent. Since $\mathcal{R}$ satisfies condition (B) and therefore also condition (A), it is sufficient to consider all critical pairs of $\mathcal{R}$. Let $(\boldsymbol{t}_0, \boldsymbol{t}, \boldsymbol{t}_1) \in \mathrm{CS}(\mathcal{R})$ be a critical situation. Lemma 3.5 implies that $\boldsymbol{t}$, $\boldsymbol{t}_0$, and $\boldsymbol{t}_1$ satisfy one of the six cases, listed in this lemma. This leads to the following three types of critical situations:

A critical situation of type (2) (according to the table in Lemma 3.5) results from $[B\$0\$]_I \to_{(1)} B0\$$ and $[B\$0\$]_I = [\$B0\$]_I \to_{(1)} \$0\$$. The resulting critical pair $(B0\$, \$0\$)$ is confluent by the rule in (1).

A critical situation of type (6) (according to the table in Lemma 3.5) results as follows: Let $d = ([\ell x]_I \to [r]_I)$ be an arbitrary rule from $\mathcal{R}$, where $x \in \{B, \$\}$. Then for all $m \geq 0$ we have $[\ell \overline{x}^m x 0\$]_I \to_{(1)} [\ell \overline{x}^m 0\$]_I$ and $[\ell \overline{x}^m x 0\$]_I = [\ell x \overline{x}^m 0\$]_I \to_d [r \overline{x}^m 0\$]_I$. The resulting critical pair $([\ell \overline{x}^m 0\$]_I, [r \overline{x}^m 0\$]_I)$ is confluent by the rules in (1).

The last type of possible critical situations is again of type (6). It is generated by the two main rules (3a) and (3b):

$$[A^{|w|+3} B \$^m C\$]_I \to_{(3a)} \triangleright q_0 w \triangleleft \$^m C\$ \quad \text{and}$$
$$[A^{|w|+3} B \$^m C\$]_I = [A^{|w|+3} \$^m B C\$]_I \to_{(3b)} A^{|w|+3} \$^m 0\$.$$

Since $A^{|w|+3}\$^m 0\$ \to_{(1)}^{+} 0\$$ we have to show that $\triangleright q_0 w \triangleleft \$^m C\$ \to_{\mathcal{R}}^{+} 0\$$ for all $m \geq 0$. Since the machine $\mathcal{M}$ does not terminate on the input w, we obtain

$$\triangleright q_0 w \triangleleft \$^m C\$ \to_{\mathcal{R}_{4,5}}^{*} \triangleright u q a_1 \$^{i_1} a_2 \$^{i_2} \cdots a_{l-1} \$^{i_{l-1}} a_l \$^{i_l} \triangleleft \$^{i_{l+1}} C\$,$$

where $u \in \Sigma^*$, $q \in Q$, $l \geq 0$, $a_1, \ldots, a_l \in \Sigma$, and $i_1, \ldots, i_{l+1} \in \{0, \ldots, \omega - 1\}$. Furthermore

$$\triangleright u q a_1 \$^{i_1} a_2 \$^{i_2} \cdots a_{l-1} \$^{i_{l-1}} a_l \$^{i_l} \triangleleft \$^{i_{l+1}} C\$ \to_{(2)} \triangleright u q a_1 \$^{i_1} a_2 \$^{i_2} \cdots a_{l-1} \$^{i_{l-1}} a_l \$^{i_l} 0\$ \to_{(1)}^{+} 0\$.$$

These are all possible critical situations. In particular the rules in (5a) to (5d) do not generate further critical situations, since $\mathcal{M}$ is deterministic. ∎

In the following let $Q \cup \Sigma \cup \{0, \triangleright, \triangleleft, A, C\} = \{b_1, \ldots, b_n\}$ and $\{B, \$\} = \{a, c\}$. Then $\mathcal{R}$ is a TRS over $\mathbb{M}(\Sigma_n, I_n)$. Let $\phi : \Sigma_n^* \longrightarrow \Sigma_2^*$ be defined by

$$\phi(a) = a, \quad \phi(c) = c, \quad \phi(b_i) = b_1 b_2^i b_1 b_2^{2n+1-i} \text{ for } i \in \{1, \ldots, n\}.$$

This is the morphism from Lemma 2.5 if we set $m = 2$, $a_1 = a$, and $a_2 = c$ in Lemma 2.5. Thus ϕ is injective. Since $x \, I_n \, y$ implies $\phi(x) \, I_2 \, \phi(y)$, ϕ can be extended to a monoid morphism $\sigma : \mathbb{M}(\Sigma_n, I_n) \longrightarrow \mathbb{M}(\Sigma_2, I_2)$ by $\sigma([s]_{I_n}) = [\phi(s)]_{I_2}$. Obviously the following facts hold for the TRS $\sigma(\mathcal{R})$ over $\mathbb{M}(\Sigma_2, I_2)$:

- $\sigma(\mathcal{R})$ satisfies condition (B).
- For all $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ it holds $\max(\sigma(\boldsymbol{\ell})) \subseteq \{a, c\}$ and $\sigma(\boldsymbol{r}) \neq 1$.

Furthermore if we set $\omega > 2n + 3$ for the value of the constant ω in $\mathcal{R}$ then also $\sigma(\mathcal{R})$ is length-reducing.

Lemma 3.7. $\sigma(\mathcal{R})$ is confluent if and only if $\mathcal{M}$ does not terminate on w.

This lemma immediately implies the undecidability of COLR($\mathbb{M}(\Sigma_2, I_2)$). A weaker form of this lemma is also stated in [3].

Proof. Because of Lemma 3.6 it suffices to show the following claim.

Claim. $\mathcal{R}$ is confluent if and only if $\sigma(\mathcal{R})$ is confluent.

It suffices to show that $\mathcal{R}$ and σ satisfy the four conditions from Lemma 2.4. The injectivity of σ follows immediately from the following fact together with the injectivity of ϕ.

$$\text{If } \phi(s) \equiv_{I_2} s' \text{ then there exists a } t \in \Sigma_n^* \text{ with } t \equiv_{I_n} s \text{ and } \phi(t) = s'. \tag{5}$$

This fact can be shown easily by an induction on the number of commutations that are necessary to transform the word $\phi(s)$ into the word s'. We already noted that $\sigma(\mathcal{R})$ is length-reducing and satisfies condition (B) and hence also condition (A). Thus also the second condition from Lemma 2.4 is satisfied. In order to show the third condition we will prove the following more general statement for all $u', v' \in \Sigma_2^*$, and $s, \ell \in \Sigma_n^+$:

$$\text{If } \phi(s) \equiv_{I_2} u'\phi(\ell)v' \text{ then } u' = \phi(u) \text{ and } v' = \phi(v) \text{ for some } u, v \in \Sigma_n^*. \tag{6}$$

Note that (6) does not hold for $\ell = 1$. So assume that $\phi(s) \equiv_{I_2} u'\phi(\ell)v'$ and $\ell \neq 1$. By possibly replacing s by another word that represents the same trace, we can assume by (5) that $\phi(s) = u'\phi(\ell)v'$. Now Lemma 2.5 implies the existence of $u, v \in \Sigma_n^*$ with $u' = \phi(u)$ and $v' = \phi(v)$.

Finally we have to prove the fourth condition from Lemma 2.4. Assume that $\boldsymbol{t}' \in \mathrm{CT}(\sigma(\mathcal{R}))$. We have to show that there exists a $\boldsymbol{t} \in \mathbb{M}(\Sigma_n, I_n)$ with $\sigma(\boldsymbol{t}) = \boldsymbol{t}'$. Since $\sigma(\mathcal{R})$ satisfies condition (B), it suffices to consider all six cases for $\boldsymbol{t}'$ that are enumerated in Lemma 3.5. The first three cases and case (5) are easy to check, since for these types it holds $\boldsymbol{t}' = x^\alpha \sigma(\boldsymbol{\ell}) y^\beta = \sigma(x^\alpha \boldsymbol{\ell} y^\beta)$ for some $x, y \in \{a, c\}$, $\boldsymbol{\ell} \in \mathrm{dom}(\mathcal{R})$, and $\alpha, \beta \geq 0$. Now let $\boldsymbol{t}'$ be of type (4), i.e., $\boldsymbol{t}' = \boldsymbol{p}'\boldsymbol{s}'\boldsymbol{q}'$, where $\sigma(\boldsymbol{\ell}_0) = \boldsymbol{p}'\boldsymbol{s}'$ and $\sigma(\boldsymbol{\ell}_1) = \boldsymbol{s}'\boldsymbol{q}'$. Let $\boldsymbol{\ell}_i = [\ell_i]_{I_n}$, $\boldsymbol{s}' = [s']_{I_2}$, $\boldsymbol{p}' = [p']_{I_2}$, and $\boldsymbol{q}' = [q']_{I_2}$, i.e., $\phi(\ell_0) \equiv_{I_2} p's'$, $\phi(\ell_1) \equiv_{I_2} s'q'$. Because of (5) we can assume that $\phi(\ell_0) = p's'$ and $\phi(\ell_1) = s'q'$. From statement (3) in Lemma 2.5 it follows that there exist $s, p, q \in \Sigma_n^*$ with $\phi(s) = s'$, $\phi(p) = p'$, and $\phi(q) = q'$. Thus $\boldsymbol{t}' = \boldsymbol{p}'\boldsymbol{s}'\boldsymbol{q}' = [p's'q']_{I_2} = \sigma([psq]_{I_n})$.

Finally we have to consider the case that $\boldsymbol{t}'$ is of type (6), i.e., $\boldsymbol{t}' = \boldsymbol{p}'x^\alpha\overline{x}^\beta\boldsymbol{q}'$, where $x \in \{a, c\}$, $\sigma(\boldsymbol{\ell}_0) = \boldsymbol{p}'x^\alpha$, and $\sigma(\boldsymbol{\ell}_1) = x^\alpha\boldsymbol{q}'$. Let $\boldsymbol{\ell}_i = [\ell_i]_{I_n}$, $\boldsymbol{p}' = [p']_{I_2}$, and $\boldsymbol{q}' = [q']_{I_2}$, i.e., $\phi(\ell_0) \equiv_{I_2} p'x^\alpha$ and $\phi(\ell_1) \equiv_{I_2} x^\alpha q'$. Because of (5) we can assume that $\phi(\ell_0) = p'x^\alpha$ and $\phi(\ell_1) = x^\alpha q'$. From $x \in \{a, c\}$, i.e., $\phi(x) = x$, it follows that $p' = \phi(p)$ and $q' = \phi(q)$ for some $p, q \in \Sigma_n^*$ and hence $\boldsymbol{t}' = [p'x^\alpha\overline{x}^\beta q']_{I_2} = [\phi(p)x^\alpha\overline{x}^\beta\phi(q)]_{I_2} = \sigma([px^\alpha\overline{x}^\beta q]_{I_n})$. ■

Now we can proof the following main lemma of this section:

Lemma 3.8. $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma_1, I_1))$ is undecidable.

Proof. In the proof we will use a coding function, which is similar to the coding function from Lemma 2.5. In the following we denote the symbol $b_1 \in \Sigma_1$ by b. The TRS $\sigma(\mathcal{R})$ over $\mathbb{M}(\Sigma_2, I_2)$ from Lemma 3.7 will be denoted by $\mathcal{P}$. We define an injective monoid morphism $\varphi : \Sigma_2^* \longrightarrow \Sigma_1^*$ by

$$\varphi(a) = a, \quad \varphi(c) = c, \quad \varphi(b_1) = bbab, \quad \varphi(b_2) = bbcb.$$

The injectivity of φ is obvious. More general the following cancellation property can be proven by an induction on $|t|$:

$$\text{If } \varphi(s) = \varphi(t)u \text{ (respectively } \varphi(s) = u\varphi(t)) \text{ then } s = tv \text{ (respectively } s = vt) \text{ and } u = \varphi(v) \text{ for some } v \in \Sigma_2^*. \quad (7)$$

Since $x \, I_2 \, y$ implies $\varphi(x) \, I_1 \, \varphi(y)$, we can extend φ to a monoid morphism $\tau : \mathbb{M}(\Sigma_2, I_2) \longrightarrow \mathbb{M}(\Sigma_1, I_1)$ by $\tau([s]_{I_2}) = [\varphi(s)]_{I_1}$. Obviously also for the TRS $\tau(\mathcal{P})$ over $\mathbb{M}(\Sigma_1, I_1)$ the following properties hold:

- $\tau(\mathcal{P})$ satisfies condition (B).
- For all $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{P}$ it holds $\max(\tau(\boldsymbol{\ell})) \subseteq \{a, c\}$ and $\tau(\boldsymbol{r}) \neq 1$.

Furthermore if we set $\omega > 4(2n + 3)$ for the value of the constant ω in the TRS $\mathcal{R}$ from Lemma 3.6 then also the TRS $\tau(\mathcal{P}) = \tau(\sigma(\mathcal{R}))$ is length-reducing. Because of Lemma 3.7 it suffices to show the following claim.

Claim. $\mathcal{P}$ is confluent if and only if $\tau(\mathcal{P})$ is confluent.

For the proof of this claim we will use Lemma 2.4. Thus we have to show the four conditions from Lemma 2.4 (for $\tau(\mathcal{P})$ instead of $\sigma(\mathcal{R})$). The injectivity of τ follows from the injectivity of φ and the following fact, which can be proven by an induction on the number of commutations that are necessary to transform the word $\varphi(s)$ into the word s'.

$$\text{If } \varphi(s) \equiv_{I_1} s' \text{ then there exists a } t \in \Sigma_2^* \text{ with } t \equiv_{I_2} s \text{ and } \varphi(t) = s'. \quad (8)$$

As already mentioned, $\tau(\mathcal{P})$ is length-reducing and satisfies condition (A). Instead of the third condition from Lemma 2.4 we prove the following more general statement for all $s, \ell \in \Sigma_2^*$ and $s_1, s_2 \in \Sigma_1^*$:

$$\text{If } |\varphi(\ell)| \geq 2 \text{ and } \varphi(s) \equiv_{I_1} s_1\varphi(\ell)s_2 \text{ then there exist } u_1, u_2 \in \Sigma_2^* \text{ such that } s_1 = \varphi(u_1) \text{ and } s_2 = \varphi(u_2). \quad (9)$$

This statement immediately implies the third condition from Lemma 2.4, since $\boldsymbol{\ell} \notin \{a, c\}$ and thus $|\tau(\boldsymbol{\ell})| \geq 2$ for all $\boldsymbol{\ell} \in \mathrm{dom}(\mathcal{P})$. In order to prove (9) let $|\varphi(\ell)| \geq 2$ and $\varphi(s) \equiv_{I_1} s_1\varphi(\ell)s_2$. Because of (8) we can assume that $\varphi(s) = s_1\varphi(\ell)s_2$. Let us choose the factorization of the form $s_1 = \varphi(u)t$, where u has maximal length among all such factorizations. From $\varphi(s) = s_1\varphi(\ell)s_2 = \varphi(u)t\varphi(\ell)s_2$ and the cancellation property (7) it follows that there exists a $v \in \Sigma_2^*$ with $t\varphi(\ell)s_2 = \varphi(v)$ and $v \neq 1$ (because of $\ell \neq 1$). We claim that $t = 1$ which implies $s_1 = \varphi(u)$. Assume that $t \neq 1$. We make a case distinction with respect to the first symbol of $v \neq 1$. If

$v = aw$ or $v = cw$ then also $t = a \cdots$ or $t = c \cdots$, which contradicts the maximality of u. Thus we have $v = b_1 w$ or $v = b_2 w$. W.l.o.g. we assume that $v = b_1 w$ for some $w \in \Sigma_2^*$, i.e., $t\varphi(\ell)s_2 = \varphi(v) = bbab\varphi(w)$. Because of the maximality of u, the word $t \neq 1$ must be a proper prefix of $bbab$. The case $t = b$ can be excludes, since otherwise $\varphi(\ell)s_2 = bab\varphi(w)$ and thus (because of $|\varphi(\ell)| \geq 2$) $\varphi(\ell) = ba \cdots$, which contradicts the definition of φ. If $t = bb$, then $\varphi(\ell)s_2 = ab\varphi(w)$ and thus $\varphi(\ell) = ab \cdots$. Since furthermore $\varphi(\ell) = ab$ is not possible possible, $w \neq 1$ must hold. If $w = a \cdots$ or $w = c \cdots$ then $\varphi(\ell) = aba \cdots$ or $\varphi(\ell) = abc \cdots$, which again is not possible. On the other hand if $w = b_1 \cdots$ or $w = b_2 \cdots$ then $\varphi(\ell) = abbb \cdots$, which is again impossible. Finally the case $t = bba$ can be excluded using the same arguments. Thus $s_1 = \varphi(u)$, i.e., $\varphi(s) = \varphi(u\ell)s_2$. Now (7) implies that there exists a u_2 with $s_2 = \varphi(u_2)$. Now the proof of claim (9) is complete.

Finally we have to verify the fourth condition from Lemma 2.4. Assume that $\boldsymbol{t} \in \mathrm{CT}(\tau(\mathcal{P}))$. We have to find a $\boldsymbol{t}' \in \mathbb{M}(\Sigma_2, I_2)$ with $\tau(\boldsymbol{t}') = \boldsymbol{t}$. Since $\tau(\mathcal{P})$ satisfies condition (B) it suffices to consider all cases that are listed in Lemma 3.5. The first three cases as well as case (5) and (6) from Lemma 3.5 can be dealt with the same arguments that were used in the proof of Lemma 3.7. Now assume that $\boldsymbol{t}$ is of type (4), i.e., $\boldsymbol{t} = \boldsymbol{psq}$, where $\tau(\boldsymbol{\ell}_0) = \boldsymbol{ps}$, $\tau(\boldsymbol{\ell}_1) = \boldsymbol{sq}$, and $\boldsymbol{\ell}_0, \boldsymbol{\ell}_1 \in \mathrm{dom}(\mathcal{P})$. Let $\boldsymbol{\ell}_i = [\ell_i]_{I_2}$, $\boldsymbol{s} = [s]_{I_1}$, $\boldsymbol{p} = [p]_{I_1}$, and $\boldsymbol{q} = [q]_{I_1}$, i.e., $\varphi(\ell_0) \equiv_{I_1} ps$ and $\varphi(\ell_1) \equiv_{I_1} sq$. Because of (5), we can assume that $\varphi(\ell_0) = ps$ and $\varphi(\ell_1) = sq$. Let us choose the factorization of the form $s = \varphi(u)v$, where u has maximal length among all such factorizations. It follows $sq = \varphi(u)vq = \varphi(\ell_1)$ and $vq = \varphi(w)$ for some $w \in \Sigma_2^*$. We claim that $v = 1$. Assume that $v \neq 1$ and hence $w \neq 1$. Now $w = a \cdots$ or $w = c \cdots$ implies $v = a \cdots$ or $v = c \cdots$, which contradicts the maximality of u. Therefore $w = b_1 \cdots$ or $w = b_2 \cdots$. W.l.o.g. assume that $w = b_1 \cdots$. Hence $vq = bbab \cdots$. The maximality of u implies that $v \neq 1$ is a proper prefix of $bbab$. If $v = bba$ then $\varphi(\ell_0) = ps = p\varphi(u)v = \cdots bba$, which is impossible. If $v = b$ or $v = bb$ then $\varphi(\ell_0) = \cdots b$. But this is also not possible, since $\max(\boldsymbol{\ell}_0) = \max([\ell_0]_{I_2}) \subseteq \{a, c\}$ implies $\varphi(\ell_0) = \cdots a$ or $\varphi(\ell_0) = \cdots c$. Note that this is the only point where the condition $\max(\boldsymbol{\ell}) \subseteq \{a, c\}$ for all $\boldsymbol{\ell} \in \mathrm{dom}(\mathcal{P})$ is used. Thus we have $v = 1$ and therefore $s = \varphi(u)$, i.e., $\varphi(\ell_0) = p\varphi(u)$ and $\varphi(\ell_1) = \varphi(u)q$. By (7) there exist $p', q' \in \Sigma_2^*$ with $\varphi(p') = p$ and $\varphi(q') = q$. Thus $\boldsymbol{t} = \boldsymbol{psq} = [psq]_{I_1} = \tau([p'uq']_{I_2})$. ■

3.2. The general case

A confluent semi-Thue system over an alphabet Σ remains confluent if we add an additional symbol to the alphabet Σ, which does not occur in the rules of $\mathcal{R}$. This trivial fact is in general wrong for trace rewriting systems, see the following example from [12], p 125. If (Γ, I) is an independence alphabet and $\Sigma \subseteq \Gamma$ then $(\Sigma, (\Sigma \times \Sigma) \cap I)$ is called an *induced subalphabet* of (Γ, I).

EXAMPLE 3.1. Let (Γ, J) be the following independence alphabet:

$$a — c — f — b — d$$

Let (Σ, I) be the following induced subalphabet of (Γ, J):

$$a — c \quad b — d$$

Let $\mathcal{R} = \{ab \to c,\, cd \to a\}$. If we consider $\mathcal{R}$ as a TRS over $\mathbb{M}(\Sigma, I)$ then $\mathcal{R}$ confluent, see [12]. Note that $\mathcal{R}$ does not satisfy (A), so we cannot apply Lemma 2.3. On the other hand it is easy to show by a direct application of Lemma 2.1 that $\mathcal{R}$ is confluent. But if we consider $\mathcal{R}$ as a TRS over $\mathbb{M}(\Gamma, J)$ then $\mathcal{R}$ is no longer confluent. To see this let us consider the trace $[cabfd]_J = [afcdb]_J$. It holds $[cabfd]_J \to_{\mathcal{R}} [ccfd]_J$ and $[afcdb]_J \to_{\mathcal{R}} [afab]_J$. From $[ccfd]_J = [cfcd]_J$ we can only derive the trace $[cfa]_J$, whereas from $[afab]_J$ only the trace $[afc]_J \neq [cfa]_J$ can be derived. Thus $\mathcal{R}$ is not confluent.

The example above shows that the following lemma is not trivial.

Lemma 3.9. Let (Γ, I) be an independence alphabet and $\Sigma \subseteq \Gamma$. If the problem $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Gamma, I))$ is decidable then also $\mathrm{COLR}_{\neq 1}(\mathbb{M}(\Sigma, I \cap (\Sigma \times \Sigma)))$ is decidable.

Proof. Let $\mathcal{R}$ be a length-reducing TRS over $\mathbb{M}(\Sigma, I \cap (\Sigma \times \Sigma))$ with $1 \notin \mathrm{ran}(\mathcal{R})$. We prove the lemma by constructing a length-reducing TRS $\mathcal{P}$ over $\mathbb{M}(\Gamma, I)$ such that $1 \notin \mathrm{ran}(\mathcal{P})$ and $\mathcal{R}$ is confluent if and only if $\mathcal{P}$ is confluent. The case $\Sigma = \Gamma$ is trivial. So there exists a symbol $0 \in \Gamma \backslash \Sigma$. Let

$$\mathcal{P} = \mathcal{R} \cup \{[ab]_I \to 0 \mid a \in \Gamma \backslash \Sigma \text{ or } b \in \Gamma \backslash \Sigma\}.$$

Note that $\mathcal{P}$ is length-reducing and $1 \notin \mathrm{ran}(\mathcal{P})$. Clearly every trace of length at most two which contains a symbol from $\Gamma \backslash \Sigma$ has 0 as its unique normal form. Since furthermore every trace of length less then two is irreducible ($\mathcal{R}$ is length-reducing and $1 \notin \mathrm{ran}(\mathcal{R})$) the following holds: $\mathcal{P}$ is confluent if and only if $\mathcal{P}$ is confluent on all traces s with $\mathrm{alph}(s) \subseteq \Sigma$ if and only if $\mathcal{R}$ is confluent. ■

Now we can prove the main result of this section.

Theorem 3.1. The decision problem $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is decidable if and only if $I = \emptyset$ or $I = (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$.

Proof. As already mentioned, confluence is decidable for terminating semi-Thue systems and vector replacement systems [8, 24]. So let $I \neq \emptyset$ and $I \neq (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$. We have to show that $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is undecidable.

Because of $I \neq \emptyset$ there exist $a, c \in \Sigma$ with $a\, I\, c$ (and thus also $a \neq c$). First assume that there exists a $b \in \Sigma \backslash \{a, c\}$ with $(a, b) \notin I$. Then one of the following two graphs is an induced subalphabet of (Σ, I):

$$a — c — b \qquad\qquad a — c \quad b$$

Lemma 3.9 and Lemma 3.4 or Lemma 3.8, respectively, imply that $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is undecidable. So we can assume that $a\, I\, x$ for all $x \in \Sigma \backslash \{a\}$. Since $I \neq (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$ there exist $d, e \in \Sigma$ with $(d, e) \notin I$ and $d \neq e$. Thus $d \neq a \neq e$ and (Σ, I) contains an induced subalphabet of the form

$$d — a — e$$

Lemma 3.9 and Lemma 3.4 imply that $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is again undecidable. ■

We close this section with a further sharpening of Theorem 3.1.

LEMMA 3.10. For every trace monoid $\mathbb{M}(\Sigma, I)$ the set of all trace rewriting systems over $\mathbb{M}(\Sigma, I)$, which are length-reducing but not confluent, is recursively enumerable.

Proof. A semi-algorithm, which checks whether a length-reducing TRS $\mathcal{R}$ is not confluent, can enumerate all traces $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \in \mathbb{M}(\Sigma, I)$ with $\boldsymbol{s} \to_{\mathcal{R}} \boldsymbol{t}$ and $\boldsymbol{s} \to_{\mathcal{R}} \boldsymbol{u}$ and check whether there exists a trace $\boldsymbol{v}$ with $\boldsymbol{t} \to_{\mathcal{R}}^{*} \boldsymbol{v}$ and $\boldsymbol{u} \to_{\mathcal{R}}^{*} \boldsymbol{v}$. Since $\mathcal{R}$ is terminating, this is decidable. ■

Now the following theorem is an immediate consequence of the previous lemma and Theorem 3.1.

THEOREM 3.2. If $I \neq \emptyset$ and $I \neq (\Sigma \times \Sigma)\backslash \mathrm{Id}_{\Sigma}$ then $\mathrm{COLR}(\mathbb{M}(\Sigma, I))$ is not recursively enumerable.

4. SPECIAL SYSTEMS

In [12] it was asked, whether confluence is decidable for special trace rewriting systems. In this section, we will show that this is indeed the case. In fact we will show that confluence is decidable in polynomial time for an even more general class of trace rewriting systems.

We say that a TRS $\mathcal{R}$ over the trace monoid $\mathbb{M}(\Sigma, I)$ satisfies the condition (C) it the following holds:

(C1) For all $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$, all factorizations $\boldsymbol{\ell}_0 = \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0$, $\boldsymbol{\ell}_1 = \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1$ with $\boldsymbol{s} \neq 1$, $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1$, and $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$, and all $a \in \Sigma$ it holds:
If $(a \, I \, \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_0$ or $a \, I \, \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_1)$ then $(a\boldsymbol{r}_0 = \boldsymbol{r}_0 a$ and $a \boldsymbol{r}_1 = \boldsymbol{r}_1 a)$.

(C2) For all $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and all factorizations $\boldsymbol{\ell}_0 = \boldsymbol{p}_0 \boldsymbol{q}_0$, $\boldsymbol{\ell}_1 = \boldsymbol{p}_1 \boldsymbol{q}_1$ with $\boldsymbol{p}_i \neq 1 \neq \boldsymbol{q}_i$ for $i \in \{0, 1\}$, $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1$, and $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$ it holds: There exist factorizations $\boldsymbol{r}_0 = \boldsymbol{s}_0 \boldsymbol{t}_0$ and $\boldsymbol{r}_1 = \boldsymbol{s}_1 \boldsymbol{t}_1$ such that for all $a \in \Sigma$ and all $i \in \{0, 1\}$ it holds: If $a \, I \, \boldsymbol{p}_i$ then $a \, I \, \boldsymbol{s}_i$, and if $a \, I \, \boldsymbol{q}_i$ then $a \, I \, \boldsymbol{t}_i$.

Note that a length-reducing TRS that satisfies condition (C) satisfies condition (A) as well: Condition (A2) and condition (C2) are identical and condition (C1) implies (A1) as follows: Let $(\boldsymbol{\ell}, \boldsymbol{r}) \in \mathcal{R}$ and $a \, I \, \boldsymbol{\ell}$. Since $\mathcal{R}$ is length-reducing we have $\boldsymbol{\ell} \neq 1$. Now consider the factorization $\boldsymbol{\ell} = \boldsymbol{p}_0 \boldsymbol{s} \boldsymbol{q}_0 = \boldsymbol{p}_1 \boldsymbol{s} \boldsymbol{q}_1$ with $\boldsymbol{p}_0 = \boldsymbol{q}_0 = \boldsymbol{p}_1 = \boldsymbol{q}_1 = 1$ and $\boldsymbol{s} = \boldsymbol{\ell} \neq 1$. Condition (C1) implies that $a\boldsymbol{r} = \boldsymbol{r}a$. Furthermore note that every special TRS satisfies condition (C). The following theorem generalizes Theorem 6 from [19].

THEOREM 4.1. For every trace monoid $\mathbb{M}(\Sigma, I)$ the following problem is decidable in polynomial time:
INPUT: A length-reducing TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ that satisfies condition (C).
QUESTION: Is $\mathcal{R}$ confluent?

```
Input: A length-reducing TRS R over M(Σ, I) that satisfies condition (C).
forall ((ℓ_0 → r_0), (ℓ_1 → r_1)) ∈ R × R do
    forall p_0, p_1, q_0, q_1, s with
    ℓ_0 = p_0 s q_0, ℓ_1 = p_1 s q_1, s ≠ 1, p_0 I p_1, q_0 I q_1 do
        if NF(p_0 r_1 q_0, R) ≠ NF(p_1 r_0 q_1, R) then
            return "R not confluent"    (*)
        forall a ∈ Σ with (a I p_1 s q_0) or (a I p_0 s q_1) do
            if NF(a p_1 r_0 q_1, R) ≠ NF(p_0 r_1 q_0 a, R) then
                return "R not confluent"    (**)
        endfor
    endfor
endfor
return "R confluent"    (***)
```

FIG. 3. The algorithm CONF

Note that by the results from Section 3, confluence is undecidable for length-reducing trace rewriting systems that only satisfy the weaker condition (A).

Proof. Let (Σ, I) be an independence alphabet and let $\mathcal{R}$ be a length-reducing TRS over $\mathbb{M}(\Sigma, I)$, which satisfies condition (C). Let NF be an algorithm, which calculates for a length-reducing TRS $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ and a trace $\boldsymbol{u} \in \mathbb{M}(\Sigma, I)$ an arbitrary normal form $\mathrm{NF}(\boldsymbol{u}, \mathcal{R})$ of $\boldsymbol{u}$ with respect to $\mathcal{R}$. Let CONF be the algorithm in Figure 3. First we show that $\mathcal{R}$ is not confluent if CONF returns "$\mathcal{R}$ not confluent". If CONF executes line $(*)$ then there exist rules $\boldsymbol{\ell}_0 \to \boldsymbol{r}_0$ and $\boldsymbol{\ell}_1 \to \boldsymbol{r}_1$ in $\mathcal{R}$ as well as factorizations $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0$ and $\boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1$ with $\boldsymbol{s} \neq 1$, $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1$, and $\boldsymbol{q}_0 \, I \, \boldsymbol{q}_1$. Furthermore there exist normal forms $\boldsymbol{u}_0$ of $\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0$ and $\boldsymbol{u}_1$ of $\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1$ with $\boldsymbol{u}_0 \neq \boldsymbol{u}_1$. But then $\mathcal{R}$ is indeed not confluent, since $\boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{q}_1 \to_{\mathcal{R}} \boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1 \to_{\mathcal{R}}^* \boldsymbol{u}_1$ and $\boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{q}_1 = \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{q}_0 \to_{\mathcal{R}} \boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 \to_{\mathcal{R}}^* \boldsymbol{u}_0$. Now let us assume that CONF executes line $(**)$. Then there exists an $a \in \Sigma$ with $a \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0$ or $a \, I \, \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_1$, and there exists a normal form $\boldsymbol{v}_0$ of $a\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1$ and a normal form $\boldsymbol{v}_1$ of $\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 a$ with $\boldsymbol{v}_0 \neq \boldsymbol{v}_1$. Assume that $a \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0$. Since $\mathcal{R}$ satisfies condition (C1), it follows $a\boldsymbol{r}_0 = \boldsymbol{r}_0 a$ and $a\boldsymbol{r}_1 = \boldsymbol{r}_1 a$. Thus $\boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0 a\boldsymbol{q}_1 \to_{\mathcal{R}} \boldsymbol{p}_1\boldsymbol{r}_0 a\boldsymbol{q}_1 = a\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1 \to_{\mathcal{R}}^* \boldsymbol{v}_0$ and

$$\boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0 a\boldsymbol{q}_1 = \boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0 a\boldsymbol{q}_1 = \boldsymbol{p}_0 a\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{q}_0 \to_{\mathcal{R}} \boldsymbol{p}_0 a\boldsymbol{r}_1\boldsymbol{q}_0 = \boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 a \to_{\mathcal{R}}^* \boldsymbol{v}_1.$$

Thus $\mathcal{R}$ is not confluent. The case $a \, I \, \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_1$ can be dealt analogously by considering the trace $\boldsymbol{p}_1 a\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{q}_1$ instead of $\boldsymbol{p}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0 a\boldsymbol{q}_1$.

Now we assume that CONF returns "$\mathcal{R}$ confluent" in line $(***)$. We have to show that $\mathcal{R}$ is confluent. By an induction on the length of traces, it suffices to prove the following implication for all $\boldsymbol{t} \in \mathbb{M}(\Sigma, I)$:

If $\mathcal{R}$ is confluent on all traces $\boldsymbol{t}'$ with $|\boldsymbol{t}'| < |\boldsymbol{t}|$ then $\mathcal{R}$ is confluent on $\boldsymbol{t}$.

So let $\boldsymbol{t} \in \mathbb{M}(\Sigma, I)$ and assume that $\mathcal{R}$ is confluent on all shorter traces. We have to show that all pairs $(\boldsymbol{t}_0, \boldsymbol{t}_1)$ with $\boldsymbol{t} \to_{\mathcal{R}}^* \boldsymbol{t}_0$ and $\boldsymbol{t} \to_{\mathcal{R}}^* \boldsymbol{t}_1$ for some $\boldsymbol{t} \in \mathbb{M}(\Sigma, I)$ are confluent. The case $\boldsymbol{t} = \boldsymbol{t}_0$ or $\boldsymbol{t} = \boldsymbol{t}_1$ is trivial. Now assume for the moment that we already have dealt all situations of the form $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_0$, $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{t}_1$. Then we can use the following arguments that are similarly to the usual proof of Newman's lemma: From $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{s}_0 \to_{\mathcal{R}}^* \boldsymbol{t}_0$ and $\boldsymbol{t} \to_{\mathcal{R}} \boldsymbol{s}_1 \to_{\mathcal{R}}^* \boldsymbol{t}_1$ it follows that there exists an $\boldsymbol{s}$ with $\boldsymbol{s}_0 \to_{\mathcal{R}}^* \boldsymbol{s}$ and $\boldsymbol{s}_1 \to_{\mathcal{R}}^* \boldsymbol{s}$. Because of $|\boldsymbol{s}_0| < |\boldsymbol{t}|$, $\boldsymbol{s}_0 \to_{\mathcal{R}}^* \boldsymbol{t}_0$, and $\boldsymbol{s}_0 \to_{\mathcal{R}}^* \boldsymbol{s}$ there exists a trace $\boldsymbol{u}$ with $\boldsymbol{t}_0 \to_{\mathcal{R}}^* \boldsymbol{u}$ and $\boldsymbol{s} \to_{\mathcal{R}}^* \boldsymbol{u}$. Now $|\boldsymbol{s}_1| < |\boldsymbol{t}|$, $\boldsymbol{s}_1 \to_{\mathcal{R}}^* \boldsymbol{t}_1$, and $\boldsymbol{s}_1 \to_{\mathcal{R}}^* \boldsymbol{s} \to_{\mathcal{R}}^* \boldsymbol{u}$ imply $\boldsymbol{t}_1 \to_{\mathcal{R}}^* \boldsymbol{v}$ and $\boldsymbol{u} \to_{\mathcal{R}}^* \boldsymbol{v}$, i.e., $\boldsymbol{t}_0 \to_{\mathcal{R}}^* \boldsymbol{u} \to_{\mathcal{R}}^* \boldsymbol{v}$ for some trace $\boldsymbol{v}$.

So it suffices to show that for all rules $(\boldsymbol{\ell}_0, \boldsymbol{r}_0), (\boldsymbol{\ell}_1, \boldsymbol{r}_1) \in \mathcal{R}$ and all factorizations $\boldsymbol{t} = \boldsymbol{u}_0\boldsymbol{\ell}_0\boldsymbol{v}_0 = \boldsymbol{u}_1\boldsymbol{\ell}_1\boldsymbol{v}_1$ the pair $(\boldsymbol{u}_0\boldsymbol{r}_0\boldsymbol{v}_0, \boldsymbol{u}_1\boldsymbol{r}_1\boldsymbol{v}_1)$ is confluent. Lemma 2.1 applied to the identity $\boldsymbol{u}_0\boldsymbol{\ell}_0\boldsymbol{v}_0 = \boldsymbol{u}_1\boldsymbol{\ell}_1\boldsymbol{v}_1$ gives nine traces $\boldsymbol{p}_i$, $\boldsymbol{q}_i$, $\boldsymbol{w}_i$, $\boldsymbol{y}_i$, $\boldsymbol{s}$ ($i \in \{0,1\}$) with

- $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0, \quad \boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1,$
- $\boldsymbol{u}_0 = \boldsymbol{y}_0\boldsymbol{p}_1\boldsymbol{w}_1, \quad \boldsymbol{u}_1 = \boldsymbol{y}_0\boldsymbol{p}_0\boldsymbol{w}_0, \quad \boldsymbol{v}_0 = \boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{y}_1, \quad \boldsymbol{v}_1 = \boldsymbol{w}_1\boldsymbol{q}_0\boldsymbol{y}_1,$
- $\boldsymbol{p}_0 \, I \, \boldsymbol{p}_1, \quad \boldsymbol{q}_0 \, I \, \boldsymbol{q}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{w}_1, \quad \boldsymbol{w}_0 \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0, \quad \boldsymbol{w}_1 \, I \, \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_1,$
- $\boldsymbol{t} = \boldsymbol{y}_0\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{w}_1\boldsymbol{q}_0\boldsymbol{y}_1 = \boldsymbol{y}_0\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{y}_1,$

see also the following picture:

$\boldsymbol{v}_1$	$\boldsymbol{w}_1$	$\boldsymbol{q}_0$	$\boldsymbol{y}_1$
$\boldsymbol{\ell}_1$	$\boldsymbol{p}_1$	$\boldsymbol{s}$	$\boldsymbol{q}_1$
$\boldsymbol{u}_1$	$\boldsymbol{y}_0$	$\boldsymbol{p}_0$	$\boldsymbol{w}_0$
	$\boldsymbol{u}_0$	$\boldsymbol{\ell}_0$	$\boldsymbol{v}_0$

We have to show that the pair $(\boldsymbol{y}_0\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0\boldsymbol{y}_1, \boldsymbol{y}_0\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{y}_1)$ is confluent. If $\boldsymbol{y}_0 \neq 1$ or $\boldsymbol{y}_1 \neq 1$ then for the trace $\boldsymbol{t}' = \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{w}_0\boldsymbol{q}_1$ it holds $|\boldsymbol{t}'| < |\boldsymbol{t}|$ and

$$\boldsymbol{t}' = \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{\ell}_1\boldsymbol{w}_1\boldsymbol{q}_0 \to_{\mathcal{R}} \boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0, \quad \boldsymbol{t}' = \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{\ell}_0\boldsymbol{w}_0\boldsymbol{q}_1 \to_{\mathcal{R}} \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1.$$

Hence the pair $(\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0, \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1)$ is confluent. But then also the pair $(\boldsymbol{y}_0\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0\boldsymbol{y}_1, \boldsymbol{y}_0\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1\boldsymbol{y}_1)$ is confluent. Thus we can assume that $\boldsymbol{y}_0 = \boldsymbol{y}_1 = 1$. We have to consider the pair $(\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0, \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1)$. If $\boldsymbol{s} = 1$, i.e., $\boldsymbol{\ell}_0 = \boldsymbol{p}_0\boldsymbol{q}_0$ and $\boldsymbol{\ell}_1 = \boldsymbol{p}_1\boldsymbol{q}_1$ then this pair is confluent by Lemma 2.2 (note that $\mathcal{R}$ satisfies condition (A)). So assume that $\boldsymbol{s} \neq 1$. Then we have one of the situations that are considered in the two outermost **forall**-loops of CONF. Since we assume that CONF returns "$\mathcal{R}$ confluent" there exists a trace $\boldsymbol{u}$ with $\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 \to_{\mathcal{R}}^* \boldsymbol{u}$ and $\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1 \to_{\mathcal{R}}^* \boldsymbol{u}$. Furthermore condition (C1) as well as $\boldsymbol{w}_0 \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0$ and $\boldsymbol{w}_1 \, I \, \boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_1$ imply $\boldsymbol{r}_i\boldsymbol{w}_j = \boldsymbol{w}_j\boldsymbol{r}_i$ for $i, j \in \{0,1\}$. Thus we obtain $\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0\boldsymbol{w}_0 \to_{\mathcal{R}}^* \boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}_0$ and $\boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}_0\boldsymbol{q}_1 = \boldsymbol{w}_0\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1$. It suffices to show that the pair $(\boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}_0, \boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1)$ is confluent. The case $\boldsymbol{w}_0 = 1 = \boldsymbol{w}_1$ is trivial. So assume w.l.o.g. that $\boldsymbol{w}_0 = \boldsymbol{w}a$ for some $a \in \Sigma$. Because of $\boldsymbol{w}_0 \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0$ we have $a \, I \, \boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0$. Thus $a \in \Sigma$ is one of the symbols that are considered in the innermost **forall**-loop of CONF. Thus there exists a trace $\boldsymbol{v} \in \mathrm{IRR}(\mathcal{R})$ with $a\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1 \to_{\mathcal{R}}^* \boldsymbol{v}$ and $\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 a \to_{\mathcal{R}}^* \boldsymbol{v}$. But since also $a\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1 \to_{\mathcal{R}}^* a\boldsymbol{u}$ and $\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0 a \to_{\mathcal{R}}^* \boldsymbol{u}a$ and $|a\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1| \leq |\boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1| < |\boldsymbol{w}_0\boldsymbol{w}_1\boldsymbol{p}_1\boldsymbol{\ell}_0\boldsymbol{q}_1| = |\boldsymbol{t}|$ we have $a\boldsymbol{u} \to_{\mathcal{R}}^* \boldsymbol{v}$ and $\boldsymbol{u}a \to_{\mathcal{R}}^* \boldsymbol{v}$ (note that we assumed that $\boldsymbol{v}$ is irreducible). It follows

$$\boldsymbol{w}\boldsymbol{u}a\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}\boldsymbol{v}\boldsymbol{w}_1 \quad \text{and} \quad \boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1 = \boldsymbol{w}a\boldsymbol{u}\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}\boldsymbol{v}\boldsymbol{w}_1. \tag{10}$$

Consider the trace $\boldsymbol{t}' = \boldsymbol{p}_0\boldsymbol{w}\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{p}_0\boldsymbol{w}\boldsymbol{\ell}_1\boldsymbol{w}_1\boldsymbol{q}_0$. The trace $\boldsymbol{t}'$ results from $\boldsymbol{t}$ by replacing the factor $\boldsymbol{w}_0 = \boldsymbol{w}a$ by $\boldsymbol{w}$. Thus $|\boldsymbol{t}'| < |\boldsymbol{t}|$. Furthermore, since

$\boldsymbol{w}$ is a factor of $\boldsymbol{w}_0$, $\boldsymbol{w}$ satisfies at least the same independencies as $\boldsymbol{w}_0$. Thus $\boldsymbol{t}' = \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{s}\boldsymbol{q}_0\boldsymbol{w}\boldsymbol{q}_1 = \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{\ell}_0\boldsymbol{w}\boldsymbol{q}_1$ as well as $\boldsymbol{w}\boldsymbol{r}_0 = \boldsymbol{r}_0\boldsymbol{w}$ and $\boldsymbol{w}\boldsymbol{r}_1 = \boldsymbol{r}_1\boldsymbol{w}$ (because of condition (C1) and $\boldsymbol{p}_1\boldsymbol{s}\boldsymbol{q}_0\, I\, \boldsymbol{w}$). Hence we obtain

$$\boldsymbol{t}' \to_{\mathcal{R}} \boldsymbol{p}_0\boldsymbol{w}\boldsymbol{r}_1\boldsymbol{w}_1\boldsymbol{q}_0 = \boldsymbol{w}_1\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0\boldsymbol{w} \to_{\mathcal{R}}^* \boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w} \quad \text{and}$$
$$\boldsymbol{t}' \to_{\mathcal{R}} \boldsymbol{p}_1\boldsymbol{w}_1\boldsymbol{r}_0\boldsymbol{w}\boldsymbol{q}_1 = \boldsymbol{w}\boldsymbol{p}_1\boldsymbol{r}_0\boldsymbol{q}_1\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}\boldsymbol{u}\boldsymbol{w}_1.$$

It follows $\boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w} \to_{\mathcal{R}}^* \boldsymbol{x}$ and $\boldsymbol{w}\boldsymbol{u}\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{x}$ for some trace $\boldsymbol{x}$. Therefore

$$\boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}_0 = \boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}a \to_{\mathcal{R}}^* \boldsymbol{x}a \quad \text{and} \quad \boldsymbol{w}\boldsymbol{u}a\boldsymbol{w}_1 = \boldsymbol{w}\boldsymbol{u}\boldsymbol{w}_1 a \to_{\mathcal{R}}^* \boldsymbol{x}a. \tag{11}$$

From $\boldsymbol{w}\boldsymbol{u}a\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}\boldsymbol{v}\boldsymbol{w}_1$ (10), $\boldsymbol{w}\boldsymbol{u}a\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{x}a$ (11), and $|\boldsymbol{w}\boldsymbol{u}a\boldsymbol{w}_1| = |\boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1| \leq |\boldsymbol{w}_0\boldsymbol{p}_0\boldsymbol{r}_1\boldsymbol{q}_0\boldsymbol{w}_1| < |\boldsymbol{p}_0\boldsymbol{w}_0\boldsymbol{\ell}_1\boldsymbol{w}_1\boldsymbol{q}_0| = |\boldsymbol{t}|$ (where the strict inequality follows from $|\boldsymbol{r}_1| < |\boldsymbol{\ell}_1|$), we obtain $\boldsymbol{w}\boldsymbol{v}\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{z}$ and $\boldsymbol{x}a \to_{\mathcal{R}}^* \boldsymbol{z}$ for some trace $\boldsymbol{z}$. Therefore $\boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{w}\boldsymbol{v}\boldsymbol{w}_1 \to_{\mathcal{R}}^* \boldsymbol{z}$ by (10) and $\boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}_0 \to_{\mathcal{R}}^* \boldsymbol{x}a \to_{\mathcal{R}}^* \boldsymbol{z}$ by (11). Hence the pair $(\boldsymbol{w}_0\boldsymbol{u}\boldsymbol{w}_1, \boldsymbol{w}_1\boldsymbol{u}\boldsymbol{w}_0)$ is confluent. This concludes the proof of the correctness of CONF.

Finally we claim that the algorithm CONF works in polynomial time. This follows from two facts:

- For a fixed independence alphabet (Σ, I), the number of different factorizations $\boldsymbol{\ell} = \boldsymbol{p}\boldsymbol{s}\boldsymbol{q}$ of a trace $\boldsymbol{\ell} \in \mathbb{M}(\Sigma, I)$ is bounded by a polynomial in $|\boldsymbol{\ell}|$. This follows from the fact that the number of prefixes of a trace $\boldsymbol{\ell}$ is bounded by a polynomial in $|\boldsymbol{\ell}|$ [6].
- A normal form of a trace $\boldsymbol{t}$ with respect to a length-reducing TRS $\mathcal{R}$ can be calculated in time, bounded polynomially in $|\boldsymbol{t}|$ and $\|\mathcal{R}\|$. The problem of calculating normal forms for length-reducing trace rewriting systems was considered for instance in [11, 13, 4, 5, 3]. The algorithms that are presented in these papers are all non-uniform, i.e., the TRS is not part of the input. But it is easy to see that these algorithms also work in polynomial time in the case that the TRS is part of the input.

■

From Theorem Theorem 4.1 we obtain the following corollary.

Corollary 4.2. COSP(M) is in P for every trace monoid M.

If the independence alphabet (Σ, I) is also part of the input then the algorithm CONF does not work in polynomial time. In fact it is open whether confluence for special TRSs is also in P if the independence alphabet is also part of the input.

5. CONCLUSION

In this paper we have investigated the confluence problem for length-reducing trace rewriting systems. We have shown that confluence is decidable for length-reducing trace rewriting systems over a trace monoid M if and only if M is a free or free commutative monoid. Furthermore we have shown that confluence for special trace rewriting systems is decidable in polynomial time. We would like to close this paper with a list of several questions that remain unsolved.

• Is confluence decidable for monadic trace rewriting systems, where monadic means that all right-hand sides have length at most one? This question was already asked in [12].

• Is confluence decidable for trace rewriting systems that contain only one rule. It is known that confluence is decidable for one-rule semi-Thue systems [25]. Moreover in [26] it was shown that confluence is decidable for a large subclass of one-rule trace rewriting systems. But the general case is still open. In particular it is an open question, whether confluence is decidable for a rule of the form $1 \to \boldsymbol{r}$.

Acknowledgments I would like to thank Volker Diekert, Anca Muscholl, Friedrich Otto, and the referees for valuable comments.

REFERENCES

1. G. Bauer and F. Otto. Finite complete rewriting systems and the complexity of the word problem. *Acta Informatica*, 21:521–540, 1984.
2. J. Berstel and D. Perrin. *Theory of Codes.* Academic Press, 1985.
3. M. Bertol. *Effiziente Normalform–Algorithmen für Ersetzungssysteme über partiell kommutativen Monoiden.* PhD thesis, Universität Stuttgart, 1996.
4. M. Bertol and V. Diekert. On efficient reduction-algorithms for some trace rewriting systems. In H. Common and J.-P. Jouannaud, editors, *Term Rewriting*, number 909 in Lecture Notes in Computer Science, pages 114–126, Berlin-Heidelberg-New York, 1995. Springer.
5. M. Bertol and V. Diekert. Trace rewriting: Computing normal forms in time $\mathcal{O}(n \log n)$. In C. Puech and R. Reischuk, editors, *Proceedings of the13th Annual Symposium on Theoretical Aspects of Computer Science 1996*, number 1046 in Lecture Notes in Computer Science, pages 269–280, Berlin-Heidelberg-New York, 1996. Springer.
6. A. Bertoni, G. Mauri, and N. Sabadini. Membership problems for regular and context free trace languages. *Information and Computation*, 82:135–150, 1989.
7. R. Book and F. Otto. *String–Rewriting Systems.* Springer, 1993.
8. R. V. Book and C. P. O'Dunlaing. Testing for the Church–Rosser property (note). *Theoretical Computer Science*, 16:223–229, 1981.
9. R. Cori and D. Perrin. Automates et commutations partielles. *R.A.I.R.O. — Informatique Théorique et Applications*, 19:21–32, 1985.
10. V. Diekert. On the Knuth-Bendix completion for concurrent processes. In T. Ottmann, editor, *Proceedings of the14th International Colloquium on Automata, Languages and Programming (ICALP 87), Karlsruhe (FRG) 1987*, number 267 in Lecture Notes in Computer Science, pages 42–53, Berlin-Heidelberg-New York, 1987. Springer.
11. V. Diekert. Combinatorial rewriting on traces. In C. Choffrut et al., editors, *Proceedings of the7th Annual Symposium on Theoretical Aspects of Computer Science (STACS 90), Rouen (France) 1990*, number 415 in Lecture Notes in Computer Science, pages 138–151, Berlin-Heidelberg-New York, 1990. Springer.
12. V. Diekert. *Combinatorics on Traces.* Number 454 in Lecture Notes in Computer Science. Springer, 1990.
13. V. Diekert. Word problems over traces which are solvable in linear time. *Theoretical Computer Science*, 74:3–18, 1990.
14. V. Diekert and G. Rozenberg, editors. *The Book of Traces.* World Scientific, Singapore, 1995.
15. G. Huet and D. Lankford. On the uniform halting problem for term rewriting systems. Report Lab. Report No. 283, INRIA, Le Chesnay, France, 1978.
16. M. Jantzen. Confluent string rewriting. In *EATCS Monographs on theoretical computer science*, volume 14. Springer, 1988.
17. D. Kapur, M. S. Krishnamoorthy, R. McNaughton, and P. Narendran. An $O(|T|^3)$ algorithm for testing the Church-Rosser property of Thue systems. *Theoretical Computer Science*, 35(1):109–114, 1985.

18. M. Lohrey. On the confluence of trace rewriting systems. In V. Arvind and R. Ramanujam, editors, *Proceedings of the 18th Conference on Foundations of Software Technology and Theoretical Computer Science, (FSTTCS'98), Chennai (India)*, number 1530 in Lecture Notes in Computer Science, pages 319–330. Springer, 1998.
19. M. Lohrey. Complexity results for confluence problems. In M. Kutylowski and L. Pacholski, editors, *Proceedings of the 24th International Symposium on Mathematical Foundations of Computer Science (MFCS'99), Szklarska Poreba (Poland)*, number 1672 in Lecture Notes in Computer Science, pages 114–124. Springer, 1999.
20. A. Mazurkiewicz. Concurrent program schemes and their interpretations. DAIMI Rep. PB 78, Aarhus University, Aarhus, 1977.
21. P. Narendran and F. Otto. Preperfectness is undecidable for Thue systems containing only length-reducing rules and a single commutation rule. *Information Processing Letters*, 29:125–130, 1988.
22. M. H. A. Newman. On theories with a combinatorial definition of "equivalence". *Annals Mathematics*, 43:223–243, 1943.
23. M. Nivat and M. Benois. Congruences parfaites et quasi–parfaites. *Seminaire Dubreil*, 25(7–01–09), 1971–1972.
24. R. M. Verma, M. Rusinowitch, and D. Lugiez. Algorithms and reductions for rewriting problems. In *Proceedings of the 9th Conference on Rewriting Techniques and Applications (RTA-98), Tsukuba (Japan)*, number 1379 in Lecture Notes in Computer Science, pages 166–180. Springer, 1998.
25. C. Wrathall. Confluence of one–rule Thue systems. In K. U. Schulz, editor, *Word Equations and Related Topics*, number 572 in Lecture Notes in Computer Science, pages 237–246. Springer, 1992.
26. C. Wrathall and V. Diekert. On confluence of one-rule trace-rewriting systems. *Mathematical Systems Theory*, 28:341–361, 1995.