Free Shuffle Algebras with Involution

Definition 0.1 *An* **involuted bimonoid** $M = (M, \cdot, \otimes, {}^{\smile}, 1)$ *is a bimonoid* $(M, \cdot, \otimes, 1)$ *equipped with an unary operation* ${}^{\smile}$*, called involution, satisfiing the following axioms, for all* $a, b \in M$*:*

$$(a \cdot b)^{\smile} = b^{\smile} \cdot a^{\smile} \tag{1}$$

$$(a \otimes b)^{\smile} = a^{\smile} \otimes b^{\smile} \tag{2}$$

$$a^{\smile\smile} = a. \tag{3}$$

Suppose that $(M, {}^{\smile})$ *and* $(M', {}^{\smile})$ *are involuted bimonoids. An involuted bimonoid morphism*

$$h : (M, {}^{\smile}) \to (M', {}^{\smile})$$

is a bimonoid morphism $h : M \to M'$ *which preserves the involution operation, i.e.,* $(a^{\smile})\, h = (ah)^{\smile}$*, for all* $a \in M$*.*

Example 0.2 *The bimonoid* $\mathcal{L}_\Sigma$ *equipped with the reversal (or mirror image) operation defined by*

$$(\sigma_1 \dots \sigma_n)^R = \sigma_n \dots \sigma_1, \tag{4}$$

for all $\sigma_1 \dots \sigma_n \in \Sigma^*$*, where* $\sigma_i \in \Sigma$*,* $i \in [n]$*, and*

$$a^R = \{w^R \mid w \in a\} \tag{5}$$

for all $a \subseteq \Sigma^*$*. This involuted bimonoid is denoted by* $\mathcal{L}^R_\Sigma$*.*

Example 0.3 *The monoid* $\mathbf{Pos}(\Sigma^*)$ *equipped with the following involution operation is an involuted bimonoid. Suppose* $P = (P, \leq, \ell)$ *is a* Σ^**-labeled poset. We define* $P^R = (P, \leq^R, \ell^R)$*, where*

$$u \leq^R v \iff v \leq u \tag{6}$$

and

$$u(\ell^R) = (u\ell)^R, \tag{7}$$

for all $u, v \in P$*. The resulting involuted bimonoid is denoted* $\mathbf{Pos}^R(\Sigma)$*.*

Note that when $a \subseteq \Sigma^*$ is finite, or regular, then so is a^R. Thus $\mathcal{F}^R_\Sigma = (\mathcal{F}_\Sigma, {}^R)$ and $\mathcal{R}^R_\Sigma = (\mathcal{R}_\Sigma, {}^R)$ are involuted bimonoids. If $P \in \mathbf{SP}(\Sigma)$, then P^R is also a series-parallel Σ-labeled poset. Thus $\mathbf{SP}^R(\Sigma) = (\mathbf{SP}(\Sigma), {}^R)$ is an involuted bimonoid.

When $\Delta = \Sigma \cup \Sigma'$ and the function $\sigma \mapsto \sigma'$ is a bijection $\Sigma \to \Sigma'$, there is another way to define an involution operation on the bimnoids $\mathcal{L}_\Delta$ and $\mathbf{Pos}(\Delta^*)$.

Example 0.4 *Suppose that $w = x_1 \dots x_n \in \Delta^*$. We define*

$$w^\smile \quad = \quad x_n{}^\smile \dots x_1{}^\smile,$$

where $x_i \in \Delta$ and $x_i{}^\smile = \sigma_i'$ if $x_i = \sigma_i$, and $x_i{}^\smile = \sigma_i$ if $x_i = \sigma_i'$, $i \in [n]$. The involuted bimonoid $\mathcal{L}_\Sigma^\smile$ is the pair $(\mathcal{L}_\Delta, {}^\smile)$ with involution given by

$$a^\smile \quad = \quad \{w^\smile \mid w \in a\},$$

for all $a \subseteq \Delta^*$.

Example 0.5 *Suppose that $P \in \mathbf{Pos}(\Delta^*)$, say $P = (P, \leq, \ell)$. We define $P^\smile := (P, \leq^R, \ell^\smile)$, where $\leq^R$ is the partial order (6) and*

$$u(\ell^\smile) \quad = \quad (u\ell)^\smile,$$

for all $u \in P$. Equipped with the operation ${}^\smile$, $\mathbf{Pos}(\Delta^*)$ is an involuted bimonoid denoted $\mathbf{Pos}^\smile(\Sigma^*)$.

Note that the series-parallel Σ-labeled posets determine an involuted bimonoid $\mathbf{SP}^\smile(\Sigma)$.

Given the set Σ, define

$$\Gamma \quad := \quad \{(\sigma, i), (\sigma', i) \mid \sigma \in \Sigma, i \geq 1\}.$$

Lemma 0.6 *There is an injective involuted bimonoid morphism $\mathcal{L}_\Sigma^\smile \to \mathcal{L}_\Gamma^R$.*

Proof. For each letter $\sigma \in \Sigma$, define

$$\begin{array}{rcl} L_\sigma & := & \{(\sigma, i)(\sigma', i) \mid i \geq 1\} \\ L_{\sigma'} & := & \{(\sigma', i)(\sigma, i) \mid i \geq 1\} \quad = \quad L_\sigma^R. \end{array}$$

When $w = x_1 \dots x_n \in (\Sigma \cup \Sigma')^*$ with $x_i \in \Sigma \cup \Sigma'$, $i \in [n]$, define

$$L_w \quad := \quad L_{x_1} \cdot \dots \cdot L_{x_n}.$$

Finally, for each $L \subseteq (\Sigma \cup \Sigma')^*$, set

$$L\varphi \quad = \quad \bigcup \{L_w \mid w \in L\}.$$

The function φ is an involuted bimonoid morphism. To prove that φ is injective, note that L is determined by $L\varphi$ in the following way. Let K denote the set of words u in Γ^* such that each letter (σ, i) or (σ', i) occurs at most once in u. Note that if $z \in K \cap L\varphi$, then for each $\sigma \in \Sigma$ and $i \geq 1$, the letter (σ, i) occurs in w iff so does the letter (σ', i). Let $z\psi$ denote the word obtained from z by deleting each letter (σ', i) preceded by an occurrence of the letter (σ, i), and deleting each letter (σ, i) preceded by an occurrence of the letter (σ', i). Then

$$L \quad = \quad \{z\psi \mid z \in K \cap L\varphi\}. \quad \square$$

Remark 0.7 *For the involuted bimonoid morphism φ we have*

$$L_1 \subseteq L_2 \quad \Longleftrightarrow \quad L_1\varphi \subseteq L_2\varphi$$

and

$$(\bigcup_{i \in I} L_i)\varphi \quad = \quad \bigcup_{i \in I} L_i\varphi,$$

for all $L_1, L_2, L_i \subseteq \Sigma^*$, $i \in I$.

Lemma 0.8 *There is an injective involuted bimonoid morphism* $\mathbf{Pos}^{\smile}(\Sigma^*) \to \mathbf{Pos}^R(\Delta^*)$*, where* Δ *is the set* $\Sigma \cup \Sigma'$*.*

Proof. Define the map

$$\varphi : \mathbf{Pos}(\Delta^*) \to \mathbf{Pos}(\Delta^*)$$

in the following way. For each $P \in \mathbf{Pos}(\Delta^*)$, let $P\varphi$ be the poset obtained from P by relabeling each vertex of P labeled w by the word $w\psi$, where ψ is the monoid homomorphism $\Delta^* \to \Delta^*$ with $\sigma\psi = \sigma\sigma'$ and $\sigma'\psi = \sigma'\sigma$, for each $\sigma \in \Sigma$. □

Lemma 0.9 *There is an injective involuted bimonoid morphism* $\mathbf{Pos}^{\smile}(\Sigma) \to \mathbf{Pos}^R(\Delta)$*, where* Δ *is the set* $\Sigma \cup \Sigma'$*.*

Proof. Given a poset $P \in \mathbf{Pos}^{\smile}(\Sigma)$, let $P\varphi$ be the poset obtained from P by replacing each vertex v with the 2-element chain $v_1 \leq v_2$ with v_1 labeled σ and v_2 labeleled σ' provided the label of v is $\sigma \in \Sigma$. If v is labeled by $\sigma' \in \Sigma'$, then label v_1 by σ' and v_2 by σ. □

Let $\mathbf{Lg}^{\smile}$ denote the variety of all involuted bimonoids.

Theorem 0.10 *The involuted bimonoid* $\mathbf{SP}^{\smile}(A)$ *is freely generated in* $\mathbf{Lg}^{\smile}$ *by the map* η *which maps a letter* $a \in A$ *to the poset which has a single vertex labeled* a*.*

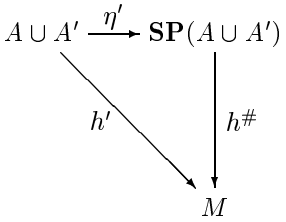
Proof. Recall that $\mathbf{SP}^{\smile}(A) = (\mathbf{SP}(A \cup A'), ^{\smile})$, where $A' = \{a' \mid A \in A\}$ is a disjoint copy of the set A. The function η has a factorization

$$A \hookrightarrow A \cup A' \xrightarrow{\eta'} \mathbf{SP}(A \cup A'),$$

where the first arrow is the inclusion of A into $A \cup A'$ and η' assigns to each letter in $A \cup A'$ the corresponding labeled poset. Suppose that $M = (M, ^{\smile})$ is an involuted bimonoid and that h is a function $A \to M$. Extend h to a function $h' : A \cup A' \to M$ by defining

$$ah' = ah \tag{8}$$
$$a'h' = (ah)^{\smile}, \tag{9}$$

for all $a \in A$. Since the bimonoid $\mathbf{SP}(A \cup A')$ is freely generated by η', there is a bimonoid morphism $h^{\#} : \mathbf{SP}(A \cup A') \to M$ with $\eta \circ h^{\#} = h'$, i.e., such that the triangle



commutes. Each poset in $\mathbf{SP}(A \cup A')$ can be obtained from the posets $a\eta'$ and $a'\eta'$, where $a \in A$, and the poset 1 by applying the operations of series and parallel composition. It follows by a straight forward induction argument on the structure of $P \in \mathbf{SP}(A \cup A')$ that

$$P^{\smile}h^{\#} = (Ph^{\#})^{\smile}. \tag{10}$$

Indeed, when $P = 1$, (10) holds obviously, and when $P = a\eta'$ or $P = a'\eta'$, (10) holds by (8) and (9) and since $\eta' \circ h^{\#} = h'$. Suppose that $P = P_1 \cdot P_2$ and that (10) holds for the posets P_1 and P_2. Then:

$$\begin{aligned} P^{\smile}h^{\#} &= (P_1 \cdot P_2)^{\smile}h^{\#} \\ &= ({P_2}^{\smile} \cdot {P_1}^{\smile})h^{\#} \\ &= {P_2}^{\smile}h^{\#} \cdot {P_1}^{\smile}h^{\#} \\ &= (P_2h^{\#})^{\smile} \cdot (P_1h^{\#})^{\smile} \\ &= (P_1h^{\#} \cdot P_2h^{\#})^{\smile} \\ &= ((P_1 \cdot P_2)h^{\#})^{\smile} \\ &= (Ph^{\#})^{\smile}, \end{aligned}$$

by (1) and since $h^{\#}$ is a bimonoid morphism. When $P = P_1 \otimes P_2$ the argument using (2) is similar. □

Corollary 0.11 *The variety* $\mathbf{Lg}^{\smile}$ *is generated by each of the following classes of involuted bimonoids.*

- *The class consisting of the structures* $\mathcal{L}^R_\Sigma$.
- *The class of the involuted bimonoids* $\mathbf{Pos}^R(\Sigma)$ *or* $\mathbf{Pos}^R(\Sigma^*)$.
- *The class of the involuted bimonoids* $\mathbf{Pos}^{\smile}(\Sigma)$ *or* $\mathbf{Pos}^{\smile}(\Sigma^*)$.

Remark 0.12 $\mathbf{Lg}^{\smile}$ *is also generated by the language structures* $\mathcal{F}^R_\Sigma$, $\mathcal{R}^R_\Sigma$, $\mathcal{F}^{\smile}_\Sigma$ *or* $\mathcal{R}^{\smile}_\Sigma$, *or the poset structures* $\mathbf{SP}^R(\Sigma^*)$, $\mathbf{SP}^R(\Sigma)$, $\mathbf{SP}^{\smile}(\Sigma^*)$ *or* $\mathbf{SP}^{\smile}(\Sigma)$.